Journal of Computational Mathematics, Vol.12, No.3, 1994, 224–234.

A BETTER DIFFERENCE SCHEME WITH FOUR NEAR-CONSERVED QUANTITIES FOR THE KdV EQUATION*1)

Han Zhen
(*Computing Center, Academia Sinica, Beijing, China*)
Shen Long-jun
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

Abstract

In this paper, we present a new semi-discrete difference scheme for the KdV equation, which possesses the first four near-conserved quantities. The scheme is better than the past one given in [4], because its solution has a more superior estimation. The convergence and the stability of the new scheme are proved.

1. Introduction

The numerical studies of the KdV equation have been largely developed since Zabusky and Kruskal used the second order accuracy Leap-Frog scheme to solve the KdV equation and revealed its important properties [8]. Recently, computational instabilities such as sideband and modulational ones using finite difference approximating were observed by the several scholars[1][6]. Not only that, other numerical methods for the KdV equation also cause the troubles about computational blow-up or numerical spurious solutions when computing time is long. The point is that, even though the KdV equation has infinite conserved quantities, it is very difficult to seek a discretization with more than two discrete conserved quantities.

Consider the periodic boundary problem of the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \quad -\infty < x < +\infty,\, t > 0 \tag{1.1}$$

$$u(x+1,t) = u(x,t), \quad -\infty < x < +\infty,\, t > 0 \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{1.3}$$

where $u_0(x)$ has period 1 and satisfies adequate smoothness requirements.

* Received February 24, 1993.
1) The Project Supported by National Natural Science Fundation of China.

In our previous paper [3], we studied the semi-discrete centered difference scheme

$$V_{jt}+\frac{1}{2}\Delta_0 V_j^2+\Delta_0\Delta_+\Delta_- V_j=0 \tag{1.4}$$

and proved that the solution of (1.4) has the first three near-conserved quantities and uniform converges to the solution of (1.1) if the initial value $u_0(x)$ is sufficiently smooth. Here, Δ_0, Δ_+ and Δ_- denote, respectively, the centered, the forward and the backward difference quotient operators with respect to space variable x. V_j takes a value on the net point $x_j=jh$ where h is the spatial mesh length such that $Jh=1$ with a positive integer J. And the meanings of the other symbols in this paper are the same as those in [3] and [9].

In authors' another paper [4], a semi-discrete difference scheme with the first four near-conserved quantities was presented. It can be written as follows:

$$V_{jt}+\frac{1}{2}\Delta_0 V_j^2+\Delta_0\Delta_+\Delta_- V_j+\frac{1}{6}h^2 V_j\Delta_0\Delta_+\Delta_- V_j-\frac{1}{36}h^4\Delta_+\Delta_- V_j\,\Delta_0\Delta_+\Delta_- V_j=0. \tag{1.5}$$

We have proved that the scheme (1.5) is stable and its solution converges to the solution of (1.1) in Sobolev space $\mathbf{L}_\infty(0,T;\mathbf{H}^3)$ for any $T>0$ if $u_0\in\mathbf{H}^3$. The four discrete near-conserved quantities are

$$F_0^h(V_h)=\sum_{j=1}^{J}3V_j h=\text{Const.}, \tag{1.6}$$

$$F_1^h(V_h)=\sum_{j=1}^{J}\frac{1}{2}V_j^2 h=\text{Const.}+O(h^2t), \tag{1.7}$$

$$F_2^h(V_h)=\sum_{j=1}^{J}\left(\frac{1}{6}V_j^3-\frac{1}{2}|\Delta_+V_j|^2\right)h=\text{Const.}+O(h^2t), \tag{1.8}$$

$$\begin{aligned}F_3^h(V_h)&=\sum_{j=1}^{J}\Big\{\frac{5}{72}V_j^4-\frac{5}{6}V_j(\Delta_+V_j)^2-\frac{5}{36}(\Delta_+V_j)^3+\frac{1}{2}(\Delta_+\Delta_-V_j)^2\Big\}h\\&=\text{Const.}+O(h^2t).\end{aligned} \tag{1.9}$$

In this paper, we give a new scheme which possesses the first four near-conserved (1.6)-(1.8) and (1.10), and a better priori estimate than (1.5).

$$\begin{aligned}F_3^h(V_h)&=\sum_{j=1}^{J}\{\frac{5}{72}V_j^4-\frac{5}{12}V_j[(\Delta_+V_j)^2+(\Delta_-V_j)^2]+\frac{1}{2}(\Delta_+\Delta_-V_j)^2\}h\\&=\text{Const.}+O(h^2t).\end{aligned} \tag{1.10}$$

Applying the theory of discrete functional analysis due to Zhou[9] and the technique of coupled priori estimating by the authors[2], we prove the convergence and the stability of the new scheme.

