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MULTISTEP METHODS FOR A CLASS OF HIGHER ORDER DIFFERENTIAL PROBLEMS: CONVERGENCE AND ERROR BOUNDS*1)

Lucas jódar José luis Morera Gregorio Rubio
(*Department of Applied Mathematics, Polytechnical University of Valencia, Spain*)

Abstract

In this paper multistep methods for higher order differential systems of the type $Y^{(r)} = f(t,Y)$ are proposed. Such methods permit the numerical solutions of initial value problems for such systems, providing error bounds and avoiding the increase of the computational cost derived from the standard approach based on the consideration of an equivalent extended first order system.

1. Introduction

Higher order differential systems of the form

$$Y^{(r)}(t) = f(t, Y(t)), \quad a \le t \le b,$$
$$Y^{(i)}(a) = \Omega_i \in \mathbb{C}^{p\times q}, \quad 0 \le t \le r-1, \quad r \ge 2 \tag{1.1}$$

are frequent in a variety of models in physics. These systems arise for example modeling the motion of a system of particles as determined from the laws of classical mechanics such as the interaction of atoms and molecules[2,16,17], the motion of the solar system and space capsules[3,21] and the evolution of star cluster[5]. Other situations where systems of the type (1.1) appear in a natural way may be found in optics[8], quantum theory of scattering[7] or celestial mechanics[3]. Apart from these problems, systems of the type (1.1) arise using the method of lines for solving higher order scalar partial differential systems[20].

(1.1) can be written as an extended first order problem[4]; however, there are advantages in studying methods for problems of the type (1.1) for several reasons:

(a) the transformation of system (1.1) into an extended first order problem increases the computational cost;

(b) the physical meaning of the original magnitudes is lost with the transformation of the system;

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(c) by requiring less generality we may able to produce more efficient algorithms;

(d) useful concepts may be identified, leading to a better understanding of what we require of a numerical method for problems in our chosen class.

Systems of the type (1.1) with $r = 1$ have been treated in [13] for the vector case and in [15] for the matrix case. The special problem

$$Y^{(r)}(t) = f(t), \quad Y^{(i)}(a) = \Omega_i, \quad 0 \leq i \leq r-1, \quad a \leq t \leq b$$

has been treated in [14] for the scalar case.

In this paper, we consider problems of the type (1.1) where f is a bounded, continuous function $f : [a,b] \times \mathbb{C}^{p\times q} \to \mathbb{C}^{p\times q}$ satisfying the Lipschitz condition

$$\|f(t,P) - f(t,Q)\| \leq L\|P-Q\|, \quad P,Q \in \mathbb{C}^{p\times q}. \tag{1.2}$$

This paper is organized as follows. In section 2 some preliminaries about rational matrix functions are included. In section 3 multistep matrix methods for problems of the type (1.1)–(1.2) are introduced and concepts of consistency, zero-stability and convergence are defined. A family of examples is given. Section 4 deals with the study of the discretization error of multistep methods, in particular it is proved that consistent and zero-stable methods are convergent.

If A is a matrix in $\mathbb{C}^{p\times p}$, we denote by $\|A\|$ its 2-norm, defined in [10]. If B is a matrix in $\mathbb{C}^{p\times q}$, we denote by $\sigma(B)$ the set of all the eigenvalues of B and its spectral radiius $\rho(B)$ is the maximum of the set $\{|z|; z \in \sigma(B)\}$. If $z \in \sigma(B)$, the index of z considered as an eigenvalue of B, denoted by $Ind(z,B)$ is the smallest non-negative integer n such that $Ker(B-zI)^n = Ker(B-zI)^{n+1}$, [6]. The number $Ind(z,B)$ coincides with the dimension of the bigest Jordan block of B in which the eigenvalue z appears in the Jordan canonical form of B. An efficient algorithm for computing $Ind(z,B)$ can be found in [1].

In an analogous way to the definition of matrices of class M, given in [9] , we say that a matrix $B \in \mathbb{C}^{p\times p}$ is of class r if for every eigenvalue $z \in \sigma(B)$ such that $|z| = \rho(B)$, every Jordan block of B associated with z has size $s \times s$ with $s \leq r$. Finally, from formulae 0.121 of [11], if q is a positive integer it follows that

$$\sum_{k=1}^{n} k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2}\begin{bmatrix} q \\ 1 \end{bmatrix} B_2 n^{q-1} + \frac{1}{4}\begin{bmatrix} q \\ 3 \end{bmatrix} B_4 n^{q-2} + \frac{1}{6}\begin{bmatrix} q \\ 5 \end{bmatrix} B_6 n^{q-5} + \cdots \tag{1.3}$$

where last term contains either n or n^2 and B_m denotes the m-th Bernouilli number.

2. Preliminaries About Rational Matrix Functions

We begin this section with a result that generalizes lemmas 5.5 and 6.2 of [12].

Theorem 2.1. *Let the polynomial* $p(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_0$ *has only zeros on the unit disk* $|z| \leq 1$ *and those with modulus 1 are of multiplicity not exceding*

r, where r is an integer $r \geq 1$. Let the coefficients γ_n for $n = 0, 1, 2, \cdots$, be defined by

$$\left[\alpha_k + \alpha_{k-1}z + \cdots \alpha_0 z^k\right]^{-1} = \sum_{n\geq 0} \gamma_n z^n, \quad |z| < 1. \tag{2.1}$$

Then there exist constants $\Gamma_0, \Gamma_1, \cdots, \Gamma_{r-1}$ such that

$$|\gamma_n| \leq \Gamma_0 + n\Gamma_1 + n^2\Gamma_2 + \cdots n^{r-1}\Gamma_{r-1}, \quad n \geq 0. \tag{2.2}$$

Proof. We use an induction argument. If $r = 1$ the result coincides with lemma 5.5 of [12]. Let us assume that the result is true for $r = 2, 3, \cdots, r-1$ and let $z_1, \cdots, z_j$ be all the distinct roots of multiplicity r of $p(z)$ with $|z| = 1$ for $1 \leq i \leq j$, and let us consider the factorization

$$p(z) = p_1(z)p_2(z), \quad p_2(z) = \prod_{i=1}^{j}(z - z_i). \tag{2.3}$$

Note that $p_2(z)$ has only j simple roots $z_1, \cdots, z_j$ with $|z_i| = 1$. If $\hat{p}(z) = \alpha_k z^k + \alpha_{k-1}z^{k-1} + \cdots \alpha_0 = z^k p(z^{-1})$ note that $\hat{p}(z)$ can be written in the form

$$\begin{aligned} &\hat{p}(z) = z^k p(z^{-1}) = \hat{p}_1(z)\hat{p}_2(z), \quad s + t = k, \\ &\hat{p}_1(z) = z^s p_1(z^{-1}), \quad \hat{p}_2(z) = z^t p_2(z^{-1}) \end{aligned} \tag{2.4}$$

