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MODIFIED SHAPE FUNCTIONS FOR SPECHT'S PLATE BENDING ELEMENT*1)

Gao Jun-bin
(*Department of Mathematics, Huazhong University of Science and Technology, Wuhan, China*)

T.M. Shih
(*Department of Applied Mathematics, Hong Kong Polytechnic, Kowloon, Hong Kong*)

Abstract

In this paper we discuss Specht's plate bending element, give the relationships between $\int_{F_\rho} w\,ds$ or $\int_{F_\rho} \frac{\partial w}{\partial n}\,ds$ and the nodal parameters (or freedoms of degrees), further light on the construction methods for that element and at last introduce a new plate bending element with good convergent properties (passing F-E-M-Test[11]).

1. Introduction

The solution of the C^1-continuity requirement of Kirchhoff bending with finite element models results in complicated higher elements[2],[4],[7]. Besides the large number of unknowns, difficulties may also arise from mixed second derivatives at the vertices taken as nodal variable[8]. To overcome such difficulties, a splitting spline element method is introduced[5],[9], but this always causes complicated computation. From the practical point of view lower-degree polynomial finite elements are more desirable. Unfortunately, the simple elements based on lower degree polynomials for the displacement field are non-conforming (not C^1 compatible). This may cause convergent problems and unreliable finite approximations. For non-conforming finite elements, there are some relaxed sufficient convergent conditions, such as the well-known patch test, the interpolation test, the generalized patch tests and the F-E-M-Test, instead of the strong C^1 continuity.

Consider the simple triangular plate bending element whose nodal variables (or freedoms of degree) are the deflection and two rotations at the vertices. Based on the quadratic displacement expansion proposed early by Zienkiewicz, this element is nonconforming because the normal slopes do not match continously along the interelement boundaries. As this element fails in the (generalized) patch test[10], Bergan in

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[1] proposed a modified displacement basis, but the modified version does not satisfy the patch test either. Later, by the aid of the interpolation test, B. Specht[13] constructed an appropriate polynomial displacement basis. This modified element passes the (generalized) patch test ensuring the convergence.

Specht's construction is based on the requirement of weak continuity, i.e., the displacement w and the normal slope $\frac{\partial w}{\partial n}$ (and tangent slope $\frac{\partial w}{\partial \tau}$) are continuous in the sense of intergral along the interelement boundaries. The intention of this article is to derive the relationships between $\int_{F_\rho} w\, ds$ as well as $\int_{F_\rho} \frac{\partial w}{\partial n} ds$ and the nodal variables, further to light on construction method for Specht's plate bending element and to introduce a new plate bending element with convergence by the aid of Shi's F-E-M-Test[5].

To facilitate our presentation, we must agree on certain notations. Given a triangle K with the vertices $P_i = (x_i, y_i)(i = 1, 2, 3)$ in counterclockwise order and the area Δ, we put

$$\xi_1 = x_2 - x_3, \qquad \xi_2 = x_3 - x_1, \qquad \xi_3 = x_1 - x_2,$$

$$\eta_1 = y_2 - y_3, \qquad \eta_2 = y_3 - y_1, \qquad \eta_3 = y_1 - y_2,$$

$$l_{12}^2 = \xi_3^2 + \eta_3^2, \quad l_{23}^2 = \xi_1^2 + \eta_1^2, \quad l_{31}^2 = \xi_2^2 + \eta_2^2,$$

$$r_1 = \frac{1}{\Delta}(\xi_2\xi_3 + \eta_2\eta_3), \ r_2 = \frac{1}{\Delta}(\xi_3\xi_1 + \eta_3\eta_1), \ r_3 = \frac{1}{\Delta}(\xi_1\xi_2 + \eta_1\eta_2),$$

$$t_1 = \frac{1}{\Delta}(\xi_1^2 + \eta_1^2), \ t_2 = \frac{1}{\Delta}(\xi_2^2 + \eta_2^2), \ t_3 = \frac{1}{\Delta}(\xi_3^2 + \eta_3^2).$$

Denote by F_i the edge of K opposite to the vertex P_i, and by τ_i and n_i the unit tangent and outward normal on $F_i(i = 1, 2, 3)$, respectively. Now we let λ_i denote the area coordinates relative to the vertices P_i, i.e.,

$$\begin{cases} x = x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3, \\ y = y_1\lambda_1 + y_2\lambda_2 + y_3\lambda_3, \\ 1 = \lambda_1 + \lambda_2 + \lambda_3 \end{cases}$$

such that the triangle K is transformed into the standard simplex $K^* = \{(\lambda_1, \lambda_2, \lambda_3) \,|\lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \geq 0\}$.

