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OPTIMUM MODIFIED EXTRAPOLATED JACOBI METHOD FOR CONSISTENTLY ORDERED MATRICES*

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Abstract

This paper is concerned with the investigation of a two-parametric linear stationary iterative method, called Modified Extrapolated Jacobi (MEJ) method, for solving linear systems $Ax = b$, where A is a nonsingular consistently ordered 2-cyclic matrix. We give sufficient and necessary conditions for strong convergence of the MEJ method and we determine the optimum extrapolation parameters and the optimum spectral radius of it, in the case where all the eigenvalues of the block Jacobi iteration matrix associated with A are real. In the last section, we compare the MEJ with other known methods.

1. Introduction and Preliminaries

We consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathrm{R}^{n,n}$, $b \in \mathrm{R}^n$ and $\det(A) \neq 0$. We also assume that A has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.2}$$

where A_{11}, A_{22} are square nonsingular (usually diagonal) matrices. As is know [6], A is a consistently ordered 2-cyclic matrix.

For solving (1.1) we intend to use the following simple iterative method:

$$x^{(m+1)} = L_{\omega_1,\omega_2}x^{(m)} + \Omega D^{-1}b, \quad m = 0, 1, 2, \ldots, \tag{1.3}$$

where

$$\Omega = \begin{bmatrix} \omega_1 I_1 & 0 \\ 0 & \omega_2 I_2 \end{bmatrix}, \tag{1.4}$$

$$D = \mathrm{diag}\,(A_{11}, A_{22}) \tag{1.5}$$

and

$$L_{\omega_1,\omega_2} = \begin{bmatrix} (1-\omega_1)I_1 & -\omega_1 A_{11}^{-1}A_{12} \\ -\omega_2 A_{22}^{-1}A_{21} & (1-\omega_2)I_2 \end{bmatrix}, \tag{1.6}$$

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In (1.4) and (1.6), ω_1, ω_2 are nonzero parameters (extrapolation parameters) and I_1, I_2 are identity matrices of the same sizes as A_{11} an A_{22} respectively. The construction of method (1.3) is based on the splitting $A = M - N$, where

$$M = D\Omega^{-1} = \Omega^{-1}D \tag{1.7}$$

and therefore, in the sequel, we will call it Modified Extrapolated Jacobi (MEJ) method for the system (1.1). Obviously, for the iteration matrix we have

$$L_{\omega_1,\omega_2} = I - \Omega D^{-1}A = I - \Omega + \Omega T, \tag{1.8}$$

where $T = I - D^{-1}A = L + U$ is the block Jacobi iteration matrix associated with A ($A = D(I - L - U)$, where L and U strictly lower and strictly upper triangular matrices, respectively). It is clear that, if $\omega_1 = \omega_2 = \omega$, then the MEJ method reduces to the known Extrapolated (Block) Jacobi (EJ) or (Block) Jacobi Overrelaxation (JOR) method (see e.g. [5], [8]) for A. It must also be noted that MEJ is a special case of the recently introduced ([2], [3]) Block Modified Accelerated Overrelaxation (MAOR) method, applied to (1.1), which has the form

$$x^{(m+1)} = L_{R,\Omega}x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \tag{1.9}$$

where

$$L_{R,\Omega} = (I - RL)^{-1}[I - \Omega + (\Omega - R)L + \Omega U] = I - (I - RL)^{-1}\Omega D^{-1}A \tag{1.10}$$

and

$$c = (I - RL)^{-1}\Omega D^{-1}b. \tag{1.11}$$

The matrices R and Ω appeared in (1.10)–(1.11) are defined by

$$R = \begin{bmatrix} r_1 I_1 & 0 \\ 0 & r_2 I_2 \end{bmatrix}, \tag{1.12}$$

and (1.4), with r_1, r_2 the acceleration parameters. If $r_1 = r_2 = 0$, then (1.9) reduces to MEJ method.