2. Difference Scheme and Several Lemmas

Our new scheme is:

$$V_{jt}+\frac{1}{2}\Delta_0 V_j^2+\Delta_0\Delta_+\Delta_- V_j+\frac{1}{12}h^2\Delta_0 V_j\Delta_+\Delta_- V_j$$
$$-\frac{1}{18}h^4\Delta_+\Delta_- V_j\Delta_0\Delta_+\Delta_- V_j=0 \tag{2.1}$$

with the periodic boundary condition

$$V_{j+J}(t)=V_j(t),\quad \text{for any } j \text{ and } t>0 \tag{2.2}$$

and the initial value

$$V_j(0)=u_0(x_j). \tag{2.3}$$

(2.1) is also a five-point scheme like (1.5) and can be rewritten as an equivalent form:

$$V_{jt}+\Delta_0\Delta_+\Delta_- V_j+\Big(\frac{25}{36}V_{j+1}+\frac{25}{36}V_{j-1}-\frac{7}{18}V_j\Big)\Delta_0 V_j \tag{2.4}$$
$$-\frac{1}{18}(V_{j+1}-2V_j+V_{j-1})\Delta_0(V_{j+1}+V_{j-1})=0.$$

To get the priori estimations of the discrete solution of (2.1), we require several lemmas which can be found in [4], [9] and [10].

Lemma 1[9,10]. *Let V_h is a discrete function. For any constants p, q, r and integers k, n which satisfy*

$$1\le q,r\le\infty,\quad 0\le k<n\quad,\quad -(n-k-\frac{1}{r})\le\frac{1}{p}\le 1$$

there exists an constant K such that the following interpolation formula holds

$$\|\Delta_+^k V_h\|_p\le K(\|V_h\|_q^{1-\alpha}\|\Delta_+^n V_h\|_r^\alpha+\|V_h\|_q) \tag{2.5}$$

where the constant α is fixed by

$$\frac{1}{p}-k=\frac{1-\alpha}{q}+\alpha\Big(\frac{1}{r}-n\Big)$$

and the norm

$$\|\Delta_+^k V_h\|_p=\Big(\sum_{j=1}^{J}|\Delta_+^k V_j|^p h\Big)^{1/p},\quad 1\le p\le\infty.$$

Lemma 2[4]. *Suppose $z(t)$ is a non-negative function on $[0,T]$ and satisfies the inequality :*

$$z(t)\le A_1+B_1\int_0^t|z(s)|^{8/3}ds\quad,\quad \forall t\in[0,T] \tag{2.6}$$

where A_1, $B_1 > 0$. *If* B_1 *is small enough, then*

$$\frac{5}{3}A_1^{5/3}B_1T \le \frac{1}{4}, \tag{2.7}$$

(2.6) *implies the estimate*

$$z(t) \le 2A_1 \quad , \quad \forall t \in [0,T] \quad . \tag{2.8}$$

Lemma 3[4]. *Set* A_h *and* B_h *are any discrete functions. There are relationships:*

$$\begin{aligned}\Delta_+\Delta_-A_h^2 = \Delta_+(A_h\Delta_-A_h) + \Delta_-(A_h\Delta_+A_h) = 2A_h\Delta_+\Delta_-A_h \\ + (\Delta_+A_h)^2 + (\Delta_-A_h)^2,\end{aligned} \tag{2.9}$$

$$\Delta_+A_h\Delta_+B_h + \Delta_-A_h\Delta_-B_h = 2\Delta_0A_h\Delta_0B_h + \frac{1}{2}h^2\Delta_+\Delta_-A_h\Delta_+\Delta_-B_h, \tag{2.10}$$

