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A HOMOTOPY ALGORITHM FOR SOLVING THE INVERSE EIGENVALUE PROBLEM FOR COMPLEX SYMMETRIC MATRICES*1)

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Abstract

A homotopy algorithm for solving the inverse eigenvalue problem for complex symmetric matrices is suggested. Some numerical examples are presented.

§1. Introduction

In this paper we shall consider the following inverse eigenvalue problem.

Problem SCG. Given $n+1$ complex $n \times n$ symmetric matrices $A_0, A_1, \cdots, A_n$, and n complex numbers $\lambda_1, \cdots, \lambda_n$, find n complex numbers $c_1, \cdots, c_n$, such that the matrix $A(c) = A_0 + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Replacing "complex" by "real" in Problem SCG, we obtain the inverse eigenvalue problem for real symmetric matrices, which is called *Problem SRG* for short.

There is a large literature on numerical methods for solving Problem SRG. But, all those methods require choosing an initial value which is sufficiently close to the solution of Problem SRG so as to guarantee the iterative convergence (see [4] and its references for details). In many cases, it often leads to the failure of the algorithms since it is hard to select a valid initial value. Therefore, how to select a valid initial value becomes a very important problem. However, so far as we know, there is no literature on this problem. In this paper, we propose a homotopy algorithm for solving Problem SCG. Theoretically, this method is independent of the selection of an initial value. Numerical experiments also show its handiness in selecting a valid initial value.

The paper is organized as follows. In §2 we construct a homotopy for solving Problem SCG and prove the existence of homotopy paths by using some results in differential topology and topological degree theory. In §3 we devise a homotopy algorithm for solving Problem SCG by following the homotopy paths . In §4 we give some numerical examples.

Notation and Definitions. Throughout this paper we use the following notation. $C^{m\times l}$ is the set of all $m\times l$ complex matrices. C^m is the set of all m-dimensional complex

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column vectors and $C^1 = C$. R^m is the set of all m-dimensional real column vectors and $R^1 = R$. $SC^{n\times n}$is the set of all $n \times n$ complex symmetric matrices. The norm $\|\ \ \|$ stands for both arbitrary vector norm and compatible matrix norm. The superscript T is for transpose. I is the $n \times n$ identity matrix. e_i is the ith column of I. S_n denotes the set of all permutations of $\{1,\cdots,n\}$.

For arbitrary $x = (x_1,\cdots,x_n)^T \in C^n$, we use $D(x)$ to denote the diagonal matrix diag $(x_1,\cdots,x_n)$, i.e., $D(x) = \text{diag}\,(x_1,\cdots,x_n)$. For arbitrary $c = (c_1,\cdots,c_n)^T \in C^n$ and $A_k \in SC^{n\times n}$, $k = 0,1,\cdots,n$, we define

$$A(c) = A_0 + \sum_{k=1}^{n} c_k A_k.$$

§2. The Construction of Homotopy and Its Properties

Let $A^0,\cdots,A^n \in SC^n$ and $\lambda = (\lambda_1,\cdots,\lambda_n)^T \in C^n$, with $\lambda_i \neq \lambda_j \quad i \neq j$. Define $f : C^{n\times n} \times C^n \longrightarrow C^{n^2+n}$ by

$$f(X,c) = \begin{pmatrix} f_1(X,c) \\ \vdots \\ f_n(X,c) \end{pmatrix} \text{ with } f_i(X,c) = \begin{pmatrix} (A(c) - \lambda_i I)x_i \\ \\ \frac{1}{2}(x_i^T x_i - 1) \end{pmatrix}$$

and $g : C^{n\times n} \times C^n \times C^n \times C^n \longrightarrow C^{n^2+n}$

$$g(X,c,d,\omega) = \begin{pmatrix} g_1(X,c,d,\omega) \\ \vdots \\ g_n(X,c,d,\omega) \end{pmatrix} \text{ with } g_i(X,c,d,\omega) = \begin{pmatrix} (DC - \omega_i I)x_i \\ \\ \frac{1}{2}(x_i^T x_i - 1) \end{pmatrix}$$

where $X = (x_1,\cdots,x_n) \in C^{n\times n}$, $x_i \in C^n$, $i = 1,\cdots,n$, $c = (c_i), d = (d_i), \omega = (\omega_i) \in C^n$, and $D = D(d), C = D(c)$.

A classical result on diagonalizable complex symmetric matrices states that if $B \in SC^{n\times n}$, then B is diagonalizable if and only if there exists a $Q \in C^{n\times n}$ such that

$$B = Q\Lambda Q^T \text{ and } Q^T Q = I$$

where Λ is an $n \times n$ diagonal matrix[5].