Our purpose in this paper is to investigate the MEJ method, in order to find sufficient and necessary conditions for strong convergence of it, as well as to determine the optimal values of the extrapolation parameters and the optimal virtual spectral radius (in the sense of [8]) of it, under the further assumption that all the eigenvalues μ of T are real. A basic reason motivating the investigation of MEJ method is the known result that under the above mentioned assumptions the optimum EJ method coincides with the Jacobi method, that is the optimum extrapolation factor is $\omega_{\text{opt}} = 1$. We show that the same is not true for the optimum MEJ method. It must be noted that the obtained results are new, since, as it seems, similar ones are not appeared in the literature, and generalize previous ones related to EJ method. In section 4, we also compare the optimum MEJ method with the following methods: Jacobi, Gauss-Seidel, optimum

EJ, optimum Extrapolated Gauss-Seidel, optimum Extrapolated Double Jacobi, SOR and Modified SOR (MSOR).

2. Convergence Analysis of the MEJ Method

It has been proved (see (4.1) for $r_2 = 0$ in [2]), that the eigenvalues λ and μ of L_{ω_1,ω_2} and T, respectively, are connected by the following functional relationship:

$$(\lambda + \omega_1 - 1)(\lambda + \omega_2 - 1) = \omega_1\omega_2\mu^2. \tag{2.1}$$

Setting

$$\alpha = \omega_1 + \omega_2, \quad \gamma = \omega_1\omega_2, \quad \alpha, \gamma \in \mathrm{R}, \tag{2.2}$$

we can write (2.1) as follows:

$$\lambda^2 - b\lambda + c = 0, \tag{2.3}$$

where

$$\begin{aligned} b &= 2 - \alpha \\ c &= 1 - \alpha + \gamma(1 - \mu^2). \end{aligned} \tag{2.4}$$

Because of (2.2), if α and γ are known, then ω_1 and ω_2 are determined by the two roots of

$$z^2 - \alpha z + \gamma = 0 \tag{2.5}$$

and can be real or complex conjugate numbers.

We assume that all the eigenvalues μ of T are real, i.e., $\mu \in \sigma(T) \subset \mathrm{R}$ where $\sigma(T)$ is the spectrum of T. Since A is a consistently ordered 2-cyclic matrix we have

$$\begin{aligned} &-\overline{\mu} \le \mu \le \overline{\mu} \equiv \rho(T) \quad (\rho(T) \text{ denotes the spectral radius of } T) \text{ and} \\ &\min_{\mu\in\sigma(T)} |\mu| \equiv \underline{\mu} \le |\mu| \le \overline{\mu}. \end{aligned} \tag{2.6}$$

It is noted that $\underline{\mu}$, $\overline{\mu}$ are eigenvalues of T. By (2.6) we have $\underline{\mu}^2 \le \mu^2 \le \overline{\mu}^2$ and consequently the MEJ method is strongly convergent if and only if the two roots of (2.3) are less than one in modulus $\forall \mu^2 \in \left[\underline{\mu}^2, \overline{\mu}^2\right]$. We can prove the following theorem.

Theorem 2.1. *Let A in (1.1) be a matrix of the form (1.2). If all the eigenvalues of the block Jacobi iteration matrix T associated with A are real, then the MEJ method is strongly convergent if and only if exactly one of the following two statements holds:*

$$\text{i) } \overline{\mu} < 1,\ 0 < \alpha < \frac{4(1-\underline{\mu}^2)}{1+\overline{\mu}^2 - 2\underline{\mu}^2} \equiv M_1 \text{ and } \max\left\{0, \frac{2(\alpha-2)}{1-\overline{\mu}^2}\right\} < \gamma < \frac{\alpha}{1-\underline{\mu}^2} \tag{2.7}$$

$$\text{ii) } \underline{\mu} > 1,\ 0 < \alpha < \frac{4(1-\overline{\mu}^2)}{1+\underline{\mu}^2 - 2\overline{\mu}^2} \equiv M_2 \text{ and } \frac{\alpha}{1-\overline{\mu}^2} < \gamma < \min\left\{0, \frac{2(\alpha-2)}{1-\underline{\mu}^2}\right\}, \tag{2.8}$$

where α, γ are defined by (2.2) and $\underline{\mu}$, $\overline{\mu}$ by (2.6).