$$\Delta_+A_h\Delta_+B_h - \Delta_-A_h\Delta_-B_h = h\{\Delta_0A_h\Delta_+\Delta_-B_h + \Delta_0B_h\Delta_+\Delta_-A_h\}, \tag{2.11}$$

$$\begin{aligned}\Delta_+\Delta_+A_h\Delta_+B_h + \Delta_-\Delta_-A_h\Delta_-B_h = 2\Delta_+\Delta_-A_h\Delta_0B_h \\ + h^2[\Delta_0\Delta_+\Delta_-A_h\Delta_+\Delta_-B_h + \Delta_0B_h\Delta_+\Delta_-\Delta_+\Delta_-A_h].\end{aligned} \tag{2.12}$$

Lemma 4[4]. *Set* A_h *and* B_h *are the periodic discrete functions. The inner product satisfies the formulas:*

$$(A_h\,,\,\Delta_+\Delta_-B_h) = -(\Delta_+A_h\,,\,\Delta_+B_h) = -(\Delta_-A_h\,,\,\Delta_-B_h), \tag{2.13}$$

$$(A_h\,,\,\Delta_0B_h) = -(\Delta_0A_h\,,\,B_h), \tag{2.14}$$

$$(A_h\,,\,B_h\Delta_0B_h) = -\frac{1}{2}(\Delta_0A_h\,,\,B_h^2) + \frac{1}{4}h^2(\Delta_+A_h\,,\,|\Delta_+B_h|^2) \tag{2.15}$$

and

$$\begin{aligned}((\Delta_+A_h)^2 + (\Delta_-A_h)^2\,,\,\Delta_0\Delta_+\Delta_-A_h) = \frac{4}{5}(\Delta_+\Delta_-A_h^2\,,\,\Delta_0\Delta_+\Delta_-A_h) \\ + \frac{1}{5}h^2(\Delta_0A_h\Delta_+\Delta_-A_h\,,\,\Delta_+\Delta_-\Delta_+\Delta_-A_h) \\ - \frac{2}{15}h^4(\Delta_+\Delta_-A_h\Delta_0\Delta_+\Delta_-A_h\,,\,\Delta_+\Delta_-\Delta_+\Delta_-A_h)\end{aligned} \tag{2.16}$$

where the inner product is

$$(A_h\,,\,B_h) = \sum_{j=1}^{J} A_jB_jh = \sum_{j=k}^{J+k-1} A_jB_jh \qquad \text{for any } k.$$

3. The Priori Estimates and the Near-Conserved Quantities

We set

$$Q_j = \frac{1}{12}h^2 \Delta_0 V_j \Delta_+\Delta_- V_j - \frac{1}{18}h^4 \Delta_+\Delta_- V_j \Delta_0 \Delta_+\Delta_- V_j \tag{3.1}$$

and

$$g_j = \Delta_+\Delta_-\Delta_+\Delta_- V_j + \frac{5}{6}\Delta_+\Delta_- V_j^2 - \frac{5}{12}[(\Delta_+ V_j)^2 + (\Delta_- V_j)^2] + \frac{5}{18}V_j^3. \tag{3.2}$$

Then, the scheme (2.1) can be rewritten in the form

$$V_{jt} + \frac{1}{2}\Delta_0 V_j^2 + \Delta_0\Delta_+\Delta_- V_j + Q_j = 0. \tag{3.3}$$

Multiplying (3.3) by g_j and summing them up for j from 1 to J, we have

$$(V_{ht}, g_h) + \frac{1}{2}(\Delta_0 V_h^2, g_h) + (\Delta_0\Delta_+\Delta_- V_h, g_h) + (Q_h, g_h) = 0. \tag{3.4}$$

