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AN IMPROVED EIGENVALUE PERTURBATION BOUND ON A NONNORMAL MATRIX AND ITS APPLICATION IN ROBUST STABILITY ANALYSIS*

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Abstract

This paper presents an improved estimation of the eigenvalue perturbation bound developed by the author. The result is useful for robust stability analysis of linear control systems.

§1. Introduction

The research on robust control systems has become one of the most attractive areas in recent studies [1],[4],[5]. An important aspect in this field is robust stability analysis. The problem under consideration can usually be reduced as an estimation of the eigenvalue perturbation bound for a prescribed matrix. Mathematically, the proposed problem is also important, especially in numerical analysis. Therefore, many contributions (such as Bauer and Fike [3], Kahan et al. [6]) have been made for this problem. The main purpose of this paper is to improve the estimation of their results, because in control theory, to get a more accurate estimate on robust stability is very important.

§2. Main Results

Theorem 1. *Suppose that $A = Q^{-1}JQ \in C^{n*n}$, and J is a Jordan matrix with the order of its largest block being m. Then for an arbitrary $u \in \lambda(B)$, where $B = A + E$, there must exist a $\lambda \in \lambda(A)$ such that, if m is an even number,*

$$\frac{|\lambda - u|^m}{(1 + 2|\lambda - u| + \cdots + \underbrace{\frac{m}{2}|\lambda - u|^{\frac{m}{2}-1} + \cdots + \frac{m}{2}|\lambda - u|^{2m-3-\frac{m}{2}}}_{\text{coefficients in these items are all } \frac{m}{2}} + \cdots + 2|\lambda - u|^{2m-3} + |\lambda - u|^{2m-2})^{\frac{1}{2}}} \leq \|Q^{-1}EQ\|_2 \tag{1}$$

and if m is an odd number,

$$\frac{|\lambda - u|^m}{(1 + 2|\lambda - u| + \cdots + \underbrace{\frac{m+1}{2}|\lambda - u|^{\frac{m+1}{2}-1} + \cdots + \frac{m+1}{2}|\lambda - u|^{2m-3-\frac{m+1}{2}}}_{\text{coefficients in these items are all } \frac{m+1}{2}} + \cdots + 2|\lambda - u|^{2m-3} + |\lambda - u|^{2m-2})^{\frac{1}{2}}}$$

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$$\leq \|Q^{-1}EQ\|_2. \tag{2}$$

Proof. If $u \in \lambda(A)$, the result is obvious. Therefore, we only consider $u \notin \lambda(A)$. According to the proof of Theorem 8 in [6], we have

$$\|(J-uI)^{-1}\|_2^{-1} \leq \|QEQ^{-1}\|_2. \tag{3}$$

Since $\sigma_{\min}(J-uI) = \|(J-uI)^{-1}\|_2^{-1}$, where $\sigma_{\min}(J-uI)$ represents the minimal singular value of $(J-uI)$, Kahan, Parlett, and Jiang suggested estimating the minimum eigenvalue for

$$T_1 = J_1J_1^H = \operatorname{diag}\left(\begin{pmatrix} 1+|\delta|^2 & (\delta)^* \\ \delta & 1+|\delta|^2 \end{pmatrix} \cdots \begin{pmatrix} 1+|\delta|^2 & \delta^* \\ \delta & 1+|\delta|^2 \end{pmatrix}, \begin{pmatrix} 1+|\delta|^2 & \delta^* \\ \delta & |\delta|^2 \end{pmatrix}\right)$$

where $\delta = \lambda - u, \lambda \in \lambda(A), \lambda \neq u$, and J_1 is an arbitrary kth order Jordan block of the matrix $(J-uI)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be the eigenvalues of T_1. Since

$$\lambda_k = \frac{\lambda_1 \cdots \lambda_k}{\lambda_1 \cdots \lambda_{k-1}} = \frac{det(T_1)}{\lambda_1 \cdots \lambda_{k-1}} = \frac{|\delta|^{2k}}{\lambda_1 \cdots \lambda_{k-1}}, \tag{4}$$

the main task is to estimate the product of $\lambda_1 \cdots \lambda_{k-1}$.

Let $C_r(T)$ be the rth compound or the rth adjugate of T, and $\lambda_1^{(r)}$ be the maximal eigenvalue of $C_r(T)$. Then, according to [2], we have

$$\lambda_1^{(k-1)} = \lambda_1 \cdots \lambda_{k-1}. \tag{5}$$

Therefore, it is necessary to estimate merely $\lambda_1^{(k-1)}$. Denote the element of the ith row and the jth column of $C_{k-1}(T_1)$ by $C_{k-1}(T_1)_{ij}$. Then

$$\lambda_1^{(k-1)} \leq \max_{1\leq i\leq k} \sum_{j=1}^{k} |C_{k-1}(T_1)_{ij}| \tag{6}$$