So, from the definition of f , it follows that

(1) c^* is a solution of Problem SCG if and only if there exists an $X^* \in C^{n\times n}$ such that $f(X^*,c^*) = 0$.

On the other hand, from the definition of g, we know that

(2) for any $d,\omega \in C^n$ with $d_i \neq 0, i = 1,\cdots,n$, if we define

$$\Gamma_0(d,\omega) = \{(X,c) \in C^{n\times n} \times C^n \quad | g(X,c,d,\omega) = o\},$$

then

$$\Gamma_0(d,\omega) = \{(P_\pi E, \quad c_\pi) | \pi \in S_n, \quad E = \text{diag}\,(\varepsilon_i) \\ \varepsilon_i = 1 \text{ or } -1, \quad 1 \leq i \leq n\} \tag{2.1}$$

where $P_\pi e_i = e_{\pi(i)}, c_\pi = (c_i^\pi) \in C^n, c_{\pi(i)}^\pi = \dfrac{\omega_i}{d_{\pi(i)}}, i = 1, \cdots, n.$

Now, we define a homotopy $h : C^{n\times n} \times C^n \times [0,1] \times C^n \times C^n \longrightarrow C^{n^2+n}$ by

$$h(X,c,t,d,\omega) = \begin{pmatrix} h_1(X,c,t,d,\omega) \\ \vdots \\ h_n(X,c,t,d,\omega) \end{pmatrix}$$

with

$$h_i(X,c,t,d,\omega) = tf_i(X,c) + (1-t)g_i(X,c,d,\omega). \tag{2.2}$$

Before recounting the main theorem of this section , we recall the definition of regular value [6].

Let $F : U \longrightarrow R^p$ be a smooth map where U is an open subset of R^m. Then any $y \in R^p$ is called a regular value provided that

$$\text{Range}\,(DF(x)) = R^p \text{ for all } x \in F^{-1}(y),$$

where $DF(x)$ denotes the Jacobi matrix of F at x.

Let $z = (z_1, \cdots, z_m)^T \in C^m$. Then z may be regarded as one point $(x_1, y_1, \cdots, x_m, y_m)^T$ in R^{2m}, where $z_k = x_k + iy_k$. In this sense we may regard C^m as R^{2m}. And for any map

$$F : C^m \longrightarrow C^p,$$

we may regard F as one map from R^{2m} to R^{2p}, and this is fully illustrated by the case of $m = p = 1$, if

$$F : C \longrightarrow C, \qquad F(x+iy) = u(x+iy) + iv(x+iy).$$

Then, we regard F as one map from R^2 to R^2 as follows:

$$F(x,y) = (u(x,y), v(x,y))^T$$

where $u(x,y) = u(x+iy), \quad v(x,y) = v(x+iy)$.

Therefore, here and here later, we regard a complex space as a real space of two real dimensions and a complex analytic map in m complex dimensions as a real analytic map in $2m$ real dimensions as stated above when needed.

The following theorem is the main result of this section.

Theorem 2.1. *For almost any $(d,\omega) \in C^n \times C^n$ (in the sense of Lebesgue measure), the zero set*

$$\Gamma(d,\omega) = \{(X,c,t) \in C^{n\times n} \times C^n \times [0,1) | h(X,c,t,d,\omega) = 0\}$$

consists of $2^n n!$ disjoint analytic arcs $C_i(t), i = 1, \cdots, 2^n n!$, emanating from the $2^n n!$ points of $\Gamma_0(d,\omega)$, and can be parametrized by t in the interval $[0,1)$. And, for each $m, 1 \le m \le 2^n n!$, either

(1) $\lim_{t\to 1-0} \|C_m(t)\| = \infty$, *or*

(2) $\lim_{t\to 1-0} C_m(t) = (X^{(0)}, c^{(0)}, 1)$, *where* $(X^{(0)}, c^{(0)})$ *is an isolated zero of* f. *Moreover, if* 0 *is a regular value of* f, *then any zero of* f *can be obtained in this manner.*

The remainder of this section constitutes the proof of Theorem 2.1. In fact, we only have to prove that for almost any $(d, \omega) \in C^n \times C^n$,the zero set $\Gamma(d,\omega)$ is bounded and 0 is the regular value of $h(X,c,t,d,\omega)$ on $C^{n\times n} \times C^n \times [0,1) \times C^n$. This is because if this has been done, then Theorem 2.1 will follow by a standard argument[3,6]. Therefore, we shall only prove the boundedness of $\Gamma(d,\omega)$ and the regularity of $h(X,c,t,d,\omega)$ (see Lemma 2.2 and Lemma 2.5 for details) in the remainder of this section , and the details of the proof of Theorem 2.1 will be omitted here.