For convenience, we omit the foot-symbol h of discrete functions below. Making the estimating term by term, and using Lemma 3 and Lemma 4, we get

$$(V_t, g) = \frac{d}{dt}\Big\{\frac{1}{2}(\Delta_+\Delta_- V, \Delta_+\Delta_- V) - \frac{5}{12}(V, (\Delta_+ V)^2 + (\Delta_- V)^2) + \frac{5}{72}(V^2, V^2)\Big\}d, \tag{3.5}$$

$$\frac{1}{2}(\Delta_0 V^2, g) = \frac{1}{2}(\Delta_0 V^2, \Delta_+\Delta_-\Delta_+\Delta_- V) - \frac{5}{24}(\Delta_0 V^2, (\Delta_+ V)^2 + (\Delta_- V)^2) + \frac{5}{36}(\Delta_0 V^2, V^3), \tag{3.6}$$

$$(\Delta_0\Delta_+\Delta_- V, g) = \frac{1}{2}(\Delta_+\Delta_- V^2, \Delta_0\Delta_+\Delta_- V) - (Q, \Delta_+\Delta_-\Delta_+\Delta_- V) + \frac{5}{18}(V^3, \Delta_0\Delta_+\Delta_- V) \tag{3.7}$$

and

$$(Q, g) = (Q, \Delta_+\Delta_-\Delta_+\Delta_- V) + \frac{5}{6}(Q, \Delta_+\Delta_- V^2) - \frac{5}{12}(Q, (\Delta_+ V)^2 + (\Delta_- V)^2) + \frac{5}{18}(Q, V^3). \tag{3.8}$$

Substituting (3.5)-(3.8) into (3.4), we obtain

$$\frac{d}{dt}\Big\{\frac{1}{2}\|\Delta_+\Delta_- V\|_2^2 - \frac{5}{12}(V, |\Delta_+ V|^2 + |\Delta_- V|^2) + \frac{5}{72}\|V\|_4^4\Big\}$$

$$= \frac{5}{24}(\Delta_0 V^2\,,\,(\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{5}{36}(\Delta_0 V^2\,,\,V^3) - \frac{5}{18}(V^3;\,\Delta_0\Delta_+\Delta_- V)$$

$$- \frac{5}{6}(Q\,,\,\Delta_+\Delta_- V^2) + \frac{5}{12}(Q\,,\,(\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{5}{18}(Q\,,\,V^3). \tag{3.9}$$

Applying Lemma 3 and Lemma 4 again, we have

$$(V^3,\Delta_0\Delta_+\Delta_- V) = \frac{3}{4}(\Delta_0 V^2\,,\,(\Delta_+ V)^2 + (\Delta_- V)^2)$$

$$- \frac{1}{2}h^2(\Delta_0 V^2\,,\,(\Delta_+\Delta_- V)^2) - \frac{1}{2}h^2(V\Delta_0 V\,,\,(\Delta_+\Delta_- V)^2) \tag{3.10}$$

and

$$(\Delta_0 V^2\,,\,V^3) = \frac{3}{10}h^2(\,(\Delta_+ V)^2\,,\,\Delta_+ V^3) - \frac{1}{5}h^2(\,(\Delta_+ V)^3 + (\Delta_- V)^3\,,\,V^2). \tag{3.11}$$

So, from (3.10), (3.11) and (3.9), we gain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\Delta_+\Delta_- V\|_2^2 - \frac{5}{12}(V\,,\,|\Delta_+ V|^2 + |\Delta_- V|^2) + \frac{5}{72}\|V\|_4^4\Big\}$$

$$= -\frac{1}{24}h^2(\,(\Delta_+ V)^2\,,\,\Delta_+ V^3) + \frac{1}{36}h^2(\,(\Delta_+ V)^3 + (\Delta_- V)^3\,,\,V^2)$$

$$+ \frac{5}{36}h^2(\Delta_0 V^2 + V\Delta_0 V\,,\,(\Delta_+\Delta_- V)^2)$$

$$- \frac{5}{6}(Q\,,\,\Delta_+\Delta_- V^2) + \frac{5}{12}(Q\,,\,(\Delta_+ V)^2 + (\Delta_- V)^2) - \frac{5}{18}(Q\,,\,V^3)$$

$$= \frac{5}{36}h^2(\Delta_0 V^2\,,\,(\Delta_+\Delta_- V)^2) + \frac{5}{54}h^4(V\Delta_0\Delta_+\Delta_- V\,,\,(\Delta_+\Delta_- V)^2)$$

$$+ \frac{5}{216}h^4(\,(\Delta_+ V)^2 + (\Delta_- V)^2\,,\,\Delta_+\Delta_- V\Delta_0\Delta_+\Delta_- V)$$