From lemma 5.5 of [12] applied to the polynomial $p_2(z)$, it follows that

$$\frac{1}{\hat{p}_2(z)} = \sum_{n\geq 0} \gamma_{n,2} z^n, \quad |z| < 1, \quad \psi = \sup_{n\geq 0} |\gamma_{n,2}| < +\infty. \tag{2.5}$$

Since $p_1(z)$ has all the roots on the unit disk $|z| \leq 1$, and those with modulus 1 are of multiplicity not exceding $r - 1$, by the induction hypothesis it follows that there exist constants $\psi_0, \psi_1, \cdots, \psi_{r-2}$, such that

$$\frac{1}{\hat{p}_1(z)} = \sum_{n\geq 0} \gamma_{n,1} z^n, \quad |z| < 1, \tag{2.6}$$

$$|\gamma_{n,1}| \leq \psi_0 + n\psi_1 + \cdots + n^{r-2}\psi_{r-2}, \quad n \geq 0. \tag{2.7}$$

From (2.5), (2.6) and taking into account the product of power series we have

$$\frac{1}{\hat{p}(z)} = \Big(\sum_{n\geq 0} \gamma_{n,1} z^n\Big)\Big(\sum_{n\geq 0} \gamma_{n,2} z^n\Big) = \sum_{n\geq 0} \gamma_n z^n, \quad |z| < 1 \tag{2.8}$$

where

$$\gamma_n = \sum_{k=0}^{n} \gamma_{1,k}\gamma_{2,n-k}, \quad n \geq 0. \tag{2.9}$$

From (2.5), (2.7) and (2.9) it follows that

$$|\gamma_n| \leq \psi \sum_{k=0}^{n} |\gamma_{1,k}| \leq \psi \sum_{k=0}^{n}\sum_{h=0}^{r-2} k^h \Gamma_h = \psi \sum_{h=0}^{r-2} \Gamma_h \Big(\sum_{k=0}^{n} k^h\Big). \tag{2.10}$$

From (1.3), for appropriate coefficients $c(m,n)$ expressed in terms of Bernouilli's numbers, we can write

$$\sum_{k=0}^{n} k^h = \sum_{m=0}^{h+1} c(m,h)n^m \ , \ \ 0 \le h \le r-2 \tag{2.11}$$

and from (2.10)–(2.11) it follows that

$$|\gamma_n| \le \Gamma_0 + n\Gamma_1 + \cdots + n^{r-1}\Gamma_{r-1} \ , \ \ n \ge 0$$

where

$$\Gamma_0 = \Psi\Big[\Psi_0 \ c\Big(0,0\Big) + \Psi_1 \ c(0,1) + \cdots + \Psi_{r-2} \ c(0,r-2)\Big] \, ,$$

$$\Gamma_1 = \Psi\Big[\Psi_0 \ c(1,0) + \Psi_1 \ c(1,1) + \cdots + \Psi_{r-3} \ c(1,r-3) + \Psi_{r-2} \ c(1,r-2)\Big] \, ,$$

$$\vdots$$

$$\Gamma_{r-2} = \Psi\Big[\Psi_{r-3} \ c(r-2,r-3) + \Psi_{r-2} \ c(r-2,r-2)\Big]$$

$$\Gamma_{r-1} = \Psi \ c(r-1,r-2)$$

Thus the result is established.

For the sake of clarity in the presentation we recall some concepts and properties of rational matrix polynomials that may be found in chapter seven of [9]. A rational matrix $\mathbb{C}^{p\times p}$ valued function $W(z)$ is a matrix function

$$W(z) = \Big[\frac{p_{ij}(z)}{q_{ij}(z)}\Big]_{1\le i,j\le p}$$

where $p_{ij}(z)$ and $q_{ij}(z)$ are scalar polynomials and $q_{ij}(z)$ are not identically zero. If the degree of each $p_{ij}(z)$ is less than or equal to the degree of $q_{ij}(z)$, we say that $W(z)$ is finite at infinity. If $W(z)$ is a $p \times p$ rational matrix function with $\det(W(z)) \not\equiv 0$, then, in a neighborhood of each $z_0 \in \mathbb{C}$, the function $W(z)$ admits the representation, called the local Smith form of $W(z)$ at z_0 ,

$$W(z) = E_1(z) \ \mathrm{diag}\Big[(z-z_0)^{\nu_1} \cdots (z-z_0)^{\nu_p}\Big] E_2(z) \tag{2.13}$$

where $E_1(z)$ and $E_2(z)$ are rational matrix functions that are defined and invertible at z_0 , and $\nu_1, \cdots, \nu_p$ are integers that are uniquely determined by $W(z)$ and z_0 up to permutation and do not depend on the particular choice of the local Smith form (2.13). The integers $\nu_1, \cdots, \nu_p$ are called the partial multiplicities of $W(z)$ at z_0 . The complex number z_0 is a pole of $W(z)$, i.e., a pole of at least one entry in $W(z)$ if and only if $W(z)$ has a negative partial multiplicity at z_0 . Also, $z_0 \in \mathbb{C}$ is a zero of $W(z)$ if z_0 is a pole of $[W(z)]^{-1}$ and this means that $W(z)$ has a positive partial multiplicity. In particular, for every $z_0 \in \mathbb{C}$, except for a finite number of points, all partial multiplicities are zeros.

The following result is a direct consequence of Lemma 7.1.1 and Theorem 7.2.3 of [9].

Theorem 2.2[9]. *Let $A_j \in \mathbb{C}^{p\times p}$ for $0 \le j \le k-1$, and let us consider the rational matrix function $W(z) = \left[z^k I + A_{k-1}z^{k-1} + \cdots + A_0\right]^{-1}$. A complex number z_0 is a pole of W(z) if and only if z_0 is an eigenvalue of the matrix*

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{k-1} \end{bmatrix} \tag{2.14}$$

and then the absolute values of negative partial multiplicities of W(z) at z_0 coincide with the sizes of Jordan blocks with eigenvalue z_0 in the Jordan form of C, that is, with the partial multiplicities of z_0 as an eigenvalue of C.

From the fact that for a rational matrix polynomial $W(z)$, $z_0 \in \mathbb{C}$ is a pole if z_0 is a pole of at least one entry of $W(z)$, from (2.13), Theorem, 2.1 and 2.2, the following result is proved working component-wise.