First, we cite a theorem which can be found in [3].

Theorem 2.2 (Transversality Theorem). *Let* $U \subset R^m$ *and* $V \subset R^q$ *be open sets , and*

$$F: \quad U \times V \longrightarrow R^p$$

be c^r *smooth, where* $r > \max\{0, m-p\}$. *Suppose for some set* $S \subset U$ *that* $y \in R^p$ *is a regular value of* F *on* $S \times V$. *Then for almost every* $c \in V$, y *is a regular value of* $F(.,c)$ *on* S.

Then, we prove several lemmas.

Lemma 2.1. *Let* B *be an* $n \times n$ *diagonalizable complex symmetric matrix,* λ *a simple eigenvalue of* B , *and* x *the corresponding eigenvector satisfying* $x^Tx = 1$. *Then the matrix*

$$G = \begin{pmatrix} A-\lambda I & x \\ x^T & 0 \end{pmatrix}$$

is nonsingular.

Proof. Since B is diagonalizable, there exists a matrix $Q \in C^{n\times n}$ such that

$$Q^TAQ = \Lambda, \quad Q^TQ = I$$

where $\Lambda = \text{dial}(\lambda, \lambda_2, \cdots, \lambda_n)$, and $Q = (x, Q_1)$.

Let

$$S = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

Then S is nonsingular and

$$S^TGS = \begin{pmatrix} Q^T(A-\lambda I)Q & Q^Tx \\ x^TQ & 0 \end{pmatrix} = \begin{pmatrix} \Lambda-\lambda I & e_1 \\ e_1^T & 0 \end{pmatrix}.$$

Since λ is a simple eigenvalue of B, $\lambda_i \neq \lambda, i = 2, \cdots, n$. Thus, S^TGS is nonsingular, and so is G.

Lemma 2.2. *For almost any* $\omega \in C^n$, 0 *is a regular value of*

$$h_\omega(X,c,t,d) = h(X,c,t,d,\omega)$$

on $C^{n\times n} \times C^n \times [0,1) \times C^n$.

Proof. Let

$$\Omega = \Big\{\omega = (\omega_1, \cdots, \omega_n) \in C^n, \quad |\omega_{i_0} - \omega_{j_0}| = \frac{t_0}{1-t_0}(\lambda_{i_0} - \lambda_{j_0}). \quad \text{for some} \quad 1 \le i_0 < j_0 \le n, \text{ and } 0 \le t_0 < 1\Big\},$$

and for given i, j, t, $1 \le i < j \le n$ and $0 \le t < 1$, let

$$\Omega(i,j,t) = \left\{\omega = (\omega_1, \cdots, \omega_n) \in C^m \quad |\omega_i - \omega_j = \frac{t}{1-t}(\lambda_i - \lambda_j)\right\}.$$

Then $\Omega(i,j,t)$ is a closed subspace of C^n and its complex dimension is $n-1$.

Now, let

$$\Omega(i,j) = \bigcup_{0 \le t < 1} \Omega(i,j,t).$$

Then, $\Omega(i,j)$ has real dimension $2n-1$, and hence is a zero measure subset of C^n. By

$$\Omega = \bigcup_{1 \le i < j \le n} \Omega(i,j)$$

we conclude that Ω is a zero measure closed subset of C^n.

Since

$$\frac{\partial h}{\partial(X,\omega)} = \begin{pmatrix} \dfrac{\partial h_1}{\partial X} & \dfrac{\partial h_1}{\partial \omega} \\ \vdots & \vdots \\ \dfrac{\partial h_n}{\partial X} & \dfrac{\partial h_n}{\partial \omega} \end{pmatrix}$$

where

$$\frac{\partial h_i}{\partial X} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right], \frac{\partial h_i}{\partial x_k} = 0, \quad i \ne k,$$

$$\frac{\partial h_i}{\partial x_i} = \begin{pmatrix} (1-t)[DC - \omega_i I] + t[A(c) - \lambda_i I] \\ x_i^T \end{pmatrix}, \frac{\partial h_i}{\partial \omega} = \left[\frac{\partial h_i}{\partial \omega_1}, \cdots, \frac{\partial h_i}{\partial \omega_n}\right],$$

$$\frac{\partial h_i}{\partial \omega_k} = 0, \quad i \ne k, \frac{\partial h_i}{\partial \omega_i} = \begin{pmatrix} (1-t)x_i \\ 0 \end{pmatrix}, \quad i = 1, 2, \cdots, n,$$

there exists an $(n+1)n \times (n+1)n$ permutation matrix P such that

$$\frac{\partial h}{\partial(X,\omega)} P = \operatorname{diag}(S_1, \cdots, S_n)$$

where

$$S_k = \left[\frac{\partial h_k}{\partial x_k}, \frac{\partial h_k}{\partial \omega_k}\right], \qquad 1 \le k \le n.$$