Proof. According to Lemma 2.1 [8, p.171] the MEJ method is strongly convergent if and only if

$$|c| < 1 \text{ and } |b| < 1 + c \; \forall \mu^2 \in \left[\underline{\mu}^2, \overline{\mu}^2\right], \tag{2.9}$$

where b, c are given by (2.4). It is shown that (2.9) is equivalent to

$$\begin{cases} \gamma(1-\mu^2) < \alpha \\ \gamma(1-\mu^2) > \alpha - 2 \\ \gamma(1-\mu^2) > 0 \\ \gamma(1-\mu^2) > 2(\alpha - 2), \quad \mu^2 \in \left[\underline{\mu}^2, \overline{\mu}^2\right]. \end{cases} \tag{2.10}$$

By distinguishing the following cases for α: (i) $\alpha \leq 0$ (ii) $0 < \alpha \leq 2$ (iii) $2 < \alpha < 4$ and (iv) $\alpha \geq 4$ and considering for each of the above cases the following cases for μ: (1) $\overline{\mu} \leq 1$ (2) $\underline{\mu} < 1 < \overline{\mu}$ and (3) $\underline{\mu} \geq 1$, we arrive after some manipulation to the conclusion that (2.10) is equivalent to (2.7) and (2.8). It must be noted that for the upper bounds M_1 and M_2 of α in (2.7) and (2.8), respectively, we have

$$2 < M_i \leq 4, \; i = 1,2 \text{ and } M_i = 4 \Longleftrightarrow \underline{\mu} = \overline{\mu}. \tag{2.11}$$

Remark. If we consider the special case $\omega_1 = \omega_2 = \omega$, then $\alpha = 2\omega$, $\gamma = \omega^2$ and from Theorem 2.1 we can obtain the known result that the EJ method with extrapolation factor ω converges if and only if $\overline{\mu} < 1$ and $0 < \omega < \dfrac{2}{1+\overline{\mu}}$. This means that the MEJ method converges for $\underline{\mu} > 1$ only if $\omega_1 \neq \omega_2$.

3. Determination of the Optimum Extrapolation Factors and Spectral Radius of the MEJ Method

We now solve the problem of choosing α and γ and hence ω_1 and ω_2 to minimize the virtual spectral radius $\overline{\rho}(L_{\omega_1,\omega_2})$ of L_{ω_1,ω_2} by proving the following theorem.

Theorem 3.1. *Under the hypothesis of Theorem 2.1, the optimal values ω_1^* and ω_2^* of ω_1 and ω_2 ($\omega_1 \neq \omega_2$), respectively, and the corresponding optimal spectral radius of the MEJ method are given in the following table:*

Table 1

Case	μ	ω_1^*	ω_2^*	$\rho(L_{\omega_1^*,\omega_2^*})$
1	$0 \leq \underline{\mu} \leq \overline{\mu} < 1$	$1 + i\sqrt{\dfrac{\underline{\mu}^2 + \overline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2}}$	$1 - i\sqrt{\dfrac{\underline{\mu}^2 + \overline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2}}$	$\sqrt{\dfrac{\overline{\mu}^2 - \underline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2}}$
2	$1 < \underline{\mu} \leq \overline{\mu}$	$1 + \sqrt{\dfrac{\underline{\mu}^2 + \overline{\mu}^2}{\underline{\mu}^2 + \overline{\mu}^2 - 2}}$	$1 - \sqrt{\dfrac{\underline{\mu}^2 + \overline{\mu}^2}{\underline{\mu}^2 + \overline{\mu}^2 - 2}}$	$\sqrt{\dfrac{\overline{\mu}^2 - \underline{\mu}^2}{\underline{\mu}^2 + \overline{\mu}^2 - 2}}$