Lemma 2.3. *Let us consider a matrix polynomial $L(z) = z^k I + z^{k-1}A_{k-1} + \cdots + A_0$ where $A_j \in \mathbb{C}^{p\times p}$, $0 \le j \le p-1$, are such that the matrix C defined by (2.14) is of class r and $\rho(C) = 1$. Let the matrix coefficients $C_n \in \mathbb{C}^{p\times p}$ be defined by*

$$[L(z)]^{-1} = \left[I + A_{k-1}z + \cdots + A_0 z^k\right]^{-1} = \sum_{n\ge 0} C_n z^n \ , \ \ |z| < 1. \tag{2.15}$$

Then there exist constants $\rho_0, \rho_1, \rho_2, \cdots, \rho_{r-1}$ such that

$$|C_n| \le \rho_0 + n\rho_1 + n^2\rho_2 + \cdots + n^{r-1}\rho_{r-1} \ , \ \ n \ge 0 \tag{2.16}$$

3. Multistep Matrix Methods

Multistep methods with matrix coefficients have been considered in [15], [18] to solve first order differential systems. Let us consider the initial value problem (1.1) under the hypothesis (1.2) and let $k \ge r+1$. A linear k-step matrix method for problem (1.1) is a relationship of the form

$$Y_{k+n} + A_{k-1}Y_{k+n-1} + \cdots + A_0 Y_n = h^r\left\{B_k f_{n+k} + \cdots + B_0 f_n\right\}, \quad k \ge r+1 \tag{3.1}$$

where $h > 0$, $A_j \in \mathbb{C}^{p\times p}$, $B_i \in \mathbb{C}^{p\times p}$, $t_n = a + nh$ and $f_m = f(t_m, Y_m)$, for $0 \le i \le k$, $0 \le j \le k-1$, $m \ge 0$.

The method (3.1) is said to be consistent if the matrix coefficients A_j, B_i satisfy the conditions

$$\left.\begin{aligned} A_0 + A_1 + A_2 + \cdots + A_{k-1} + I = 0 \\ A_1 + 2A_2 + \cdots + (k-1)A_{k-1} + kI = 0 \\ A_1 + 2^2 A_2 + \cdots + (k-1)^2 A_{k-1} + k^2 I = 0 \\ \vdots \\ A_1 + 2^{r-1} A_2 + \cdots + (k-1)^{r-1} A_{k-1} + k^{r-1} I = 0 \\ \tfrac{1}{r!}\Big[A_1 + 2^r A_2 + \cdots + (k-1)^r A_{k-1} + k^r I\Big] - \Big[B_0 + B_1 + \cdots + B_k\Big] = 0 \end{aligned}\right\} \qquad (3.2)$$

We say that the method (3.1) is zero-stable if the matrix C defined by (2.14) is of class r and its spectral radius is one, $\rho(C) = 1$.

Remark 1. For the scalar case, i.e., $p = q = 1$, the previous definition of zero-stability coincides with those given in [15], or in chapter 6 of [12] for the case $r = 2$. In fact, from [19], for the scalar case, the eigenvalue z of the corresponding matrix C defined by (2.14), are such that if the multiplicity of z is m, then there is only one Jordan block associated with z, and it is of dimension m. Thus for the scalar case the above definition of zero-stability means that all the roots of $p(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$ are on the unit disk $|z| \leq 1$ and those with modulus 1 are of multiplicity not exceding r.

Let us associate to the method (3.1) the difference operator $\mathcal{L}$, which for a positive number h and a high order continuously differentiable $\mathbb{C}^{p\times q}$ valued matrix function is defined by

$$\begin{aligned} \mathcal{L}[Y(t), h] &= Y(t + kh) + A_{k-1}Y(t + (k-1)h) + \cdots + A_0 Y(t) \\ &- h^r\Big\{B_k Y^{(r)}(t + kh) + B_{k-1}Y^{(r)}(t + (k-1)h) + \cdots + B_0 Y^{(r)}(t)\Big\} \end{aligned}$$

Expanding the last expression in powers of h, we obtain

$$\mathcal{L}[Y(t), h] = M_0 Y(t) + M_1 Y'(t)h + M_2 Y^{(2)}(t)h^2 + \cdots + M_s Y^s(t)h^s + \cdots \qquad (3.3)$$

where the coefficients $M_s \in \mathbb{C}^{p\times p}$, are independent of $Y(t)$ and may be computed by the equations

$$\left.\begin{aligned} M_0 &= A_0 + A_1 + A_2 + \cdots + A_{k-1} + I, \\ M_1 &= A_1 + 2A_2 + \cdots + (k-1)A_{k-1} + kI, \\ &\vdots \\ M_{r-1} &= A_1 + 2^{r-1}A_2 + \cdots + (k-1)^{r-1}A_{k-1} + k^{r-1}I, \\ M_r &= \tfrac{1}{r!}\Big[A_1 + 2^r A_2 + \cdots + (k-1)^r A_{k-1} + k^r I\Big] - \Big[B_0 + B_1 + \cdots + B_k\Big], \\ &\vdots \\ M_s &= \frac{1}{s!}\Big[A_1 + 2^s A_2 + \cdots + (k-1)^s A_{k-1} + k^s I\Big] \\ &- \frac{1}{(s-r)!}\Big[0^{s-r}B_0 + B_1 + \cdots + (k-1)^{(s-r)}B_{k-1} + k^{s-r} I\Big] \end{aligned}\right\} \qquad (3.4)$$

$$s = r, r+1, r+2, \cdots.$$

In an analogous way to the scalar case[12], the order w of the difference operator $\mathcal{L}$ and of the method (3.1), is defined as the unique integer w such that $M_s = 0,\ s = 0, 1, \cdots, w + r - 1,\ M_{w+r} \neq 0$.

The following result provides a class of (r+1)-steps consistent and zero-stable matrix methods.

Theorem 3.1. *Let A be a matrix in $\mathbb{C}^{p\times p}$ of class r such that*

$$1 \notin \sigma((-1)^k A)\,,\ \ \rho(A) \leq 1\,,\ \ k = r + 1 \tag{3.5}$$

and let us consider the method

$$Y_{n+k} + A_{k-1}Y_{n+k-1} + \cdots + A_0 Y_n = h^r\Big\{B_k f_{n+k} + B_{k-1} f_{n+k-1} + \cdots + B_0 f_n\Big\} \tag{3.6}$$

where

$$A_0 = A,$$

$$A_j = (-1)^j \binom{k-1}{k-j+1} A + (-1)^{k-j}\binom{k-1}{k-j} I\,,\ \ 1 \leq j \leq k-1 \tag{3.7}$$

and $B_i \in \mathbb{C}^{p\times p}$, $0 \leq i \leq k$, satisfy

$$B_0 + B_1 + \cdots + B_k = \frac{1}{r!}\Big\{A_1 + 2^r A_2 + 3^r A_3 + \cdots + (k-1)^r A_{k-1} + k^r I\Big\}. \tag{3.8}$$

Then the method (3.6) *is consistent and zero-stable.*

Proof. From the definition of the matrix coefficients A_j and B_i given by (3.7) and (3.8) respectively, the consistency conditions (3.2) are verified. In order to prove the zero-stability we have to show that the matrix C given by

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A & \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A - (-1)^{k-1}\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I & -\begin{bmatrix} k-1 \\ k-3 \end{bmatrix} A - (-1)^{k-2}\begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I & \cdots & -(-1)^{k-1}\begin{bmatrix} k-1 \\ 0 \end{bmatrix} A + \begin{bmatrix} k-1 \\ 1 \end{bmatrix} I \end{bmatrix}$$

has spectral radius 1 and that C is a matrix of class r. Note that the matrix polynomial

$$\begin{aligned} L(z) &= z^k I + z^{k-1} A_{k-1} + \cdots + A_0 \\ &= z^k I + \Big\{(-1)^{k-1}\begin{bmatrix} k-1 \\ 0 \end{bmatrix} A + (-1)^1 \begin{bmatrix} k-1 \\ 1 \end{bmatrix} I\Big\} z^{k-1} \\ &+ \Big\{(-1)^{k-2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A + (-1)^2 \begin{bmatrix} k-1 \\ 2 \end{bmatrix} I\Big\} z^{k-2} + \cdots \\ &\cdots + \Big\{(-1)^1\begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A + (-1)^{k-1} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I\Big\} z + A \\ &= (zI - I)^{k-1}(zI - (-1)^k A). \end{aligned}$$