is a well known result. From

$$T_1 \operatorname{adj}(T_1) = |T_1|I = |\delta|^{2k}I$$

we have

$$\operatorname{adj}(T_1) = |\delta|^{2k}T_1^{-1} \tag{7}$$

while

$$T_1^{-1} = \frac{1}{|\delta|^2}\begin{pmatrix} \begin{pmatrix} 1 & -\frac{1}{\delta} & \\ -\frac{1}{\delta} & 1+\frac{1}{|\delta|^2} & \\ (-\frac{1}{\delta^*})^2 & (-\frac{1}{\delta^*})(1+\frac{1}{|\delta|^2}) & \end{pmatrix} & \cdots \underbrace{a_{ij} = (-\frac{1}{\delta})^{j-i}(1+\frac{1}{|\delta|^2}+\cdots+\frac{1}{|\delta|^{-2(i-1)}})}_{i>j} \\ \underbrace{a_{ij} = (-\frac{1}{\delta^*})^{i-j}(1+\frac{1}{|\delta|^2}+\cdots+\frac{1}{|\delta|^{2(j-1)}})}_{j>i} & \cdots (1+\frac{1}{|\delta|^2}+\cdots+\frac{1}{|\delta|^{2(k-1)}}) \end{pmatrix}$$

$$\equiv |\delta|^{-2}F \tag{8}$$

which yields

$$\mathrm{adj}\,(T_1) = |\delta|^{2k-2} F.$$

Since

$$\sum_{i=1}^{k} |C_{k-1}(T_1)_{ir}| = \sum_{r=1}^{k} |C_{k-1}(T_1)_{ir}| \tag{9}$$

and

$$\sum_{r=1}^{k} |C_{k-1}(T_1)_{ir}| = \sum_{r=1}^{i} |C_{k-1}(T_1)_{ir} + \sum_{r=i+1}^{k} |C_{k-1}(T_1)_{ir}| \tag{10}$$

we have

$$\begin{aligned} \sum_{r=1}^{i} |C_{k-1}(T_1)_{ir}| &= \sum_{r=1}^{i} (|\delta|^{2k-i+r-2} + |\delta|^{2k-i+r-4} + \cdots + |\delta|^{2k-i-r} \\ &= |\delta|^{2k-i-1} \\ &+ |\delta|^{2k-i} + |\delta|^{2k-i-2} \\ &+ |\delta|^{2k-i+1} + |\delta|^{2k-i-1} + |\delta|^{2k-i-3} \\ &\vdots \\ &+ |\delta|^{2k-2} + |\delta|^{2k-4} + \cdots + |\delta|^{2(k-i)}, \end{aligned} \tag{11}$$

$$\begin{aligned} \sum_{r=i+1}^{k} |C_{k-1}(T_1)_{ir}| &= \sum_{r=i+1}^{k} (|\delta|^{2k-2-r+i} + |\delta|^{2k-4-r+i} + \cdots + |\delta|^{2k-i-r}) \\ &= |\delta|^{2k-3} + |\delta|^{2k-5} + \cdots + |\delta|^{2k-2i-1} \\ &+ |\delta|^{2k-4} + |\delta|^{2k-6} + \cdots + |\delta|^{2k-2i-2} \\ &+ |\delta|^{2k-5} + |\delta|^{2k-7} + \cdots + |\delta|^{2k-2i-3} \\ &\vdots \\ &+ |\delta|^{k+i-2} + |\delta|^{k+i-4} + \cdots + |\delta|^{k-i}. \end{aligned} \tag{12}$$

Denote by $[x]$ the largest integer smaller than x. Now we divide our discussion into two cases.

Case 1. $i \le [\frac{k+1}{2}]$. From (11), (12), we can directly write

$$\begin{aligned} \sum_{r=1}^{k} C_{k-1}(T_1)_{ir} &= |\delta|^{2k-2} + 2|\delta|^{2k-3} + 3|\delta|^{2k-4} + \cdots \\ &+ \underbrace{i|\delta|^{2k-i-1} + \cdots + i|\delta|^{k-2+i}}_{\text{coefficients of these items are all i}} \quad +(i-1)|\delta|^{k-3+i} \\ &+ (i-1)|\delta|^{k-4+i} + \cdots + |\delta|^{K-i+1} + |\delta|^{k-i} \end{aligned} \tag{13}$$

Case 2. $i > [\frac{k+1}{2}]$. Notice that equation (11) can be rewritten as

$$\begin{aligned}\sum_{r=1}^{i}|C_{k-1}(T_1)_{ir}| &= (|\delta|^{2k-2}+|\delta|^{2k-3}+2|\delta|^{2k-4}+2|\delta|_{2k-5} \\ &+ 3|\delta|^{2k-6}+\cdots+\frac{i-1}{2}|\delta|^{2k-i+1}+\frac{i-1}{2}|\delta|^{2k-i} \\ &+ \frac{i+1}{2}|\delta|^{2k-i-1}+\frac{i-1}{2}|\delta|^{2k-i-2}+\frac{i-1}{2}|\delta|^{2k-i-3} \\ &+ \frac{i-3}{2}|\delta|^{2k-i-4}+\cdots+|\delta|+1) \qquad (14)\end{aligned}$$