Proof. In what follows, because of (2.2), we denote by $\Lambda_{\alpha,\gamma}$ the iteration matrix of the MEJ method ($\Lambda_{\alpha,\gamma} \equiv L_{\omega_1,\omega_2}$). Because of (2.4), we can show that the root radius of (2.3) is given by

$$\Gamma(\alpha,\gamma,\mu^2) = \frac{1}{2}\left| |2-\alpha| + \sqrt{\alpha^2 - 4\gamma(1-\mu^2)} \right| \tag{2.12}$$

and the virtual spectral radius of $\Lambda_{\alpha,\gamma}$ is

$$\overline{\rho}(\Lambda_{\alpha,\gamma}) = \max_{\mu^2\in[\underline{\mu}^2,\overline{\mu}^2]} \Gamma(\alpha,\gamma,\mu^2). \tag{2.13}$$

We assume first that statment (i) of Theorem 2.1 holds. Setting $\Delta(\mu^2) \equiv \alpha^2 - 4\gamma(1-\mu^2)$, we must consider the following three cases:

$$(1)\ \Delta(\overline{\mu}^2) \le 0, \quad (2)\ \Delta(\underline{\mu}^2) \ge 0 \ \text{ and } \ (3)\ \Delta(\underline{\mu}^2) \le 0 \le \Delta(\overline{\mu}^2).$$

If case (1) holds, then we have $\Delta(\mu^2) \le 0\ \forall \mu^2 \in \left[\underline{\mu}^2, \overline{\mu}^2\right]$. Consequently,

$$\Gamma(\alpha,\gamma,\mu^2) = \sqrt{1-\alpha+\gamma(1-\mu^2)} \tag{2.14}$$

and

$$\overline{\rho}(\Lambda_{\alpha,\gamma}) = \Gamma(\alpha,\gamma,\underline{\mu}^2) = \rho(\Lambda_{\alpha,\gamma}) = \sqrt{1-\alpha+\gamma(1-\underline{\mu}^2)}. \tag{2.15}$$

Moreover, it can be shown that

$$\Delta(\overline{\mu}^2) \le 0 \ \text{ if and only if } \ 0 < \alpha < \frac{4(1-\overline{\mu}^2)}{1-\underline{\mu}^2} \ \text{ and } \quad \frac{\alpha^2}{4(1-\overline{\mu}^2)} \le \gamma < \frac{\alpha}{1-\underline{\mu}^2}. \tag{2.16}$$

Hence, using (2.16), we can prove that

$$\begin{aligned}\min_{\alpha,\gamma} \rho(\Lambda_{\alpha,\gamma}) &= \min_{\alpha}\min_{\gamma} \sqrt{1-\alpha+\gamma(1-\underline{\mu}^2)} = \min_{\alpha} \sqrt{1-\alpha+\frac{\alpha^2}{4(1-\overline{\mu}^2)}(1-\underline{\mu}^2)} \\ &= \min_{\alpha} \sqrt{\frac{(1-\underline{\mu}^2)\alpha^2 - 4(1-\overline{\mu}^2)\alpha + 4(1-\overline{\mu}^2)}{4(1-\overline{\mu}^2)}} = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}},\end{aligned}$$

implying that in this case the optimal values α^*, γ^* of α, γ are

$$\alpha^* = \frac{2(1-\overline{\mu}^2)}{1-\underline{\mu}^2}, \quad \gamma^* = \frac{\alpha^{*2}}{4(1-\overline{\mu}^2)} = \frac{1-\overline{\mu}^2}{(1-\underline{\mu}^2)^2} \tag{2.17}$$

and

$$\rho(\Lambda_{\alpha^*,\gamma^*}) = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}}. \tag{2.18}$$

If case (2) holds, i.e., $\Delta(\underline{\mu}^2) \ge 0$, then we have $\Delta(\mu^2) \ge 0\ \forall \mu^2 \in \left[\underline{\mu}^2, \overline{\mu}^2\right]$ and

$$\Gamma(\alpha,\gamma,\mu^2) = \frac{|2-\alpha| + \sqrt{\alpha^2 - 4\gamma(1-\mu^2)}}{2}. \tag{2.19}$$