For any $(X, c, t, d, \omega) \in h^{-1}(0)$ with $\omega \notin \Omega$, from Lemma 2.1, it follows that

$$\tilde{S}_i = \begin{pmatrix} (1-t)[DC - \omega_i I] + t[A(c) - \lambda_i I] & x_i \\ x_i^T & 0 \end{pmatrix}$$

is nonsingular, and so is S_i for any $t \in [0,1)$, and hence rank $\left(\frac{\partial h}{\partial(X,\omega)}\right) = n(n+1)$. Thus, 0 is a regular value of h on $C^{n\times n} \times C^n \times [0,1) \times C^n \times (C^n \setminus \Omega)$. It follows immediately from Theorem 2.2 that for almost any $\omega \in C^n$, 0 is a regular value of h_ω on $C^{n\times n} \times C^n \times [0,1) \times C^n$.

Lemma 2.3. *There exists an open dense subset U of C^n such that for $b = (b_1, \cdots, b_n)^T \in U$ and $s \in R$, the matrix*

$$C + s\sum_{k=1}^{n} b_k c_k A_k$$

has at least one nonzero eigenvalue besides $c = (c_1, \cdots, c_n)^T = 0$, where $C = D(c)$.

Proof. We can easily prove that the matrix

$$C + s\sum_{k=1}^{n} b_k c_k A_k$$

has no nonzero eigenvalue if and only if

$$E_k\left(C + s\sum_{k=1}^{n} b_k c_k A_k\right) = 0, \quad i = 1, 2, \cdots, n \tag{2.3}$$

where $E_k(B)$ denotes the sums of the $k \times k$ principle minors of an $n \times n$ matrix B.

Now we consider the system

$$E_k\left(C + \sum_{k=1}^{n} b_k c_k A_k\right) = 0, \quad k = 1, 2, \cdots, n. \tag{2.4}$$

Similarly to the proof of Theorem 3.3 in [7], we can prove that there exists a proper algebraic subset T of C^n such that for any $b = (b_1, \cdots, b_n) \in C^n \setminus T$, the system (2.4) has no solution except $c = (c_1, \cdots, c_n) = 0$.

Let $U = C^n \setminus RT$, where $RT = \{rb | b \in T,\ r \in R\}$. Since RT is closed and its real dimension is $2n-1$, U is an open dense subset of C^n. Moreover, for $b = (b_1, \cdots, b_n) \in U$, and $s \in R$, the system (2.3) has no solution besides $c = (c_1, \cdots, c_n) = 0$ since $sb \notin T$. Thus, we have proven the lemma.

Lemma 2.4. *There exists an open dense subset V of C^n such that for $d = (d_1, \cdots, d_n)^T \in V$ and $1 \neq t \in R$, the matrix*

$$(1-t)DC + t\sum_{k=1}^{n} c_k A_k$$

has at least one nonzero eigenvalue besides $c = (c_1, \cdots, c_n)^T = 0$, where $D = D(d)$, $C = D(c)$.

Proof. Let U be the set of Lemma 2.3, and let

$$V = \left\{d = (d_i) \in C^n \mid d_i \neq 0,\ 1 \leq i \leq n, \text{ and } \left(\frac{1}{d_1}, \cdots, \frac{1}{d_n}\right) \in U\right\}.$$

Then, we can easily prove that V is an open dense subset of C^n. Moreover, by Lemma 2.3, if $d=(d_i)\in V$ and $1\neq t\in R$, the matrix

$$C+\frac{t}{1-t}\sum_{k=1}^{n}\frac{1}{d_k}c_kA_k$$

has at least one nonzero eigenvalue besides $c=(c_1,\cdots,c_n)^T=0$. Hence the matrix

$$(1-t)\Big(DC+\frac{t}{1-t}\sum_{k=1}^{n}\frac{1}{d_k}d_kc_kA_k\Big)$$

has at least one nonzero eigenvalue besides $(c_1d_1,\cdots,c_nd_n)^T=0$, *i.e.*, besides $(c_1,\cdots,c_n)^T=0$. The proof of the lemma is completed.