$$- \frac{5}{288}h^2(\,(\Delta_+ V)^3 + (\Delta_- V)^3\,,\,V^2) + \frac{13}{864}h^4(\,(\Delta_+ V)^3\,,\,(\Delta_+ V)^2)$$

$$- \frac{5}{648}h^4(\Delta_+ V^3\,,\,\Delta_+\Delta_+ V\Delta_+\Delta_- V)$$

$$\le \frac{25}{54}h^2\|V\|_\infty\|\Delta_+ V\|_\infty\|\Delta_+\Delta_- V\|_2^2 + \frac{1}{8}h^2\|V\|_{2q}^2\|\Delta_+ V\|_{3p}^3, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.12}$$

Here, the constants $\frac{25}{54}$ and $\frac{1}{8}$ are respectively much smaller than those in scheme (1.5) which can be found in [4] where the constants are $\dfrac{425}{432}$ and $\dfrac{1015}{2592}$.

According to the interpolation formula (2.5), we have following inequalities:

$$\|V\|_\infty\|\Delta_+ V\|_\infty \le 2K^2(\|V\|_4^{8/7}\|\Delta_+\Delta_- V\|_2^{6/7} + \|V\|_4^2)$$

$$\|V\|_{2q}^2\|\Delta_+ V\|_{3p}^3 \le 16K^5(\|V\|_4^{22/7}\|\Delta_+\Delta_- V\|_2^{13/7} + \|V\|_4^5)$$

and

$$\frac{5}{6}|(V\,,\,(\Delta_+V)^2+(\Delta_-V)^2)| \le \frac{5}{6}\|V\|_4\|\Delta_+V\|_{8/3}^2 \le \frac{2}{72}\|V\|_4^4+\frac{1}{4}\|\Delta_+\Delta_-V\|_2^2$$
$$+\frac{3}{2}(\frac{25}{3})^{2/3}K^{8/3}\|V\|_2^{8/3}+81(\frac{25}{3}K^4)^{8/3}\|V\|_2^{14/3}$$

Making an integral of (3.12) with respect to t in $[0,t]$ and thanks to above inequalities, we get

$$\|\Delta_+\Delta_-V(t)\|_2^2+\frac{1}{6}\|V(t)\|_4^4 \le C_0+K_1\|V(t)\|_2^{8/3}+K_2\|V(t)\|_2^{14/3} \tag{3.13}$$
$$+K_3h^2\int_0^t\{\|V(s)\|_4^{22/7}\|\Delta_+\Delta_-V(s)\|_2^{13/7}+\|V(s)\|_4^5\}ds+$$
$$+K_4h^2\int_0^t\|\Delta_+\Delta_-V(s)\|_2^2\{\|V(s)\|_4^{8/7}\|\Delta_+\Delta_-V(s)\|_2^{6/7}+\|V(s)\|_4^2\}ds,$$

where and below K_i $(i=1,2,\cdots)$ are absolute constants independent of h and $V_h(t)$, and C_0 is a constant dependent on the initial value:

$$C_0=2\|\Delta_+\Delta_-u_0\|_2^2-\frac{5}{3}(u_0\,,\,(\Delta_+u_0)^2+(\Delta_-u_0)^2)+\frac{5}{18}\|u_0\|_4^4. \tag{3.14}$$

Now, multiplying (2.1) by V_j and sum them up for j from 1 to J, we get

$$(V\,,\,V_t)+\frac{1}{4}h^2(V\,,\,\Delta_0V\Delta_+\Delta_-V)-\frac{1}{18}h^4(V\,,\,\Delta_+\Delta_-V\Delta_0\Delta_+\Delta_-V)=0. \tag{3.15}$$

Because there are

$$(V\,,\,\Delta_0V\Delta_+\Delta_-V)=-\frac{1}{2}(\Delta_+V\,,\,(\Delta_+V)^2)$$

and

$$(V\,,\,\Delta_+\Delta_-V\Delta_0\Delta_+\Delta_-V)=-\frac{1}{2}(\Delta_0V\,,\,(\Delta_+\Delta_-V)^2)+\frac{1}{4}h^2(\Delta_+V\,,\,(\Delta_+\Delta_+\Delta_-V)^2),$$

(3.15) brings about the following inequality

$$|\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_2^2| \le \frac{17}{36}h^2\|\Delta_+V(t)\|_3^3 \tag{3.16}$$

where the constant $\frac{17}{36}$ is also smaller than that of scheme (1.5) which is $\frac{35}{54}$ in [4].