2. Analysis for Specht's Element

Specht's plate bending element was defined in [13] as follows.

Let K be a triangle with vertices at $P_i = (x_i, y_i), (i = 1, 2, 3)$ in counterclockwise order. Specht's element has three degrees of freedom per vertex, i.e., displacement at vertex and the two rotations expressed by the derivatives of the transverse displacement, samilar to Zienkiewicz's element,

$$D(K,w) = (w(P_1), w_x(P_1), w_y(P_1), w(P_2), w_x(P_2), w_y(P_2), \\ w(P_3), w_x(P_3), w_y(P_3))^T. \tag{2.1}$$

The shape function space of Specht's element is

$$P(K) = \Big\{ w \in R(K) \big| \int_{F_i} P_2^{(i)} \frac{\partial w}{\partial n_i}\, ds = 0, \quad 1 \le i \le 3 \Big\}, \tag{2.2}$$

where $P_2^{(i)}$ is the Legendre polynomial of order 2 on F_i and

$$R(K) = \text{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \\ \lambda_3^2\lambda_1, \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2\}. \tag{2.3}$$

It is clear that $\dim P(K) = 9$ and the interpolation problem $(P(K), D(K,w))$ is unisolvable, i.e., for any given constants $C = (c_1, c_2, \cdots, c_9)^T$ there exists unique $w \in P(K)$ such that $D(K,w) = C$. In [13], B. Specht wrote "The required three higher terms are assumed linear combinations of the following cubic and quartic terms: $\lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2$. This assumption is successful", but why did B.Specht add those terms? To explain Specht's element again, first, we introduce an interpolation theorem.

Let $\pi_k(K)$ be polynomial space of order k defined on K and denote by $\Lambda(K,w)$ the following interpolation conditions (or linear functionals defined on $\pi(K)$)

$$\Lambda(K,w) = \Big(w(P_1), w_x(P_1), w_y(P_1), w(P_2), w_x(P_2), w_y(P_2), \\ w(P_3), w_x(P_3), w_y(P_3), \int_{F_1} w\, ds, \int_{F_2} w\, ds, \\ \int_{F_3} w\, ds, \int_{F_1} \frac{\partial w}{\partial n_1}\, ds, \int_{F_2} \frac{\partial w}{\partial n_2}\, ds, \int_{F_3} \frac{\partial w}{\partial n_3}\, ds\Big)^T. \tag{2.4}$$

Theorem 2.1. *The interpolation problem $(\pi_4(K), \Lambda(K,w))$ is unisolvable, that is, for any given constants $C = (c_1, c_2, \cdots, c_{15})^T$, there exists a unique polynomial $w \in \pi_4(K)$ such that*

$$\Lambda(K,w) = C.$$

Proof. For $w \in \pi_4(K)$, by Bernstein-Bezier representation, we have

$$w = \sum_{i+j+k=4} \frac{4!}{i!j!k!} w_{ijk}\, \lambda_1^i \lambda_2^j \lambda_3^k.$$

It is not difficult to show that the coefficients $w_{ijk}(i = 0 \text{ or } j = 0 \text{ or } k = 0)$ can be represented by $w(P_1) = c_1, w_x(P_1) = c_2, w_y(P_1) = c_3$, $w(P_2) = c_4, w_x(P_2) = c_5, w_y(P_2) = c_6$, $w(P_3) = c_7, w_x(P_3) = c_8, w_y(P_3) = c_9$, $\int_{F_1} w\, ds = c_{10}, \int_{F_2} w\, ds = c_{11}, \int_{F_3} w\, ds = c_{12}$.