Lemma 2.5. *There exists an open dense subset U of C^n such that for $d=(d_i)\in U$ and $\omega=(\omega_i)\in C^n$ with $(1-t)\omega_i+t\lambda_i\neq(1-t)\omega_j+t\lambda_j,\ i\neq j,\ t\in[0,1)$, the set*

$$\{(X,c,t)\in C^{n\times n}\times C^n\times[0,t_0]\quad|h(X,c,t,d,\omega)=0\}$$

is bounded for any $t_0\in[0,1)$.

Proof. Let U be the open dense subset which is selected in Lemma 2.4 . Given $d\in U$ and $\omega=(\omega_i)\in C^n$ with $(1-t)\omega_i+t\lambda_i\neq(1-t)\omega_j+t\lambda_j,\ i\neq j,\ t\in[0,1)$, we consider the following map:

$$H(X,c,t)=h(X,c,t,d,\omega).$$

If there exists some t_0 such that the set

$$\Gamma=\{(X,c,t,)\in C^{n\times n}\times C^n\times[0,t_0]\quad|H(X,c,t)=0\}$$

is not bounded, then there exists a sequence $\left\{X^{(m)},c^{(m)},t_m\right\}\subset\Gamma$ such that

$$\lim_{m\to\infty}(\|X^{(m)}\|+\|c^{(m)}\|)=\infty. \tag{2.5}$$

Now suppose $\lim_{m\to\infty}\|c^{(m)}\|=\infty$. Without loss of generality, we may assume that

$$\lim_{m\to\infty}\frac{c^{(m)}}{\|c^{(m)}\|}=\bar{c}=(\bar{c}_i)\in C^n \quad\text{and}\quad \lim_{m\to\infty}t_m=\tau\in[0,t_0].$$

Then, $\|\bar{c}\|=1$.

Since $H(X^{(m)},c^{(m)},t_m)=0$, the matrix

$$(1-t_m)DC^{(m)}+t_mA(c^{(m)})$$

has the eigenvalues $(1-t_m)\omega_1+t_m\lambda_1,\cdots,(1-t_m)\omega_n+t_m\lambda_n$, where $C^{(m)}=(c_i^{(m)})$, $D=D(d)$. Hence, the matrix

$$\frac{1}{\|c^{(m)}\|}[(1-t_m)DC^{(m)}+t_mA(c^{(m)})]$$

has eigenvalues $\frac{1}{\|c^{(m)}\|}[(1-t_m)\omega_1+t_m\lambda_1],\cdots,\frac{1}{\|c^{(m)}\|}[(1-t_m)\omega_n+t_m\lambda_n]$. Let $m\to\infty$. From the continuity of eigenvalues, it follows that the matrix

$$(1-\tau)D\bar{C}+\tau\sum_{k=1}^{n}\bar{c}_kA_k$$

has no nonzero eigenvalue. Noticing that $d \in U$ and U is the open set in Lemma 2.4, we obtain $\tilde{c}_i = 0$, $i = 1, 2, \cdots, n$. This contradicts $\|\tilde{c}\| = 1$. Therefore, from (2.5), we must have

$$\lim_{m \to \infty} \|X^{(m)}\| = \infty. \tag{2.6}$$

In this case, we may assume that

$$\lim_{m \to \infty} \|c^{(m)}\| = \bar{c} = (\bar{c}_i) \in C^n \quad \text{and} \quad \lim_{m \to \infty} t_m = \tau \in [0, t_0].$$

From a similar argument, it follows that the matrix

$$(1 - \tau) D\bar{C} + \tau A(\bar{c})$$

has eigenvalues $(1-\tau)\omega_1 + \tau\lambda_1, \cdots, (1-\tau)\omega_n + \tau\lambda_n$, where $\bar{C} = D(\bar{c})$. Since $(1-\tau)\omega_i + \tau\lambda_i \neq (1-\tau)\omega_j + \tau\lambda_j$, $i \neq j$, there exist a neighborhood $\mathcal{B}(\bar{c}, \tau)$ of $(\bar{c}, \tau)$ and complex analytic functions $\lambda_1(c,t), \cdots, \lambda_n(c,t) \in C$ and $x_1(c,t), \cdots, x_n(c,t) \in C^n$ such that

$$\begin{cases} ((1-t)DC - tA(c))x_i(c,t) - \lambda_i(c,t)x_i(c,t) = 0, \\ (x_i(c,t))^T x_i(c,t) = 1, \quad (c,t) \in \mathcal{B}(\bar{c}, \tau), \\ \lambda_i(\bar{c}, \tau) = (1-\tau)\omega_i + \tau\lambda_i, \quad i = 1, 2, \cdots, n. \end{cases}$$

Hence, when m is sufficiently large, we have

$$x_i^{(m)} = \pm x_i(c^{(m)}, t_m), \qquad i = 1, 2, \cdots, n,$$

where $x_i^{(m)} = X^{(m)} e_i$.