From [9] it follows that

$$\sigma(C) = \{1\} \cup \sigma\Big((-1)^k A\Big). \tag{3.9}$$

First we prove that the index of 1 as an eigenvalue of C is smaller than or equal to r. It is straightforward to show that matrices $(C-I)^{k-1}$ and $(C-I)^k$ have a block structure of the form

$$(C-I)^{k-1} = \begin{bmatrix} (-1)^{k-1}I & (-1)^{k-2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} I & \cdots & (-1)^0\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I \\ (-1)^1 A & (-1)^2\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A & \cdots & (-1)^k\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A \\ (-1)^{k-1}A^2 & (-1)^{k-2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^2 & \cdots & (-1)^k\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^2 \\ \vdots & \vdots & & \vdots \\ (-1)^{k-1}A^{k-1} & (-1)^{k-2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^{k-1} & \vdots & (-1)^0\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^{k-1} \end{bmatrix},$$

$$(C-I)^k =$$

$$\begin{bmatrix} (-1)^k I + (-1)A & (-1)^{k-1}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} I + (-1)^2\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A & \cdots (-1)^1\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I + (-1)^k\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A \\ (-1)^2 A + (-1)^{k-1}A^2 & (-1)^3\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A + (-1)^{k-2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^2 & \cdots (-1)^{k+1}\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A + (-1)^0\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^2 \\ (-1)^k A^2 + (-1)A^3 & (-1)^{k-1}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^2 + (-1)^2\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^3 & \cdots (-1)^1\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^2 + (-1)^k\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^3 \\ \vdots & \vdots & \vdots \\ (-1)^k A^{k-1} + (-1)A^k & (-1)^{k-1}\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^{k-1} + (-1)^2\begin{bmatrix} k-1 \\ 1 \end{bmatrix} A^k & \cdots (-1)^1\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^{k-1} + (-1)^k\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} A^k \end{bmatrix}.$$

The block matrices $(C-I)^{k-1}$ and $(C-I)^k$ can be written in the following compact form

$$\left.\begin{array}{lll} (C-I)^{k-1} = \big(M_{ij}\big) & , M_{ij} \in \mathbb{C}^{p\times p} , 1 \le i,j \le k & \\ (C-I)^k = \big(P_{ij}\big) & , P_{ij} \in \mathbb{C}^{p\times p} , 1 \le i,j \le k & \\ Mi,j = (-1)^j \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^{i-1} & , 1 \le i \le k & , i \text{ even} \\ Mi,j = (-1)^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^{i-1} & , 1 \le i \le k & , i \text{ odd} \\ Pi,j = (-1)^{j+1} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^{i-1} + (-1)^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^i & , 1 \le i \le k & , i \text{ even} \\ Pi,j = (-1)^{k+1-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^{i-1} + (-1)^j \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} A^i & , 1 \le i \le k & , i \text{ odd} \end{array}\right\} \tag{3.10}$$

From (3.10) it follows that

$$P_{ij} = -\Big(I + (-1)^{k-1}A\Big)M_{ij}, \quad 1 \le i,j \le k \tag{3.11}$$

or

$$(C-I)^k = D_k(C-I)^{k-1} \tag{3.12}$$

where D_k is the block diagonal matrix in $\mathbb{C}^{kp\times kp}$ defined by

$$D_k = \operatorname{diag}\Big[-\Big(I + (-1)^{k-1}A\Big), \cdots, -\Big(I + (-1)^{k-1}A\Big)\Big]. \tag{3.13}$$

Hence

$$\operatorname{Ker}(C-I)^k = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix},\ x_i \in \mathbb{C}^p,\ 1 \le i \le k,\ D_k(C-I)^{k-1}x = 0 \right\}. \tag{3.14}$$

Since from the hypothesis (3.5) the matrix D_k is invertible, from (3.14) it follows that

$$\operatorname{Ker}(C-I)^k = \operatorname{Ker}(C-I)^{k-1}. \tag{3.15}$$

Equation (3.15) means that the index of 1 as an eigenvalue of C is smaller than or equal to $k-1=r$.

Let w be an eigenvalue of the matrix $(-1)^k A$ with $w \neq 1$ and $|w| \neq 1$. Since $(-1)^k A$ is a matrix of class r and $\rho((-1)^k A) \le 1$, it follows that $\operatorname{Ind}(w, (-1)^k A) \le r = k-1$, and

$$\operatorname{Ker}(wI + (-1)^{k-1}A)^{k-1} = \operatorname{Ker}(wI + (-1)^{k-1}A)^k. \tag{3.16}$$

Let us denote by $f_i^{(s)}, 1 \le i \le k$, the i-th block-row of the $k \times k$ block matrix $(C-wI)^s$, where s is a positive integer, and let us consider the matrix in $\mathbb{C}^{p \times kp}$ defined by

$$F_k^{(s)} = \begin{bmatrix} k-1 \\ 0 \end{bmatrix} f_k^{(s)} + (-1)^1 \begin{bmatrix} k-1 \\ 1 \end{bmatrix} f_{k-1}^{(s)} + \cdots + (-1)^{k-1} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} f_1^{(s)}. \tag{3.17}$$

Using an induction argument we show that

$$F_k^{(s)} = (-1)^{k+s-1}(Iw + (-1)^{k-1}A)^s \left[I, -\begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I, \begin{bmatrix} k-1 \\ k-3 \end{bmatrix} I, \cdots, (-1)^{k-1} \begin{bmatrix} k-1 \\ 0 \end{bmatrix} I \right],$$
$$s \ge 1 \tag{3.18}$$

In order to prove (3.18) for s=1, note that C-wI takes the form

$$C - wI =$$

$$\begin{bmatrix} -wI & I & \cdots & 0 & 0 \\ 0 & -wI & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ -A & \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A + (-1)^k \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I & \cdots & (-1)^{k-1} \begin{bmatrix} k-1 \\ 1 \end{bmatrix} A - (-1)^1 \begin{bmatrix} k-1 \\ 0 \end{bmatrix} I & (-1)^k \begin{bmatrix} k-1 \\ 0 \end{bmatrix} A + \begin{bmatrix} k-1 \\ 1 \end{bmatrix} I - wI \end{bmatrix}.$$