By the aid of the barycentric coordinates with respect to K, we obtain

$$l_{12}\frac{\partial w}{\partial n_3} = -\frac{1}{2}\Big(r_2\frac{\partial w}{\partial \lambda_1} + r_1\frac{\partial w}{\partial \lambda_2} + t_3\frac{\partial w}{\partial \lambda_3}\Big) = -2\Bigg(r_2 \sum_{i+j+k=3} \frac{3!}{i!j!k!}\, w_{i+1\,jk}\lambda_1^i\lambda_2^j\lambda_3^k$$

$$+ r_1 \sum_{i+j+k=3} \frac{3!}{i!j!k!}\, w_{ij+1\,k}\lambda_1^i\lambda_2^j\lambda_3^k + t_3 \sum_{i+j+k=3} \frac{3!}{i!j!k!}\, w_{ijk+1}\lambda_1^i\lambda_2^j\lambda_3^k\Bigg).$$

Substituting $\lambda_3 = 0$ on the edge P_1P_2 yields the following relation

$$l_{12}\frac{\partial w}{\partial n_3}\Big|_{\lambda_3=0} = -2\Big(r_2 \sum_{i+j=3} \frac{3!}{i!j!}\, w_{i+1\,j0}\lambda_1^i\lambda_2^j + r_1 \sum_{i+j=3} \frac{3!}{i!j!}\, w_{ij+1\,0}\lambda_1^i\lambda_2^j$$

$$+ t_3 \sum_{i+j=3} \frac{3!}{i!j!}\, w_{ij1}\lambda_1^i\lambda_2^j\Big).$$

We integrate the above equation on the edge F_3, then

$$2l_{12}\int_{F_3}\frac{\partial w}{\partial n_3}\, ds = -\Big(r_2 \sum_{i+j=3} w_{i+1j0} + \sum_{i+j=3} w_{ij+10} + t_3 \sum_{i+j=3} w_{ij1}\Big).$$

Thus we have

$$t_3(w_{211} + w_{121}) = -2l_{12}\int_{F_3}\frac{\partial w}{\partial n_3}\, ds - t_3(w_{301} + w_{031})$$

$$- r_2 \sum_{i+j=3} w_{i+1j0} - r_1 \sum_{i+j=3} w_{ij+10}.$$

Similarily, the following relations are derived

$$t_(w_{121} + w_{112}) = -2l_{23}\int_{F_1}\frac{\partial w}{\partial n_1}\, ds - t_1(w_{130} + w_{103})$$

$$- r_3 \sum_{j+k=3} w_{0j+1k} - r_2 \sum_{j+k=3} w_{0jk+1},$$

$$t_2(w_{211} + w_{112}) = -2l_{31}\int_{F_2}\frac{\partial w}{\partial n_2}\, ds - t_2(w_{301} + w_{103})$$

$$- r_1 \sum_{i+k=3} w_{i0k+1} - r_3 \sum_{i+k=3} w_{i+10k}.$$

Hence the coefficients w_{211}, w_{121} and w_{112} can be represented by $C = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15})^T$.

Now we consider the following interpolation problem: Find a 9-dimensional subspace $Q(K)$ of $\pi_4(K)$ such that for any given values $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ there exists a unique $w \in Q(K)$ satisfying

$$\begin{cases} w(P_1) = c_1, & w_x(P_1) = c_2, \quad w_y(P_1) = c_3, \\ w(P_2) = c_4, & w_x(P_2) = c_5, \quad w_y(P_2) = c_6, \\ w(P_2) = c_7, & w_x(P_3) = c_8, \quad w_y(P_3) = c_9, \end{cases} \tag{2.5}$$

$$\left\{\begin{aligned}\int_{F_3} w\,ds &= \frac{l_{12}}{2}[w(P_1)+w(P_2)] + \frac{l_{12}^2}{12}\left[\frac{\partial w}{\partial \tau_3}(P_1) - \frac{\partial w}{\partial \tau_3}(P_2)\right]\\ &= \frac{l_{12}}{2}[c_1+c_2] + \frac{l_{12}}{12}\left[\xi_3(c_5-c_2)+\eta_3(c_6-c_3)\right],\\ \int_{F_1} w\,ds &= \frac{l_{23}}{2}[w(P_2)+w(P_3)] + \frac{l_{23}^2}{12}\left[\frac{\partial w}{\partial \tau_1}(P_2) - \frac{\partial w}{\partial \tau_1}(P_3)\right]\\ &= \frac{l_{23}}{2}[c_2+c_3] + \frac{l_{23}}{12}\left[\xi_1(c_8-c_5)+\eta_1(c_9-c_6)\right],\\ \int_{F_2} w\,ds &= \frac{l_{31}}{2}[w(P_3)+w(P_1)] + \frac{l_{31}^2}{12}\left[\frac{\partial w}{\partial \tau_2}(P_3) - \frac{\partial w}{\partial \tau_2}(P_1)\right]\\ &= \frac{l_{31}}{2}[c_3+c_1] + \frac{l_{31}}{12}\left[\xi_2(c_2-c_8)+\eta_2(c_3-c_9)\right],\end{aligned}\right. \tag{2.6}$$