Thus, for sufficiently large m, we obtain

$$\|X^{(m)}\| \leq \|X(\bar{c}, \tau)\| + 1$$

where $X(\bar{c}, \tau) = [x_1(\bar{c}, \tau), \cdots, x_n(\bar{c}, \tau)]$. This contradicts (2.6) and thus the lemma is proven.

§3. Homotopy Algorithm

In this section we are going to describe a homotopy algorithm by following the homotopy paths with Algorithm L and Algorithm N.

From Theorem 2.1, it follows that if 0 is a regular value of f, then for almost all parameters $(d, \omega) \in C^n \times C^n$, all the solutions of $f(X, c) = 0$ are obtained by following $2^n n!$ paths. And we also know that for every $\pi \in \mathcal{S}_n$, there are 2^n paths emanating from the 2^n points $(P_\pi E, c_\pi)$, $E = \operatorname{diag}(\varepsilon_i), \varepsilon_i = 1$, or $-1, i = 1, \cdots, n$ (see (2.1)), and if some one of these paths terminates at one zero of f, every one of these paths terminates at one zero of f. Now assume these terminal points are $(X_\pi^{(k)}, c_\pi^{(k)})$, $i = 1, 2, \cdots, 2^n$. Then we can easily prove that $c_\pi^{(k)} = c_\pi^{(1)}$, $k = 1, \cdots, 2^n$. Therefore, we only have to follow one of the 2^n paths with the purpose of finding the solution of Problem SCG.

Now assume 0 is a regular value of f. Firstly, we randomly choose a parameter $(d,\omega) \in C^n \times C^n$ with $d_i \neq 0$ and $(1-t)(\omega_i - \omega_j)+, t(\lambda_i - \lambda_j) \neq 0,\ i,j = 1,2,\cdots,n$. Secondly, we choose $n!$ representatives from $\Gamma_0(d,\omega)$ as follows:

$$(P_\pi, c_\pi), \qquad \pi \in S_n$$

where P_π and π are defined by (2.1). Then, we numerically follow the $n!$ paths with a starting point (P_π, c_π) by a predictor-corrector method. Here we use a locally linearly convergent algorithm, i.e., Algorithm L, for a "predictor step" and a locally quadratically convergent algorithm, i.e., Algorithm N, for a "corrector step". Now we describe the homotopy algorithm for solving Problem SCG as follows:

Algorithm H. (1) Given $\pi \in S_n$, let $c^{(0)} = c_\pi$, $X^{(0)} = P_\pi$. Choose step size h (generally, $h \geq 0.01$).

(2) Compute $t_{m+1} = t_m + h$ $(t_0 = 0)$ and form matrices $A_o(t_{m+1}) = t_{m+1}A_0$, $A_k(t_{m+1}) = t_{m+1}A_k + (1-t_{m+1})d_k e_k e_k^T$, $k = 1,2,\cdots,n$, and vector $\lambda(t_{t_{m+1}}) = \omega + t_{m+1}(\lambda - \omega)$.

(3) Compute $\bar{c}^{(m+1)}$ and $\bar{X}^{(m+1)}$ by using Algorithm L to iterate once for matrices $A_k(t_{m+1})$, $k = 0,1,2,\cdots,n$, and vector $\lambda(t_{m+1})$ with the starting point $(c^{(m)}, X^{(m)})$.

(4) Compute $c^{(m+1)}$ and $X^{(m+1)}$ by using Algorithm N to iterate for matrices $A_k(t_{m+1}), k = 0,1,2,\cdots,n$, and vector $\lambda(t_{m+1})$ with the starting point $(\bar{c}^{(m+1)}, \bar{X}^{(m+1)})$.

(5) Stop if $t_{m+1} = 1$.

§4. Numerical Examples

We have tested Algorithm H described in this paper on IBM 4341. Single precision arithmetic was used throughout. Here, we give two numerical examples.

In the examples we use $\bar{c}$ to denote the computing solution of Problem SCG and $\bar{\lambda}_1,\cdots,\bar{\lambda}_n$ the corresponding eigenvalues of $A(\bar{c})$.