From (3.16), we immediately gain

$$\|V(t)\|_2^2 \le \|u_0\|_2^2+\frac{17}{36}h^2\int_0^t\|\Delta_+V(s)\|_3^3\,\mathrm{d}s. \tag{3.17}$$

Therefore, as similar as doing in [4], we get the following inequality for any $t\le T$ from (3.13), (3.17) standing by Lemma 1.

$$\|\Delta_+\Delta_-V(t)\|_2^2+\frac{1}{6}\|V(t)\|_4^4 \le C_1+K_5Th^2[1+(Th^2)^{1/3}+(Th^2)^{4/3}] \tag{3.18}$$

$$+ K_6h^2[1 + (Th^2)^{1/3} + (Th^2)^{4/3}] \int_0^t \{\|\Delta_+\Delta_- V(s)\|_2^2 + \frac{1}{6}\|V(s)\|_4^4\}^{8/3} ds$$

where

$$C_1 = C_0 + +6\Big(\frac{25}{3}K^4\Big)^{2/3} \sqrt[3]{2}\,\|u_0\|_2^{4/3} + 324\Big(\frac{25}{3}K^4\Big)^{8/3} \sqrt[3]{8}\,\|u_0\|_2^{7/3}$$

is only dependent on the initial value.

According to Lemma 2, for any $T > 0$, if h is small enough there is an estimate

$$\max_{0\le t\le T} \{\|\Delta_+\Delta_- V(t)\|_2^2 + \frac{1}{6}\|V(t)\|_4^4\} \le C_2 \tag{3.19}$$

where C_2 and below C_i $(i = 3, 4, \cdots)$ are the constants independent of h and T.

Furthermore, using interpolation formula, from (3.19) we get

$$\max_{0\le t\le T} \{\|V(t)\|_\infty + \|\Delta_+ V(t)\|_\infty + \|\Delta_+\Delta_- V(t)\|_2\} \le C_3. \tag{3.20}$$

Hence, we gain the forth near-conserved quantity from (3.12) and (3.20), which is written as (1.10). The first conserved quantity (1.6) of scheme (2.1) is proved immediately because $\sum_{j=1}^{J} V_{jt} = 0$ and the second (1.7) is obtained from (3.16) and (3.20). To derive the third one (1.8), we make the inner product of $\Delta_+\Delta_- V_h + \frac{1}{2}V_h^2$ and equation (2.1). There is

$$\begin{aligned}\Big|\frac{d}{dt}\Big\{\frac{1}{6}(V^2, V) - \frac{1}{2}(\Delta_+ V, \Delta_+ V)\Big\}\Big| &= \Big|(Q, \Delta_+\Delta_- V + \frac{1}{2}V^2)\Big| \\ &\le K_7h^2[\|\Delta_+ V\|_\infty \|\Delta_+\Delta_- V\|_2^2 + \|V\|_\infty \|\Delta_+ V\|_2 \|\Delta_+\Delta_- V\|_2] \\ &\le C_4h^2\end{aligned}$$

and (1.8) is true.

Next, we estimate V_{ht} and $\Delta_0\Delta_+\Delta_- V_h$. Set $V_j' = V_{jt}$. Making the derivation of (2.1) with respect to t, we have

$$V_{jt}' + \Delta_0(V_j'V_j) + \Delta_0\Delta_+\Delta_- V_j' + Q_{jt} = 0. \tag{3.21}$$

Multiplying (3.21) by V_{jt} and summing them up for j from 1 to J, we get

$$\frac{1}{2}\frac{d}{dt}(V', V') - (VV', \Delta_0 V') + (Q_t, V') = 0. \tag{3.22}$$

Because there is

$$(VV', \Delta_0 V') = -\frac{1}{2}\sum_{j=1}^{J} V_j'V_{j+1}'\Delta_+ V_j h, \tag{3.23}$$

the following inequality is obtained from (3.22)

$$\frac{d}{dt}\|V_t(t)\|_2^2 \le K_8\|\Delta_+ V\|_\infty \|V_t(t)\|_2^2 \le C_5\|V_t(t)\|_2^2$$

or, by Gronwall's inequality,

$$\max_{0\le t\le T}\|V_t(t)\|_2^2 \le \|V_t(0)\|_2^2 \exp(C_5T) \equiv \tilde{C}_1 \tag{3.24}$$

for $u_0 \in \mathbf{H}^3$. The constants $\tilde{C}_i$ $(i=1,2,\cdots)$ are dependent on T.