and

$$\left\{\begin{aligned}\int_{F_3} \frac{\partial w}{\partial n_3}\,ds &= \frac{l_{12}}{2}\left[\frac{\partial w}{\partial n_3}(P_1)+\frac{\partial w}{\partial n_3}(P_2)\right]\\ &= \frac{1}{2}[-\xi_3(c_2+c_5)+\eta_3(c_3+c_6)],\\ \int_{F_1} \frac{\partial w}{\partial n_1}\,ds &= \frac{l_{23}}{2}\left[\frac{\partial w}{\partial n_1}(P_2)+\frac{\partial w}{\partial n_1}(P_3)\right]\\ &= \frac{1}{2}[-\xi_1(c_5+c_8)+\eta_1(c_6+c_9)],\\ \int_{F_2} \frac{\partial w}{\partial n_2}\,ds &= \frac{l_{31}}{2}\left[\frac{\partial w}{\partial n_2}(P_3)+\frac{\partial w}{\partial n_2}(P_1)\right]\\ &= \frac{1}{2}[-\xi_2(c_8+c_2)+\eta_2(c_9+c_3)].\end{aligned}\right. \tag{2.7}$$

Denote by Q_1 the coefficient matrix, with respect to $C = (c_1, c_2, \cdots, c_9)^T$, of the right hand sides of (2.6) and (2.7), and let $Q = \begin{pmatrix} I \\ Q_1 \end{pmatrix}$, then (2.5), (2.6) and (2.7) can be written as

$$\Lambda(K,w) = Q\,D(K,w). \tag{2.8}$$

Let the interpolation polynomial be

$$w = \sum_{i+j+k=4} \frac{4!}{i!j!k!}\, w_{ijk}\, \lambda_1^i \lambda_2^j \lambda_3^k. \tag{2.9}$$

Substituting (2.9) into (2.4) yields the following relationship

$$\Lambda(K,w) = G\,X, \tag{2.10}$$

where $X = (w_{ijk})^T_{i+j+k=4}$.

Clearly G is a certain nonsingular matrix of order 15×15 in view of Theorem 2.1. Then according to (2.8) and (2.10), we have

$$G\,X = Q\,D(K,w).$$

Defining

$$Q(K) = \left\{ w = \sum_{i+j+k=4} \frac{4!}{i!j!k!} w_{ijk} \lambda_1^i \lambda_2^j \lambda_3^k \in \pi_4(K) \mid GX = QD(K,w) \right\},$$

we obtain the following results.

Theorem 2.2. *With assumptions as above we have $Q(K) = P(K)$ (The shape function space of Specht's element) and $(Q(K), D(K,w), K)$ is just the Specht's plate bending element.*

Proof. It is necessary to show that for any polynomial $w \in P(K)$, (2.6) and (2.7) are valid. In [12], Shi and Chen have showed that the integrals of normal slopes of Specht's element on each edge of K are discretized by linear integral formula. Thus (2.7) is valid for Specht's element. Let $w \in P(K)$, then from [13] w is a polynomial of order 3 on each edge of K. Hence equations (2.6) are also valid for w. This completes the proof.

By (2.6) and (2.7), element $(Q(K), D(K,w), K)$ ($K \in \Delta$ a triangulation) passes the strong F1 and strong F2 test[11] which ensure convergence.

3. A New Plate Bending Triangular Element

It is known that the strong F1 and the strong F2 tests ensure the Patch Test for the plate bending problem, but the strong F1 and the strong F2 tests are indeed stronger conditions for convergence of finite element. In general, F1 test (not strong F1 test) can be satisfied when the displacement values at the vertices of the triangular element are used as the freedoms of degree (or parameters) of the finite element[11]. Thus it is not essential that how to discretise integrals $\int_{F_p} w\,ds$ (such as (2.6) in the construction of Specht's element). It is important to keep the strong F2 test.