Example 4.1. Here $n = 8$,

$$A_0 = \begin{pmatrix} 0 &&&&&&& \\ 4 & 0 &&&&&& \\ -1 & -1 & 0 &&&&& \\ 1 & 2 & 3 & 0 &&&& \\ 1 & 1 & 1 & 1 & 0 &&& \\ 5 & 4 & 3 & 2 & 1 & 0 && \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix},$$

$$A_k = \sum_{j=1}^{k-1} 0.1 \times b_{kj}(e_k e_j^T + e_j e_k^T) + e_k e_k^T, \quad k = 1,\cdots,8,$$

$$\lambda = (-8.06, 4.48, 7.78, 16.5, 17.3, 25.2, 31.98, 45.8)^T$$

where

$$B=(b_{ij})=\begin{pmatrix} 0 &&&&&&& \\ 1 & 0 &&&&&& \\ -1 & 1 & 0 &&&&& \\ 1 & -1 & 1 & 0 &&&& \\ -1 & 1 & -1 & 1 & 0 &&& \\ 1 & -1 & 1 & -1 & 1 & 0 && \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

We have calculated this example with Algorithm L, Algorithm N and Algorithm H. We choose $\varepsilon = 10^{-6}$ as the termination scalar and the starting point $(c^{(0)}, X^{(0)})$ with $c^{(0)} = (1, 4.5, 9.5, 16.5, 19, 25, 31, 36)^T$ and $X^{(0)}$ a matrix of eigenvectors of $A(c^{(0)})$. With Algorithm L, after 33 iterations we obtain a convergent solution. With Algorithm N, after four iterations we also obtain a convergent solution. The numerical results are shown in Table 1 and Table 2, respectively.

Table 1

i	$\bar{c}_i$	$\lambda_i - \bar{\lambda}_i$
1	1.050880	0.022952
2	4.955760	0.006259
3	9.958759	−0.029887
4	15.518641	0.003754
5	19.330276	−0.001343
6	25.067154	−0.002838
7	30.131516	−0.000153
8	34.986633	0.000031

Table 2

i	$\bar{c}_i$	$\lambda_i - \bar{\lambda}_i$
1	0.990784	0.000079
2	4.597589	−0.000084
3	10.594908	−0.000084
4	15.555343	−0.000137
5	19.023209	−0.000076
6	25.136169	−0.000290
7	30.111267	−0.000366
8	34.990723	−0.000229

However, if we choose the starting point $(c^{(0)}, X^{(0)})$ with $c^{(0)}$ = (-5, 4.5, 8, 16.5, 19, 25, 32, 45)^T and $X^{(0)}$ a matrix of eigenvectors of $A(c^{(0)})$, both algorithms are not yet convergent after 55 iterations . But, with Algorithm H, we choose the same termination scalar and

$$d=(1,1,1,1,1,1,1,1)^T,\quad w=(-5,4.5,8,16.5,19,25,32,45)^T,$$

$$\pi:\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&3&4&5&6&7&8\end{pmatrix},\ h=0.2$$

and obtain a convergent solution shown in Table 3. The number of iterations in each corrector step is at most 5.

Table 3

i	$\bar{c}_i$	$\lambda_i - \bar{\lambda}_i$
1	0.990762	0.000155
2	4.597559	-0.000083
3	10.594975	-0.000079
4	15.555345	-0.000153
5	19.023178	-0.000107
6	25.136154	-0.000275
7	30.111282	-0.000336
8	34.990707	-0.000275

Example 4.2. Here $n = 4$,

$$A_0 = \begin{pmatrix} 0 & & & \\ 2 & 0 & & \\ 3 & 2 & 0 & \\ 1 & 2 & 3 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & & & \\ 0.001 & 0 & & \\ 0.001 & 0.001 & 0 & \\ 0 & 0.001 & 0.001 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & & & \\ -0.001 & 1 & & \\ 0 & -0.001 & 0 & \\ 0 & 0 & -0.001 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & & & \\ 0.002 & 0 & & \\ 0.002 & 0.002 & 1 & \\ 0.002 & 0.002 & 0.002 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & & & \\ 0.002 & 0 & & \\ 0.001 & 0.002 & 0 & \\ 0 & 0.001 & 0.002 & 1 \end{pmatrix}, \qquad \lambda = (-30, -10, 10, 30)^T.$$

We have calculated this example with Algorithm H and obtained its two computing solutions.

We choose $\varepsilon = 10^{-6}$ as the termination scalar and

$$d = (1,1,1,1)^T, \quad \omega = (-30, -10, 10, 30)^T, \quad h = 0.2,$$

$$\pi_1 : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \pi_2 : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

We obtain the numerical results as follows.

Table 4

i	$\bar{c}_i$	$\lambda_i - \bar{\lambda}_i$
1	-29.585205	0.000061
2	-9.862612	-0.000001
3	10.100522	-0.000123
4	29.347290	-0.000061

Table 5

i	$\bar{c}_i$	$\lambda_i - \bar{\lambda}_i$
1	29.525787	0.000061
2	9.947137	0.000076
3	-9.980934	0.000021
4	-29.491989	-0.000061

Appendix

In this appendix we shall state only two iterative methods, which are used in Algorithm H, for solving the problem SCG; the details will be omitted.