Finally, from (2.1) and (3.24), we see

$$\max_{0\le t\le T}\|\Delta_0\Delta_+\Delta_- V(t)\|_2 \le \tilde{C}_2. \tag{3.25}$$

Thus, we have

Theorem 1. *For any given $T>0$, if h is small enough, the solution $V_h(t)$ of the difference scheme* (2.1)–(2.3) *satisfies the priori estimates:*

$$\max_{0\le t\le T}\|V(t)\|_{\mathbf{H}^2} \le C_6, \quad \max_{0\le t\le T}\|V(t)\|_{\mathbf{H}^3} \le \tilde{C}_3, \quad \max_{0\le t\le T}\|V_t(t)\|_2 \le \tilde{C}_4$$

Meanwhile, scheme (2.1) *has the first four near-conserved quantities* (1.6)–(1.8) *and* (1.10).

4. Convergence and Stability

Having the priori estimations in Theorem 1, basing on the framework about convergence of discrete solutions given by Zhou[9], we get following convergence theorem[4].

Theorem 2. *Suppose that $u_0 \in \mathbf{H}^3$. For any $T>0$, the difference solution $V_h(x,t)$ of scheme* (2.1)–(2.3) *converges to the differential solution $u(x,t)$ of* (1.1)–(1.3) *in the Sobolev space $\mathbf{L}_\infty(0,T;\mathbf{H}^3)$ as $h\to 0$.*

Now, we turn to analysis the stability of the difference scheme (2.1), or say, how the discrete solutions of (2.1) depend on the initial values.

Suppose $\tilde{V}_h(t)$ is the solution of scheme (2.1), (2.2) with another initial $\tilde{u}_0(x) \in \mathbf{H}^3$. Let $W_h(t) = V_h(t) - \tilde{V}_h(t)$ and $\bar{V}_h(t) = V_h(t) + \tilde{V}_h(t)$, then $W_h(t)$ satisfies the difference equations :

$$\begin{aligned} W_{jt} &+ \frac{1}{2}\Delta_0[\bar{V}_j W_j] + \Delta_0\Delta_+\Delta_- W_j + \frac{1}{24}h^2[\Delta_0 W_j \Delta_+\Delta_-\bar{V}_j + \Delta_0\bar{V}_j\Delta_+\Delta_- W_j] \quad (4.1)\\ &- \frac{1}{36}h^4[\Delta_+\Delta_- W_j\Delta_0\Delta_+\Delta_-\bar{V}_j + \Delta_+\Delta_-\bar{V}_j\Delta_0\Delta_+\Delta_- W_j] = 0,\\ W_j(0) &= u_0(x_j) - \tilde{u}_0(x_j) \end{aligned}$$

and the periodic boundary condition.

Multiplying (4.1) by W_j and summing them up, using formula (3.23) and Theorem 1, we have

$$\frac{d}{dt}\|W(t)\|_2^2 \le C_7\|W(t)\|_2^2, \quad 0\le t\le T \tag{4.3}$$

therefore

$$\max_{0\le t\le T}\|W(t)\|_2 \le \tilde{C}_5\|u_0 - \tilde{u}_0\|_2. \tag{4.4}$$