Now let us discuss another interpolation problem as follows: Find a polynomial subspace $R(K)$ such that for any given constants $C = (c_1, c_2, \cdots, c_{12})^T$ there exists a unique polynomial $w \in R(K)$ satisfying the following interpolation conditions

$$\begin{cases} w(P_1) = c_1, \quad w_x(P_1) = c_2, \quad w_y(P_1) = c_3, \\ w(P_2) = c_4, \quad w_x(P_2) = c_5, \quad w_y(P_2) = c_6, \\ w(P_3) = c_7, \quad w_x(P_3) = c_8, \quad w_y(P_3) = c_9, \\ \int_{F_1} \frac{\partial w}{\partial n_1} ds = c_{10}, \int_{F_2} \frac{\partial w}{\partial n_2} ds = c_{11}, \int_{F_3} \frac{\partial w}{\partial n_3} ds = c_{13}. \end{cases} \tag{3.1}$$

Let

$$F(K,w) = \Big(w(P_1), w_x(P_1), w_y(P_1), w(P_2), w_x(P_2), w_y(P_2), w(P_3), \\ w_x(P_3), w_y(P_3), \int_{F_1} \frac{\partial w}{\partial n_1} ds, \int_{F_2} \frac{\partial w}{\partial n_2} ds, \int_{F_3} \frac{\partial w}{\partial n_3} ds\Big)^T. \tag{3.2}$$

We will use the method introduced in [6] to find the interpolation subspace $R(K)$.

Denote

$$\begin{aligned} R(K) = \pi_3(K) \oplus \{ & d_1(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(\lambda_2^3 + \lambda_3^3 + 3\lambda_1\lambda_2\lambda_3) \\ & + d_2(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(\lambda_3^3 + \lambda_1^3 + 3\lambda_1\lambda_2\lambda_3) \\ & + d_3(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(\lambda_1^3 + \lambda_2^3 + 3\lambda_1\lambda_2\lambda_3) : \\ & t_1d_1 + t_2d_2 + t_3d_3 = 0,\ d_i \in R\}. \end{aligned} \tag{3.3}$$

Refer to [6] and the Section 4, we can prove the following

Theorem 3.1. *The interpolation problem* $(R(K), F(K, w), K)$ *is unisolvable and* $\dim R(K) = 12$.

Now let the interpolation polynomial

$$\begin{aligned} w = & \beta_1\lambda_1^3 + \beta_2\lambda_2^3 + \beta_3\lambda_3^3 + \beta_4\lambda_1^2\lambda_2 + \beta_5\lambda_2^2\lambda_1 \\ & + \beta_6\lambda_2^2\lambda_3 + \beta_7\lambda_3^2\lambda_2 + \beta_8\lambda_3^2\lambda_1 + \beta_9\lambda_1^2\lambda_3 + \beta_{10}\lambda_1\lambda_2\lambda_3 \\ & + d_1(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(\lambda_2^3 + \lambda_3^3 + 3\lambda_1\lambda_2\lambda_3) \\ & + d_2(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(\lambda_3^3 + \lambda_1^3 + 3\lambda_1\lambda_2\lambda_3) \\ & + d_3(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(\lambda_1^3 + \lambda_2^3 + 3\lambda_1\lambda_2\lambda_3) \end{aligned} \tag{3.4}$$

and $t_1d_1 + t_2d_2 + t_3d_3 = 0$.

Substituting (3.4) into (3.2), we have

$$F(K, w) = C_{12\times 13}X$$

where $X = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, d_1, d_2, d_3)^T$ and $t_1d_1 + t_2d_2 + t_3\ d_3 = 0$, or $\begin{pmatrix} F(K,w) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix} X$ where $t = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ t_1,\ \ t_2,\ \ t_3)$, then $\begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix}$ is a nonsingular matrix with Theorem 3.1.

Now we discretise the three integrals in (3.1) as in (2.7), then we have

$$F(K, w) = G\,D(K, w)$$

or

$$\begin{pmatrix} G\,D(K,w) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix} X,$$

and hence

$$X = \begin{pmatrix} C_{12\times 13} \\ t \end{pmatrix}^{-1} \begin{pmatrix} G\,D(K,w) \\ 0 \end{pmatrix}. \tag{3.5}$$

Let $P^*(K) = \{w \in R(K) \mid w$ is defined as (3.4) and (3.5) and $D(K, w)$ is the degree of freedom $\}$, then we have

Theorem 3.2. *The new finite element $(P^*(K), D(K,w), K)$ passes the F1 test and the strong F2 test and hence it converges for the plate bending problem. $P^*(K) \neq P(K)$(shape function space of Specht's element).*