If $F_k^{(s)} = \left(F_{k1}^{(s)}, F_{k2}^{(s)}, \cdots, F_{kk}^{(s)}\right)$, note that from the definition of $F_k^{(s)}$, for $s=1$ one gets

$$F_{k1}^{(1)} = -A + (-1)^{k-1} \begin{bmatrix} k-1 \\ 0 \end{bmatrix} (-wI) = -(A + (-1)^{k-1}wI) = (-1)^k (Iw + (-1)^{k-1}A)$$

$$F_{k2}^{(1)} = \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A + (-1)^k \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I + (-1)^{k-2}(-wI) + (-1)^{k-1} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I$$

$$= (-1)^{k-1} \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} (Iw + (-1)^{k-1}A),$$

$$\vdots$$

$$F_{kj}^{(1)} = (-1)^j \begin{bmatrix} k-1 \\ k-j \end{bmatrix} A + (-1)^{k-j+2} \begin{bmatrix} k-1 \\ k-j+1 \end{bmatrix} I + (-1)^{k-j} \begin{bmatrix} k-1 \\ k-j \end{bmatrix} (-wI)$$

$$+ (-1)^{k-j+1} \begin{bmatrix} k-1 \\ k-j+1 \end{bmatrix} I = (-1)^{k-j+1} \begin{bmatrix} k-1 \\ k-j \end{bmatrix} (Iw + (-1)^{k-1}A) , \ 2 \le j \le k.$$

Hence

$$F_k^{(1)} = \left(F_{k1}^{(1)}, \cdots, F_{kk}^{(1)}\right)$$

$$= (Iw + (-1)^{k-1}A)\Big[(-1)^k \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I, (-1)^{k-1} \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I, \cdots, (-1)^1 \begin{bmatrix} k-1 \\ 0 \end{bmatrix} I\Big]$$

$$= (-1)^k (Iw + (-1)^{k-1}A)\Big[I, - \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I, \cdots, (-1)^{k-1} \begin{bmatrix} k-1 \\ 0 \end{bmatrix} I\Big]$$

and the equation (3.18) holds for $s = 1$.

Let us suppose that equation (3.18) holds for $s = h$. From the relationship $(C - wI)^{h+1} = (C - wI)(C - wI)^h$, it follows that

$$f_i^{(h+1)} = wIf_i^{(h)} + f_{i+1}^{(h)}, \ 1 \le i \le k-1, \tag{3.19}$$

$$f_k^{(h+1)} = -Af_1^{(h)} + \Big\{ \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A + (-1)^k \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I \Big\} f_2^{(h)}$$

$$+ \Big\{ - \begin{bmatrix} k-1 \\ k-3 \end{bmatrix} A + (-1)^{k-1} \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I \Big\} f_3^{(h)} + \cdots + \Big\{ (-1)^i \begin{bmatrix} k-1 \\ k-i \end{bmatrix} A \tag{3.20}$$

$$+ (-1)^{k-i+2} \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} I \Big\} f_i^{(h)} + \cdots + \Big\{ (-1)^k \begin{bmatrix} k-1 \\ 0 \end{bmatrix} A + \begin{bmatrix} k-1 \\ 1 \end{bmatrix} I - wI \Big\} f_k^{(h)}.$$

Hence

$$F_k^{(h+1)} = \begin{bmatrix} k-1 \\ 0 \end{bmatrix} f_k^{(h+1)} + (-1)^1 \begin{bmatrix} k-1 \\ 1 \end{bmatrix} f_{k-1}^{(h+1)} + (-1)^2 \begin{bmatrix} k-1 \\ 2 \end{bmatrix} f_{k-2}^{(h+1)} + \cdots$$

$$+ (-1)^{k-i} \begin{bmatrix} k-1 \\ k-i \end{bmatrix} f_i^{(h+1)} + (-1)^{k-i+1} \begin{bmatrix} k-1 \\ k-i+1 \end{bmatrix} f_{i-1}^{(h+1)}$$

$$+ (-1)^{k-1} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} f_1^{(h+1)} = \begin{bmatrix} k-1 \\ 0 \end{bmatrix} \Big\{ - Af_1^{(h)} + \Big[\begin{bmatrix} k-1 \\ k-2 \end{bmatrix} A$$

$$+ (-1)^k \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} I \Big] f_2^{(h)} + \cdots + \Big[(-1)^i \begin{bmatrix} k-1 \\ k-i \end{bmatrix} A$$

$$+ (-1)^{k-i+2} \begin{bmatrix} k-1 \\ k-i+1 \end{bmatrix} I \Big] f_i^{(h)} + \cdots + \Big[(-1)^k \begin{bmatrix} k-1 \\ 0 \end{bmatrix} A$$

$$+ \begin{bmatrix} k-1 \\ 1 \end{bmatrix} I - wI \Big] f_i^{(h)} \Big\} + (-1)^1 \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \Big\{ - wIf_{k-1}^{(h)} + f_k^{(h)} \Big\} + \cdots$$

$$+(-1)^{k-i}\begin{bmatrix}k-1\\k-i\end{bmatrix}\left\{-wIf_i^{(h)}+f_{i+1}^{(h)}\right\}$$

$$+(-1)^{k-i+1}\begin{bmatrix}k-1\\k-i+1\end{bmatrix}\left\{-wIf_{i-1}^{(h)}+f_i^{(h)}\right\}+\cdots$$

$$+(-1)^{k-1}\begin{bmatrix}k-1\\k-1\end{bmatrix}\left\{-wIf_1^{(h)}+f_2^{(h)}\right\} \qquad (3.21)$$

Note that the matrix coefficient of $f_i^{(h)}$, $2 \le i \le k-1$, in (3.21) takes the form

$$(-1)^i\begin{bmatrix}k-1\\k-1\end{bmatrix}Af_i^{(h)}+(-1)^{k-i+2}\begin{bmatrix}k-1\\k-i+1\end{bmatrix}f_i^{(h)}-(-1)^{k-i}\begin{bmatrix}k-1\\k-1\end{bmatrix}wIf_i^{(h)}$$

$$+(-1)^{k-i+1}\begin{bmatrix}k-1\\k-i+1\end{bmatrix}f_i^h=(-1)^{k-i+1}[wI+(-1)^{k-1}A].$$

The coefficient of $f_1^{(h)}$ is

$$-\begin{bmatrix}k-1\\0\end{bmatrix}A+(-1)^k\begin{bmatrix}k-1\\k-1\end{bmatrix}w=(-1)^k[wI+(-1)^{k-1}A]$$

and the one of $f_k^{(h)}$ is

$$\begin{bmatrix}k-1\\0\end{bmatrix}\left[(-1)^k\begin{bmatrix}k-1\\0\end{bmatrix}A+\begin{bmatrix}k-1\\1\end{bmatrix}I-wI\right]=(-1)[wI+(-1)^{k-1}A].$$