if i is an odd number. If i is an even number, we have

$$\begin{aligned}\sum_{r=1}^{i}|C_{k-1}(T_1)_{ir}| &= (|\delta|^{2k-2}+|\delta|^{2k-3}+2|\delta|^{2k-4}+2|\delta|^{2k-5}+\cdots \\ &+ \frac{i-2}{2}|\delta|^{2k-i+1}+\frac{i}{2}|\delta|^{2k-2}+\frac{i}{2}|\delta|^{2k-i-1}+\frac{i}{2}|\delta|^{2k-i-2} \\ &+ \frac{i-2}{2}|\delta|^{2k-i-3}+\frac{i-2}{2}|\delta|^{2k-i-4}+\cdots+|\delta|+1). \qquad (15)\end{aligned}$$

If k and i are both odd or both even, equation (12) is equivalent to

$$\begin{aligned}\sum_{r=i+1}^{k}|C_{k-1}(T_1)_{ir}| &\Rightarrow |\delta|^{2k-3}+|\delta|^{2k-4}+2|\delta|^{2k-5}+2|\delta|^{2k-6}+3|\delta|^{2k-7} \\ &+ \cdots+\frac{k-i}{2}|\delta|^{k+i-1}+\frac{k-i}{2}|\delta|^{k+i-2} \\ &+ \frac{k-i-2}{2}|\delta|^{k+i-3}+\cdots+|\delta|^{k-i+1}+|\delta|^{k-i}. \qquad (16)\end{aligned}$$

If $(k-i)$ is an odd number, then

$$\begin{aligned}\sum_{r=i+1}^{k}|C_{k-1}(T_1)_{ir}| &= |\delta|^{2k-3}+|\delta|^{2k-4}+2|\delta|^{2k-5}+2|\delta|^{2k-6}+3|\delta|^{2k-7} \\ &+ \cdots+\frac{k-i-1}{2}|\delta|^{k+i}+\frac{k-i-1}{2}|\delta|^{k+i-1} \\ &+ \frac{k-i+1}{2}|\delta|^{k+i-2}+\frac{k-i-1}{2}|\delta|^{k+i-3} \\ &+ \frac{k-i-1}{2}|\delta|^{k+i-4}+\cdots+|\delta|^{k-i+1}+|\delta|^{k-i}. \qquad (17)\end{aligned}$$

Combining the above results gives the following formulas. If both i and k are odd, then

$$\begin{aligned}\sum_{r=1}^{k}|C_{k-1}(T_1)_{ir}| &= (14)+(16)\le|\delta|^{2k-2}+2|\delta|^{2k-3}+3|\delta|^{2k-4} \\ &+ \cdots+\underbrace{\frac{k+1}{2}|\delta|^{2k-3-\frac{k+1}{2}}+\cdots+\frac{k+1}{2}|\delta|^{\frac{k+1}{2}-1}}_{\text{coefficients of these items are all } \frac{k+1}{2}} \\ &+ \frac{k-1}{2}|\delta|^{\frac{k+1}{2}-2}+\cdots+2|\delta|+1\equiv f_1(\delta). \qquad (18)\end{aligned}$$

If both i and k are even, then

$$\sum_{r=1}^{k}|C_{k-1}(T_1)_{ir}| = (15)+(16) \le |\delta|^{2k-2} + 2|\delta|^{2k-3} + 3|\delta|^{2k-4} + \cdots + \underbrace{\frac{K}{2}|\delta|^{2k-3-\frac{k}{2}} + \cdots + \frac{k}{2}|\delta|^{\frac{k}{2}-1}}_{\text{coefficients of these items are all } k/2} + \frac{k-2}{2}|\delta|^{\frac{k}{2}-2} + \cdots + 2|\delta| + 1 \equiv f_2(\delta). \tag{19}$$

If i is even and k is odd, we have a similar result as follows:

$$\sum_{r=1}^{k}|C_{k-1}(T_1)_{ir}| = (15)+(17) \le f_1(\delta) \tag{20}$$

Finally, if i is odd and k is even , then

$$\sum_{r=1}^{k}|C_{k-1}(T_1)_{ir}| = (14)+(17). \tag{21}$$

Notice that under the condition $i \ge K/2+1$, for the highest power item of equation (17), we have

$$k+i-2 \ge \frac{3k}{2} - 1 \tag{22}$$

and a similar relation for the highest power of equation (14)

$$2k-i-1 \le \frac{3K}{2} - 2. \tag{23}$$

Therefore

$$K+i-2 > 2k-i-1. \tag{24}$$

From the above result, equation (21) becomes

$$\sum_{r=1}^{k}|C_{k-1}(T_1)_{ir}| = (14)+(17) \le f_2(\delta). \tag{25}$$

Since (18), (19), (20), (21) hold for arbitrary $i, i = 1, 2, \cdots, k$, and $k \le m$, combining equations (3), (4), (5), (6), (7), (8), we can directly obtain the inequalities of (1) and (2). This concludes our proof.

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