Proof. For $(P(K), D(K,w), K)$, integral $\int_{F_i} \frac{\partial w}{\partial n_i}\, ds$ depends only upon the parameters on the edge F_i in the sense of (2.7). Thus $\int_{F_i} \Delta \frac{\partial w}{\partial n_i}\, ds = 0$ along the interelement boundary F_i. On the other hand, as the values at the vertices of the triangles are degree of freedoms, we can easily prove that $\int_{F_i} \Delta\, w\, ds = o(||h||_{2,K_1 \cup K_2})$, where F_i is the common boundary of K_1 and K_2. This is just F1 test. With the conclusions of [11] the finite element $(P^*(K), D(K,w), K)$ is convergent over any regular triangulation for the fourth order elliptic problems. At last it is not difficult to show $P^*(K) \neq P(K)$ by direct computation.

4. The Proof for the Interpolation Theorem

In this section we will use the constructing method introduced in [6] to find the interpolation subspace $R(K)$ related to the interpolation conditions (3.1) or $F(K,w)$ of (3.2).

Let $w \in \pi(K)$ and $w(x,y) \equiv w(\lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3)$ is the barycentric coordinate of (x,y) with respect to the triangle K. We associate a function analytic at 0 with each interpolation condition of (3.1).

$$w(P_1) = w(1,0,0) \leftrightarrow e^{\lambda_1}, w(P_2) = w(0,1,0) \leftrightarrow e^{\lambda_2}, w(P_3) = w(0,0,1) \leftrightarrow e^{\lambda_3},$$

$$w_x(P_1) = \frac{1}{2\Delta}\left[\eta_1 \frac{\partial}{\partial \lambda_1} + \eta_2 \frac{\partial}{\partial \lambda_2} + \eta_3 \frac{\partial}{\partial \lambda_3}\right] w(1,0,0)$$

$$\leftrightarrow \frac{1}{2\Delta}(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3)e^{\lambda_1} = l_1(\lambda_1, \lambda_2, \lambda_3)e^{\lambda_1},$$

$$w_y(P_1) = -\frac{1}{2\Delta}\left[\xi_1 \frac{\partial}{\partial \lambda_1} + \xi_2 \frac{\partial}{\partial \lambda_2} + \xi_3 \frac{\partial}{\partial \lambda_3}\right] w(1,0,0)$$

$$\leftrightarrow \frac{1}{2\Delta}(\xi_1\lambda_1 + \xi_2\lambda_2 + \xi_3\lambda_3)e^{\lambda_1} = l_2(\lambda_1, \lambda_2, \lambda_3)e^{\lambda_1},$$

$$w_x(P_2) = \frac{1}{2\Delta}\left[\eta_1 \frac{\partial}{\partial \lambda_1} + \eta_2 \frac{\partial}{\partial \lambda_2} + \eta_3 \frac{\partial}{\partial \lambda_3}\right] w(0,1,0)$$

$$\leftrightarrow \frac{1}{2\Delta}(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3)e^{\lambda_2} = l_1(\lambda_1, \lambda_2, \lambda_3)e^{\lambda_2},$$

$$w_y(P_2) = -\frac{1}{2\Delta}\left[\xi_1 \frac{\partial}{\partial \lambda_1} + \xi_2 \frac{\partial}{\partial \lambda_2} + \xi_3 \frac{\partial}{\partial \lambda_3}\right] w(0,1,0)$$

$$\leftrightarrow \frac{1}{2\Delta}(\xi_1\lambda_1 + \xi_2\lambda_2 + \xi_3\lambda_3)e^{\lambda_2} = l_2(\lambda_1, \lambda_2, \lambda_3)e^{\lambda_2},$$

$$w_x(P_3) = \frac{1}{2\Delta}\left[\eta_1\frac{\partial}{\partial\lambda_1} + \eta_2\frac{\partial}{\partial\lambda_2} + \eta_3\frac{\partial}{\partial\lambda_3}\right] w(0,0,1)$$

$$\leftrightarrow \frac{1}{2\Delta}(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3)e^{\lambda_3} = l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},$$

$$w_y(P_3) = -\frac{1}{2\Delta}\left[\xi_1\frac{\partial}{\partial\lambda_1} + \xi_2\frac{\partial}{\partial\lambda_2} + \xi_3\frac{\partial}{\partial\lambda_3}\right] w(0,0,1)$$

$$\leftrightarrow \frac{1}{2\Delta}(\xi_1\lambda_1 + \xi_2\lambda_2 + \xi_3\lambda_3)e^{\lambda_3} = l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},$$