From these expressions and (3.17) we can write

$$F_k^{(h+1)}=(wI+(-1)^{k-1}A)\left[(-1)^kf_1^{(h)}+\sum_{i=2}^{k-1}(-1)^{k-i+1}\begin{bmatrix}k-1\\k-1\end{bmatrix}f_i^{(h)}+(-1)f_k^{(h)}\right]$$

$$=(-1)(wI+(-1)^{k-1}A)\left[f_k^{(h)}+(-1)\begin{bmatrix}k-1\\1\end{bmatrix}f_{k-1}^{(h)}+\cdots+(-1)^{k-1}\begin{bmatrix}k-1\\k-1\end{bmatrix}f_1^{(h)}\right]$$

$$=(-1)(wI+(-1)^{k-1}A)F_k^{(h)}=(-1)^{k+h}(wI+(-1)^{k-1}A)$$

$$\times(wI+(-1)^{k-1}A)^h\left[I,-\begin{bmatrix}k-1\\k-2\end{bmatrix}I,\cdots,(-1)^{k-1}\begin{bmatrix}k-1\\0\end{bmatrix}I\right]$$

$$=(-1)^{k-1+h+1}(wI+(-1)^{k-1}A)^{h+1}\left[I,-\begin{bmatrix}k-1\\k-2\end{bmatrix}I,\cdots,(-1)^{k-1}\begin{bmatrix}k-1\\0\end{bmatrix}I\right].$$

Thus (3.18) holds for $s=h+1$, and for every positive integer $s \ge 1$. From the definition of $F_k^{(h)}$ and $f_i^{(h)}$ we can write

$$(C-Iw)^h=\begin{bmatrix}M^{(h)}\\\cdots\\f_s^{(h)}\end{bmatrix}=\begin{bmatrix}I & \vdots & 0\\\cdots&\cdots&\cdots\\ & g & \end{bmatrix}\begin{bmatrix}M^{(h)}\\\cdots\\F_k^{(h)}\end{bmatrix}=W\begin{bmatrix}M^{(h)}\\\cdots\\F_k^{(h)}\end{bmatrix}, \qquad (2.22)$$

where g is the block matrix in $\mathbb{C}^{p\times kp}$ defined by

$$g=\left[(-1)^{k-1}I,(-1)^{k-2}\begin{bmatrix}k-1\\k-2\end{bmatrix}I,(-1)^{k-3}\begin{bmatrix}k-1\\k-3\end{bmatrix}I,\cdots,(-1)^0I\right] \qquad (3.23)$$

and M^h is a matrix in $\mathbb{C}^{(k-1)p\times kp}$.

From (3.22)–(3.23) and the invertibility of the matrix W appearing in (3.22), the system

$$0 = (C - wI)^{k-1}x = W \begin{bmatrix} M^{(k-1)} \\ F_k^{(k-1)} \end{bmatrix} x\ ,\ \ x \in \mathbb{C}^{kp} \tag{3.24}$$

can be written in the equivalent form

$$\begin{bmatrix} M^{(k-1)} \\ F_k^{(k-1)} \end{bmatrix} x = 0. \tag{3.25}$$

Considering the system

$$(C - wI)^k x = 0\,;\ x \in \mathbb{C}^{kp}\,, \tag{3.26}$$

written in the form

$$\begin{bmatrix} M^{(k)} \\ F_k^{(k)} \end{bmatrix} x = 0\,, \tag{3.27}$$

note that from the definition of $F_k^{(s)}$ given in (3.17), it follows that

$$F_k^{(k-1)} = -F_k^{(k)}. \tag{3.28}$$

From (3.16), (3.18) and (3.28) the algebraic systems

$$F_k^{(k-1)}x = 0\,,\ F_k^{(k)}x = 0\,,\ x \in \mathbb{C}^{kp}$$

are equivalent. Furthermore if T is the invertible matrix in $\mathbb{C}^{kp\times kp}$ defined by

$$T = \begin{bmatrix} -wI & I & \cdots & 0 & 0 \\ 0 & -wI & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^k I & (-1)^{k-1} I \begin{bmatrix} k-1 \\ 1 \end{bmatrix} & \cdots & \left[\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} - w\right] I & I \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{3.29}$$

one gets the equation

$$T \begin{bmatrix} M^{(k-1)} \\ F_k^{(k-1)} \end{bmatrix} = \begin{bmatrix} M^{(k)} \\ F_k^{(k)} \end{bmatrix}.$$

From the previous comments the systems (3.25) and (3.27), as well as the systems (3.24) and (3.26) are equivalent. In particular

$$Ker(C - wI)^{k-1} = Ker(C - wI)^k$$

and $\mathrm{Ind}(w, C) \le r = k - 1$. This proves that method (3.6)–(3.7) is zero-stable.

4. Convergence and Error Bounds

We begin this section with a result that will be used below to study the discretization error of multistep matrix methods.

Theorem 4.1. *Let us consider the matrix difference equation*

$$Z_{m+k} + A_{k-1}Z_{m+k-1} + \cdots + A_0 Z_m = h^r\Big\{B_{k,m}\Big\|Z_{m+k}\Big\| + B_{k-1,m}\Big\|Z_{m+k-1}\Big\|$$
$$+ \cdots + B_{0,m}\Big\|Z_m\Big\|\Big\} + \Lambda_m\,,\; m \geq 0, \tag{4.1}$$

where $A_j \in \mathbb{C}^{p\times p}$ for $0 \leq j \leq k-1$, $B_{i,m} \in \mathbb{C}^{p\times p}$ for $0 \leq i \leq k$, $\Lambda_m \in \mathbb{C}^{p\times p}$, $m \geq 0$, $h > 0$ and let N be an integer with $Nh = c$. Let B_* , B and Λ be positive constants such that

$$\Big\|B_{k,m}\Big\| + \Big\|B_{k-1,m}\Big\| + \cdots + \Big\|B_{0,m}\Big\| \leq B_*\,,\; 0 \leq m \leq N,$$
$$\Big\|B_{k,m}\Big\| \leq B\,,\; \Big\|\Lambda_m\Big\| \leq \Lambda\,,\; 0 \leq m \leq N\,;\; 0 \leq h^r < B^{-1}, \tag{4.2}$$

and let us suppose that the method (3.1) *is zero-stable.*

If $\Big\{Z_m\Big\}$ is a solution of (4.1) *for which*

$$\Big\|Z_m\Big\| \leq Z\,,\; 0 \leq m \leq k-1, \tag{4.3}$$

then

$$\Big\|Z_n\Big\| \leq K_* \exp(nh^r L_*)\,,\; 0 \leq n \leq N, \tag{4.4}$$

where

$$L_* = \frac{B\sum_{i=0}^{r-1}\rho_i N^i}{1-h^r B}, \quad A = 1 + \Big\|A_{k-1}\Big\| + \cdots + \Big\|A_0\Big\|, \tag{4.5}$$

$$K_* = \frac{(KAZ + \Lambda N)\sum_{i=0}^{r-1}\rho_i N^i}{1-h^r B}, \tag{4.6}$$

and $\rho_0, \cdots, \rho_r$ are given by lemma 2.3.