Secondly, multiplying (4.1) by $\Delta_+\Delta_- W_j$ and summing them up too, using Theorem 1 and the formula below

$$(\Delta_0[\bar{V}W],\, \Delta_+\Delta_- W) = \frac{1}{2}\sum_{j=1}^{J}[(\bar{V}_{j+1}+\bar{V}_{j-1})\Delta_0 W_j + (W_{j+1}+W_{j-1})\Delta_0\bar{V}_j]\Delta_+\Delta_- W_j h$$

$$= -\frac{1}{2}\sum_{j=1}^{J}[(W_{j+2}+W_j)\Delta_0\Delta_+\bar{V}_j + \Delta_0\bar{V}_j\Delta_+(W_{j+1}+W_{j-1})]\Delta_+ W_j h$$

$$-\frac{1}{4}\sum_{j=1}^{J}\Delta_+(\bar{V}_{j+1}+\bar{V}_{j-1})(\Delta_+ W_j)^2 h$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_+ W(t)\|_2^2 \le K_9(\|\Delta_+\bar{V}\|_\infty\|\Delta_+ W\|_2^2 + \|\Delta_+\Delta_-\bar{V}\|_2\|W\|_\infty\|\Delta_+ W\|_2) \tag{4.5}$$
$$\le C_8(\|\Delta_+ W(t)\|_2^2 + \|W(t)\|_2^2).$$

From (4.3) and (4.5), according to the Gronwall's inequality, we obtain the estimate:

$$\max_{o\le t\le T}\|W(t)\|_{\mathbf{H}^1} \le \tilde{C}_6\|u_0 - \tilde{u}_0\|_{\mathbf{H}^1}. \tag{4.6}$$

At the last, we multiply (4.1) by $\Delta_+\Delta_-\Delta_+\Delta_- W_j$ and sum them up. Because that

$$\|(\Delta_0(\bar{V}W),\, \Delta_+\Delta_-\Delta_+\Delta_- W)\| \le K_{10}\{\|\Delta_+\bar{V}\|_\infty\|\Delta_+\Delta_- W\|_2^2$$
$$+\|\Delta_+\Delta_-\bar{V}\|_2\|\Delta_+ W\|_\infty\|\Delta_+\Delta_- W\|_2 + \|\Delta_0\Delta_+\Delta_-\bar{V}\|_2\|W\|_\infty\|\Delta_+\Delta_- W\|_2\},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_+\Delta_- W(t)\|_2^2 \le \tilde{C}_7(\|\Delta_+\Delta_- W(t)\|_2^2 + \|\Delta_+ W(t)\|_2^2 + \|W(t)\|_2^2). \tag{4.7}$$

From (4.3)(4.5) and (4.7), we get directly

$$\max_{0\le t\le T}\|W(t)\|_{\mathbf{H}^2} \le \tilde{C}_8\|u_0 - \tilde{u}_0\|_{\mathbf{H}^2}. \tag{4.8}$$

We end our article by following theorem.

Theorem 3. *The scheme* (2.1) *is stable about the initial values in the sense of* (4.8) *for small h, where $u_0(x)$, $\tilde{u}_0(x)\in\mathbf{H}^3$.*

References

[1] A.Aoyagi and K.Abe, Parametric Excitation of Computational Modes Inherent to Leap-Frog Scheme Applied to the KdV Equation, *J. Comp. Phys.*, 83 (1989), 447–462.

[2] Han Zhen, Economical Difference Methods and Parallel Algorithms for Parabolic Partial Differential Equations (Systems), Ph.D. Thesis, IAPCM, Beijing, 1992. (in chinese)

[3] Han Zhen, Shen Longjun and Fu Hongyuan, Uniform Convergence of A Difference Solution for the KdV Equation, to be published.

[4] Han Zhen, Shen Longjun and Fu Hongyuan, A Difference Scheme with Four Near-Conserved Quantities for the KdV Equation, to be published.

[5] P.D.Lax, Almost Periodic Solutions of the KdV Equation, *SIAM Review,* 18 (1976), 351–375.

[6] D.M.Sloan, On Modulational Instabilities in Discretizations of the KdV Equation, *J. Comp. Phys.,* 79 (1988), 167–183.

[7] A.Stuart, Nonlinear Instability in Dissipative Finite Difference Schemes, *SIAM Review,* 31 (1989), 191–220.

[8] N.J.Zabusky and M.D.Kruskal, Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States, *Phys. Rev. Letters,* 15 (1965), 240–243.

[9] Zhou Yulin, Applications of Discrete Functional Analysis to the Finite Difference Method, International Academic Publisher, Beijing, China, 1990.

[10] Zhou Yulin, On the General Interpolation Formulas for Discrete Functional Spaces (I), *J. Comp. Math.,* 11 (1993), 188–192.