$$\int_{F_3}\frac{\partial w}{\partial n_3}\,ds = l_{12}\frac{\partial w}{\partial n_3}(P_1) + \frac{l_{12}^2}{2}\frac{\partial^2 w}{\partial\tau_3^2\partial n_3}(P_1) + \frac{l_{12}^3}{6}\frac{\partial^3 w}{\partial\tau_3^2\partial n_3}(P_1)$$

$$+ \frac{l_{12}^4}{24}\frac{\partial^4 w}{\partial\tau_3^3\partial n_3}(P_1) + \cdots$$

$$= \left(1 + \frac{1}{2}(\frac{\partial}{\partial\lambda_2} - \frac{\partial}{\partial\lambda_1}) + \frac{1}{6}(\frac{\partial}{\partial\lambda_2} - \frac{\partial}{\partial\lambda_1})^2 + \frac{1}{24}(\frac{\partial}{\partial\lambda_2} - \frac{\partial}{\partial\lambda_1})^3 + \cdots\right)$$

$$\times\left(r_2\frac{\partial}{\partial\lambda_1} + r_1\frac{\partial}{\partial\lambda_2} + t_3\frac{\partial}{\partial\lambda_3}\right) w(1,0,0)$$

$$\leftrightarrow \left[1 + \frac{1}{2}(\lambda_2 - \lambda_1) + \frac{1}{6}(\lambda_2 - \lambda_1)^2 + \frac{1}{24}(\lambda_2 - \lambda_1)^3 + \cdots\right]$$

$$\times (r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)e^{\lambda_1} \equiv p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}.$$

Similarily,

$$\int_{F_1}\frac{\partial w}{\partial n_1}\,ds \leftrightarrow \left[1 + \frac{1}{2}(\lambda_3 - \lambda_2) + \frac{1}{6}(\lambda_3 - \lambda_2)^2 + \frac{1}{24}(\lambda_3 - \lambda_2)^3 + \cdots\right]$$

$$\times (r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)e^{\lambda_2} \equiv p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2},$$

$$\int_{F_2}\frac{\partial w}{\partial n_2}\,ds \leftrightarrow \left[1 + \frac{1}{2}(\lambda_1 - \lambda_3) + \frac{1}{6}(\lambda_1 - \lambda_3)^2 + \frac{1}{24}(\lambda_1 - \lambda_3)^3 + \cdots\right]$$

$$\times (r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)e^{\lambda_3} \equiv p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}.$$

Define

$$H = \text{span}\{e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}, l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}, l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}, l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2},$$

$$l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}, l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}, l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3},$$

$$p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1}, p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}, p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}\}$$

and $H_\downarrow = \text{span}\{f_\downarrow \mid f \in H\}$ where $f_\downarrow$ is the leading term of the Taylor's series of f in H, then from the conclusions of [6] $H_\downarrow$ is an interpolation polynomial space with respect to $F(K,w)$.

Let f be any function in H,

$$f = c_1e^{\lambda_1} + c_2e^{\lambda_2} + c_3e^{\lambda_3} + c_4l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1} + c_5l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}$$
$$+ c_6l_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3} + c_7l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1} + c_8l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}$$
$$+ c_9l_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3} + c_{10}p_1(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_1} + c_{11}p_2(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_2}$$
$$+ c_{12}p_3(\lambda_1,\lambda_2,\lambda_3)e^{\lambda_3}.$$

We expand f as the power series at $(\lambda_1,\lambda_2,\lambda_3) = (0,0,0)$ and let the coefficients of all of the cubic terms be zero, then we have a linear system of ten equations about $c_1,\cdots,c_{12}$ which yields the following relations

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0,$$
$$c_4 = -\frac{1}{3\Delta}[\eta_3c_{10} + \eta_2c_{12}], \quad c_7 = -\frac{1}{3\Delta}[\xi_3c_{10} + \xi_2c_{12}],$$
$$c_5 = -\frac{1}{3\Delta}[\eta_3c_{10} + \eta_1c_{11}], \quad c_8 = -\frac{1}{3\Delta}[\xi_3c_{10} + \xi_1c_{11}],$$
$$c_6 = -\frac{1}{3\Delta}[\eta_1c_{11} + \eta_2c_{12}], \quad c_9 = -\frac{1}{3\Delta}[\xi_1c_{11} + \xi_2c_{12}],$$
$$t_3c_{10} + t_1c_{11} + t_2c_{12} = 0.$$