Proof. Let us write equation (4.1) for $m = n-k-j$

$$Z_{n-j} + A_{k-1}Z_{n-j-1} + A_{k-2}Z_{n-j-2} + \cdots + A_0 Z_{n-k-j} = h^r\Big\{B_{k,n-k-j}\Big\|Z_{n-j}\Big\|$$
$$+ B_{k-1,n-k-j}\Big\|Z_{n-k-1}\Big\| + \cdots + B_{0,n-k-j}\Big\|Z_{n-k-j}\Big\|\Big\} + \Lambda_{n-k-j}. \tag{4.7}$$

If $C_j \in \mathbb{C}^{p\times p}$ is defined by Lemma 2.3, premultiplying equation (4.7) by C_j , for $j = 0, 1, \cdots, n-k$ and adding the resulting equations, it follows that the sum of the left hand side takes the form

$$S_n = C_0 Z_n + [C_1 + C_0 A_{k-1}]Z_{n-1} + [C_2 + C_1 A_{k-1} + C_0 A_{k-2}]Z_{n-2} + \cdots$$
$$+ [C_{k+1} + C_k A_{k-1} + \cdots + C_0 A_0]Z_{n-k-1} + \cdots + [C_{n-k} + C_{n-k-1}A_{k-1} + \cdots$$
$$+ C_{n-2k}A_0]Z_k + +[C_{n-k}A_{k-1} + \cdots + C_{n-2k+1}A_0]Z_{k-1} + \cdots + C_{n-k}A_0 Z_0.$$

Taking into account that $C_0 = I$, we can write

$$S_n = Z_n + \Big\{C_{n-k}A_{k-1} + \cdots + C_{n-2k}A_0\Big\}Z_k$$

$$\Big\{C_{n-k}A_{k-1} + \cdots + C_{n-2k+1}A_0\Big\}Z_{k-1} + \cdots + C_{n-k}A_0Z_0. \tag{4.8}$$

From (4.8) and Lemma 2.3 it follows that

$$\Big\|S_n - Z_n\Big\| \le kAZ\sum_{i=0}^{r-1}\rho_i N^i. \tag{4.9}$$

The sum of the right hand side of the resulting equations after premultiplying equation (4.7) by C_j , for $j = 0, 1, \cdots, n-k$, takes the form,

$$T_n = h^r\Big\{B_{k,n-k}\Big\|Z_n\Big\| + [B_{k-1,n-k} + C_1B_{k,n-k-1}\Big\|Z_{n-1}\Big\|$$

$$+ \cdots + [B_{0,n-k} + \cdots + C_kB_{k,n-2k}]\Big\|Z_{n-k}\Big\| + \cdots + C_{n-k}B_{0,0}\Big\|Z_0\Big\|\Big\}$$

$$+ \cdots + \Lambda_{n-k} + C_1\Lambda_{n-k-1} + \cdots + C_{n-k}\Lambda_0. \tag{4.10}$$

From lemma 2.3 we can write

$$\Big\|\Lambda_{n-k} + C_1\Lambda_{n-k-1} + \cdots + C_{n-k}\Lambda_0\Big\| \le \Lambda\sum_{j=0}^{n-k}\Big[\sum_{i=0}^{r-1}\rho_i j^i\Big]$$

$$\le \Lambda\sum_{j=0}^{n-k}\Big(\rho_0 + \rho_1 j + \rho_2 j^2 + \cdots + \rho_{r-1}j^{r-1}\Big)$$

$$\le \Lambda N\sum_{i=0}^{r-1}\rho_i N^i. \tag{4.11}$$

Since expressions S_n and T_n given by (4.8) and (4.10) are coincident, it follows that

$$Z_n = (Z_n - S_n) + T_n. \tag{4.12}$$

Taking norms in (4.10) and using (4.11) we have

$$\Big\|T_n\Big\| \le h^rB_*\Big\|Z_n\Big\| + h^rB_*\Big(\sum_{i=0}^{r-1}\rho_iN^i\Big)\Big(\sum_{m=0}^{n-1}\Big\|Z_m\Big\|\Big) + \Lambda N\sum_{i=0}^{r-1}\rho_iN^i. \tag{4.13}$$

From (4.9), (4.12) and (4.13) it follows that

$$\Big\|Z_n\Big\| \le h^rB\Big|Z_n\Big\| + h^rB_*\Big(\sum_{i=0}^{r-1}\rho_iN^i\Big)\Big(\sum_{m=0}^{n-1}\Big\|Z_m\Big\|\Big) + kAZ\sum_{i=0}^{r-1}\rho_iN^i + N\Lambda\sum_{i=0}^{r-1}\rho_iN^i,$$

$$(1 - h^rB)\Big\|Z\Big\| \le h^rB_*\Big(\sum_{i=0}^{r-1}\rho_iN^i\Big)\Big(\sum_{m=0}^{n-1}\Big\|Z_m\Big\|\Big) + (kAZ + N\Lambda)\sum_{i=0}^{r-1}\rho_iN^i,$$

$$\Big\|Z_n\Big\| \le \frac{h^rB_*\Big(\sum_{i=0}^{r-1}\rho_iN^i\Big)}{1 - h^rB}\Big(\sum_{m=0}^{n-1}\Big\|Z_m\Big\|\Big) + \frac{(kAZ + \Lambda N)\Big(\sum_{i=0}^{r-1}\rho_iN^i\Big)}{1 - h^rB}. \tag{4.14}$$

From (4.5), (4.6) and (4.14) it follows that

$$\left\|Z_n\right\| \leq h^r L_* \sum_{m=0}^{n-1} \left\|Z_m\right\| + K_* \tag{4.15}$$

from (4.15) and [12], the scalars $\left\{\left\|Z_n\right\|\right\}$ satisfy

$$\left\|Z_n\right\| \leq K_*(1+h^r L_*)^n\,,\ 0 \leq n \leq N, \tag{4.16}$$

and using the inequality $(1+h^r L_*)^n \leq \exp(nh^r L_*)$, from (4.16) one gets (4.4). Thus the result is established.

The global truncation error of the method (3.1) at the point $t_n = a + nh$ denoted by e_n is the difference $e_n = Y(t_n) - Y_n$, where $Y(t_n)$ is the value of the theoretical solution Y(t) of (1.1) at t_n , and Y_n is the approximate value provided by the method (3.1). We discuss a bound for the discretization error under the assumption that the exact theoretical solution Y(t) has $w+r-times$ continuous derivatives in [a,b], where w is the order of the method (3.1).