Given $A_k = (a_{(ij)}^{(k)}) \in C^{n\times n}$, $k = 0, 1, \cdots, n$, and $\lambda = (\lambda_1, \cdots, \lambda_n) \in C^n$ with $\lambda_i \neq \lambda_j, i \neq j$, assume that the corresponding Problem SCG has a solution $c = (c_1^*, \cdots, c_n^*)^T$.

Algorithm L. Choose a starting value $c^{(0)}$. Form $A(c^{(0)})$ and find its eigenvectors $q_1^{(0)}, \cdots, q_n^{(0)}$. Compute

$$\beta_i^{(0)} = (q_i^{(0)})^T q_i^{(0)}, \quad i = 1, \cdots, n.$$

For $m = 0, 1, 2, \cdots$,

(1) Compute

$$b^{(m)} = \Big(\sum_{j=1}^{n} \lambda_j (q_{1j}^{(m)})^2 / \beta_j^{(m)} - a_{11}^{(0)}, \cdots, \sum_{j=1}^{n} \lambda_j (q_{nj}^{(m)})^2 / \beta_j^{(m)} - a_{nn}^{(0)}\Big)^T \in C^n$$

where $q_{ij}^{(m)} = e_i^T q_j^{(m)}$.

(2) Compute $c^{(m+1)}$ by solving

$$M c^{(m+1)} = b^{(m)}$$

where the matrix $M = (m_{ij})$ is defined by

$$m_{ij} = a_{ii}^{(j)}, \quad i, j = 1, \cdots, n.$$

(3) Form $A(c^{(m+1)})$. Solve the n linear systems

$$(A(c^{(m+1)}) - \lambda_i I) y_i = q_i^{(m)}, \quad i = 1, \cdots, n$$

and compute

$$q_i^{(m+1)} = y_i / \|y_i\|_\infty$$

and

$$\beta_i^{(m+1)} = (q_i^{(m+1)})^T q_i^{(m+1)}, \quad i = 1, \cdots, n.$$

Algorithm L is a generalization of Algorithm A for solving the additive inverse eigenvalue problem in [8]. In some conditions we can prove that this algorithm is locally linearly convergent.

Algorithm N. Choose a starting value $(c^{(0)}, x_1^{(0)}, \cdots, x_n^{(0)})$, where $c^{(0)}, x_i^{(0)} \in C^n$.

For $m = 0, 1, \cdots$,

(1) Compute $x_i^{(m+1)}, \alpha_i^{(m+1)}, i = 1, \cdots, n$, by solving

$$\begin{cases} (A(c^{(m)}) - \lambda_i I) x_i^{(m+1)} + \alpha_i^{(m+1)} x_i^{(m)} = 0 \\ (x_i^{(m)})^T x_i^{(m+1)} = \frac{1}{2}((x_i^{(m)})^T x_i^{(m)} + 1) \\ i = 1, \cdots, n, \end{cases}$$

where $\alpha_i^{(m+1)} \in C$, $i = 1, \cdots, n$.

(2) Form

$$b^{(m)} = \left(\alpha_1^{(m+1)}\left(x_1^{(m)}\right)^T x_1^{(m)}, \cdots, \alpha_n^{(m+1)}\left(x_n^{(m)}\right)^T x_n^{(m)}\right)^T$$

and

$$J^{(m)} = \begin{pmatrix} x_1^{(m)T} A_1 x_1^{(m)} & \cdots & x_1^{(m)T} A_n x_1^{(m)} \\ \vdots & \ddots & \vdots \\ x_n^{(m)T} A_1 x_n^{(m)} & \cdots & x_n^{(m)T} A_n x_n^{(m)} \end{pmatrix}.$$

(3) Compute $c^{(m+1)}$ by solving

$$J^{(m)}(c^{(m+1)} - c^{(m)}) = b^{(m)}.$$

The basic idea of this algorithm is applying Newton's method to the equivalent nonlinear system

$$f(X, c) = 0,$$

where f is defined in §2. By a standard argument (see [4]), it follows that the iterates $\{c^{(m)}\}$ generated by Algorithm N converge quadratically to c^* when $(c^{(0)}, x_1^{(0)}, \cdots, x_n^{(0)})$ is sufficiently close to the solution $(c^*, x_1^*, \cdots, x_n^*)$, where x_i^* is the eigenvalue of $A(c^*)$ associated with λ_i.

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