Then we can prove that the coefficients of $1, \lambda_1, \lambda_2, \lambda_3, \lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1$ are also zero and that f is of the following form (here only the quartic terms are written):

$$f = \frac{1}{72}c_{10}(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_1^3 - \lambda_2^3)$$
$$+ \frac{1}{72}c_{11}(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(3\lambda_2^2\lambda_3 + 3\lambda_2\lambda_3^2 - \lambda_2^3 - \lambda_3^3)$$
$$+ \frac{1}{72}c_{12}(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(3\lambda_3^2\lambda_1 + 3\lambda_3\lambda_1^2 - \lambda_3^3 - \lambda_1^3) + \cdots$$

where $t_3c_{10} + t_1c_{11} + t_2c_{12} = 0$. Hence we have, noting that $\lambda_1 + \lambda_2 + \lambda_3 \equiv 1$,

$$H_\downarrow = \pi_3 \oplus \{c_{10}(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_1^3 - \lambda_2^3)$$
$$+ c_{11}(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(3\lambda_2^2\lambda_3 + 3\lambda_2\lambda_3^2 - \lambda_2^3 - \lambda_3^3)$$
$$+ c_{12}(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(3\lambda_3^2\lambda_1 + 3\lambda_3\lambda_1^2 - \lambda_3^3 - \lambda_1^3) :$$
$$| \; t_3c_{10} + t_1c_{11} + t_2c_{12} = 0\}$$
$$\equiv \pi \oplus \{d_1(r_3\lambda_2 + r_2\lambda_3 + t_1\lambda_1)(\lambda_2^3 + \lambda_3^3 + 3\lambda_1\lambda_2\lambda_3)$$
$$+ d_2(r_1\lambda_3 + r_3\lambda_1 + t_2\lambda_2)(\lambda_3^3 + \lambda_1^3 + 3\lambda_1\lambda_2\lambda_3)$$

$$+ d_3(r_2\lambda_1 + r_1\lambda_2 + t_3\lambda_3)(\lambda_1^3 + \lambda_2^3 + 3\lambda_1\lambda_2\lambda_3) :$$

$$| \ t_1d_1 + t_2d_2 + t_3d_3 = 0 : d_i \in R\} = P^*(K).$$

Thus the interpolation problem $(P^*(K), F(K,w), K)$ is unisovable.

References

[1] P.G. Bergan, Finite elements based on energe orthogonal functions , *Int. J. Numer. Methods in Eng.*, **15** (1980), 1541–1555.

[2] C. Caramanlian, A solution to the C^1-continuity problem in plate bending, *Int. J. Numer. Methods in Eng.*, **19** (1983), 1291–1371.

[3] S.C. Chen and Z.C. Shi, Double set parameter method of constructing stiffness matrices, *Numer. Math. Sinica*, **13** (1991), 486–496.

[4] G. Farin, Triangular Bernstein-Bezier patches, *Computer Aided Geometric Design*, **3** (1986), 83–127.

[5] J.B. Gao, A new finite element of C^1 cubic splines, *J. Comput. and Appl. Math.*, **40** (1992), 305–312.

[6] J.B. Gao, Interpolation methods for construction of shape function space of nonconforming finite elements, to appear in *Computer Methods in Applied Mechanics and Engineering.*

[7] B.M. Irons, A conforming quartic triangular element for plate bending, *Int. J. Numer. Methods in Eng.*, **1** (1969), 29–45.

[8] C. Lawson, Software for C^1 surface interpolation, in *Math. Software* III, J.R.Rice eds., Academic Press, New York, 1977, 161–194.

[9] M.J.D. Powell and M.A. Sabin, Piecewise quadratic approximation on triangles, *ACM Trans. on Math. Software*, **3** (1977), 316–325.

[10] Z.C. Shi, The generalized patch test for Zienkiewicz's triangles, *J. Comput. Math.*, **2** (1984), 286–279.

[11] Z.C. Shi, The F-E-M-Test for convergence of nonconforming finite element, *Math. Comput.*, **49** (1987), 391–405.

[12] Z.C. Shi and S.C. Chen, Analysis for Specht's 9-parameter plate triangular element, *Numer. Math. Sinica*, **10** (1988), 312–318.

[13] B. Specht, Modified shape functions for the three-node plate bending element passing the patch test, *Int. J. Numer. Methods in Eng.*, **26** (1988), 705–715.