Thinking of applications, we shall drop the assumption that the sequence $\left\{Y_n\right\}$ is an exact solution of the difference equation (3.1). Instead, we shall assume that $\left\{Y_n\right\}$ satisfies

$$Y_{n+k} + A_{k-1}Y_{n+k-1} + \cdots + A_0 Y_n = h^r\left\{B_k f_{n+k} + \cdots + B_0 f_n\right\} + \theta_n K_1 h^{s+r}, \tag{4.17}$$

where K_1 and s are non-negative constants and $\theta_n \in \mathbb{C}^{p\times q}$ with $\left\|\theta_n\right\| \leq 1$. We shall assume that the starting values $Y_n = \Omega_n(h)$, $0 \leq n \leq k-1$ are matrices in $\mathbb{C}^{p\times q}$ such that

$$\left\|Y_n - Y(t_n)\right\| \leq h^{r-1}\delta\,,\ n = 0,1,2,\cdots,k-1. \tag{4.18}$$

Substracting from (4.17) the quantity $\mathcal{L}[Y(t_n),h]$ defined by (3.3) one gets that $e_n = Y(t_n) - Y_n$ verifies

$$\begin{aligned} &e_{n+k} + A_{k-1}e_{n+k-1} + \cdots + A_0 e_n - h^r\Big\{B_k\Big[Y^{(r)}(t_n+kh) - f_{n+k}\Big] + \cdots \\ &\quad + B_0\Big[Y^{(r)}(t_n) - f_n\Big]\Big\} = e_{n+k} + A_{k+1}e_{n+k-1} + \cdots \\ &\quad + A_0 e_n - h^r\Big\{B_k\Big[f\Big(t_n+kh, Y(t_{n+k})\Big) - f_{n+k}\Big] + \cdots \\ &\quad + B_0\Big[f\Big(t_n, Y(t_n)\Big) - f\Big]\Big\} = \mathcal{L}[Y(t_n),h] - \theta_n K_1 h^{s+r}. \end{aligned} \tag{4.19}$$

Now let us consider the matrix sequence in $\mathbb{C}^{p\times q}$ defined by

$$P_n = \begin{cases} [f(t_n, Y(t_n))]\|e_n\|^1\,, & \text{if } e_n \neq 0, \\ 0 & , \text{if } e_n = 0. \end{cases} \tag{4.20}$$

From (4.19)–(4.20), we can write

$$e_{n+k} + A_{k-1}e_{n+k-1} + \cdots + A_0 e_n = h^r\Big\{B_k P_{n+k}\left\|e_{n+k}\right\| + \cdots$$

$$+ B_0 P_n \left\| e_n \right\| \Big\} + \mathcal{L}[Y(t_n), h] - \theta_n K_1 h^{s+r} \tag{4.21}$$

Let us denote $Z_n = e_n$, $\Lambda_n = \mathcal{L}[Y(t_n), h] - \theta K_1 h^{s+r}$, and let us suppose that the method (3.1) is of order $w \geq 1$. In an analogous way to the scalar case[12], it is easy to show that

$$\left\| \mathcal{L}[Y(t_n), h] \right\| \leq h^{w+r} GD, \tag{4.22}$$

where G and D are positive constants satisfying

$$D \geq \max \left\{ \left\| Y^{r+w}(t) \right\|,\ a \leq t \leq b \right\},\quad G = \left\| M_{r+w} \right\|, \tag{4.23}$$

and M_{r+w} is defined by (3.4). Hence,

$$\left\| \mathcal{L}[Y(t_n), h] - \theta_n K_1 h^{r+s} \right\| \leq h^{w+r} GD + K_1 h^{s+r}. \tag{4.24}$$

From (1.2) and (4.20) one gets $\left\| P_n \right\| \leq L$, $\left\| B_k P_{n+k} \right\| \leq L \left\| B_k \right\|$. Taking $Z = h^{r-1}\delta$ and $\Lambda = h^{w+r} GD + K_1 h^{s+r}$, by application of Theorem 4.1 with $h > 0$, $Nh = (t_n - a)/h$, it follows that

$$\left\| e_n \right\| \leq \frac{\left(\sum_{i=0}^{r-1} \rho_i N^i \right) \left[KAh^{r-1}\delta + N(h^{w+r} GD + K_1 h^{s+r}) \right]}{1 - h^r \left\| B_k \right\| L} \exp(nh^r L_*), \tag{4.25}$$

where

$$L_* = \frac{L\overline{B} \left(\sum_{i=0}^{r-1} \rho_i N^i \right)}{1 - h^r \left\| B_k \right\| L},\quad \overline{B} = \left\| B_0 \right\| + \left\| B_1 \right\| + \cdots + \left\| B_k \right\|. \tag{4.26}$$

Note that from the equations $Nh = t_n - a$, $0 \leq n \leq N$, it follows that $nh \leq t_n - a$, and

$$nh^r L_* = \frac{L\overline{B} nh \left[\rho_0 h^{r-1} + \rho_1 h^{r-2}(t_n - a) + \cdots + \rho_{r-1}(t_n - a)^{r-1} \right]}{1 - h^r \left\| B_k \right\| L}$$

$$\leq \frac{L\overline{B} [t_n - a] \left[\rho_0 h^{r-1} + \rho_1 h^{r-2}(t_n - a) + \cdots + \rho_{r-1}(t_n - a)^{r-1} \right]}{1 - h^r \left\| B_k \right\| L}.$$

Thus as $h \to 0$, $nh^r L_*$ tends to $L\overline{B}(t_n - a)^r \rho_{r-1} \leq L\overline{B}(b - a)^r \rho_{r-1}$ and the following result has been established.

Theorem 4.2. *Let us consider a consistent and zero-stable method* (3.1) *of order* $w \geq 1$ *and let* $h > 0$ *with* $1 - h^2 L \left\| B_k \right\| > 0$ *where* L *is given by* (1.2), $a \leq t \leq b$, $N = (t_n - a)/h$ *integer. Then the discretization error* $e_n = Y(t_n) - Y_n$ *satisfies the inequality* (4.25)–(4.26), *where* $\rho_0, \cdots, \rho_{r-1}$ *are given by lemma* 2.3, A *is given by Theorem* 4.1, *and* δ *is a positive number determined by* (4.18).

Remark 2. As a direct consequence of Theorem 4.2, it follows that a consistent and zero stable method (3.1) is convergent. From a practical point of view it is interesting to obtain the starting values $Y_0, Y_1, \cdots, Y_{k-1}$ satistying (4.18). To overcome this difficulty we transform the problem (1.1) into the equivalent problem[4]

$$Z = \begin{bmatrix} Y_1 \\ \vdots \\ Y_r \end{bmatrix}, \quad Z' = \begin{bmatrix} Y_2 \\ Y_3 \\ \vdots \\ f(t, Y_1) \end{bmatrix}, \quad Z(a) = \begin{bmatrix} \Omega_0 \\ \Omega_1 \\ \vdots \\ \Omega_{r-1} \end{bmatrix}, \quad a \leq t \leq b. \tag{4.27}$$

Then we apply the one-step method

$$\begin{bmatrix} Y_{1,n+1} \\ \vdots \\ Y_{r,n+1} \end{bmatrix} = h^{r-1} I_{p\times q} + \begin{bmatrix} Y_{2,n} \\ Y_{3,n} \\ \vdots \\ f(t_n, Y_{1,n}) \end{bmatrix}, \quad \begin{bmatrix} Y_{1,0} \\ \vdots \\ Y_{r,0} \end{bmatrix} = \begin{bmatrix} \Omega_0 \\ \Omega 1 \\ \vdots \\ \Omega_{r-1} \end{bmatrix}, \quad n \geq 0 \tag{4.28}$$

where $I_{p\times q}$ is the matrix in $\mathbb{C}^{p\times q}$ with all its entries equal to one.

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