Moreover, it can be shown that $\Delta(\underline{\mu}^2) \ge 0$ if and only if $0 < \alpha < \alpha_1$ and

$$\max\left\{0, \frac{2(\alpha-2)}{1-\overline{\mu}^2}\right\} < \gamma \le \frac{\alpha^2}{4(1-\underline{\mu}^2)}, \tag{2.20}$$

where

$$\alpha_1 = \frac{4(1-\underline{\mu}^2) - 4\sqrt{(1-\underline{\mu}^2)(\overline{\mu}^2-\underline{\mu}^2)}}{1-\overline{\mu}^2} > 2. \tag{2.21}$$

Consequently, we obtain

$$\overline{\rho}(\Lambda_{\alpha,\gamma}) = \Gamma(\alpha,\gamma,\overline{\mu}^2) = \rho(\Lambda_{\alpha,\gamma}) = \frac{|2-\alpha| + \sqrt{\alpha^2 - 4\gamma(1-\overline{\mu}^2)}}{2}, \tag{2.22}$$

and

$$\begin{aligned}\min_{\alpha,\gamma} \rho(\Lambda_{\alpha,\gamma}) &= \frac{1}{2}\min_{\alpha}\min_{\gamma}\left\{|2-\alpha| + \sqrt{\alpha^2 - 4\gamma(1-\overline{\mu}^2)}\right\} \\ &= \frac{1}{2}\min_{\alpha}\left\{|2-\alpha| + \sqrt{\alpha^2 - 4\left[\frac{\alpha^2}{4(1-\underline{\mu}^2)}\right](1-\overline{\mu}^2)}\right\} \\ &= \frac{1}{2}\min_{\alpha}\left\{|2-\alpha| + \alpha\sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}}\right\} = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}},\end{aligned}$$

implying now that

$$\alpha^* = 2, \quad \gamma^* = \frac{\alpha^{*2}}{4(1-\underline{\mu}^2)} = \frac{1}{1-\underline{\mu}^2} \tag{2.23}$$

and

$$\rho(\Lambda_{\alpha^*,\gamma^*}) = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}}. \tag{2.24}$$

We assume now that case (3) holds, i.e.,

$$\Delta(\underline{\mu}^2) \le 0 \le \Delta(\overline{\mu}^2). \tag{2.25}$$

It can be proved that (2.25) holds if and only if

$$\begin{aligned} &0 < \alpha < \frac{4(1-\underline{\mu}^2)}{1+\overline{\mu}^2-2\underline{\mu}^2} \quad \text{and} \\ &\max\left\{\frac{\alpha^2}{4(1-\underline{\mu}^2)}, \frac{2(\alpha-2)}{1-\overline{\mu}^2}\right\} \le \gamma \le \min\left\{\frac{\alpha^2}{4(1-\overline{\mu}^2)}, \frac{\alpha}{1-\underline{\mu}^2}\right\},\end{aligned} \tag{2.26}$$

where the left inequality for γ is strict if the max is equal to the second expression and similarly for the right inequality.

In this case, since there is a $\widehat{\mu} \in \left[\underline{\mu}, \overline{\mu}\right]$, such that $\Delta(\widehat{\mu}^2) \equiv \alpha^2 - 4\gamma(1-\widehat{\mu}^2) = 0$, it is easy to show that

$$\overline{\rho}(\Lambda_{\alpha,\gamma}) = \rho(\Lambda_{\alpha,\gamma}) = \max\{M(\alpha,\gamma), N(\alpha,\gamma)\}, \tag{2.27}$$

where

$$M(\alpha,\gamma) \equiv \sqrt{1-\alpha+\gamma(1-\underline{\mu}^2)}, \tag{2.28}$$

$$N(\alpha,\gamma) \equiv \frac{|2-\alpha| + \sqrt{\alpha^2 - 4\gamma(1-\overline{\mu}^2)}}{2} \tag{2.29}$$

In order to find $\min_{\alpha,\gamma} \rho(\Lambda_{\alpha,\gamma})$, we can show that

$$\begin{aligned} &M(\alpha,\gamma) = N(\alpha,\gamma) \text{ if and only if } \gamma = \rho_2(\alpha), \text{ where} \\ &\rho_2(\alpha) = \frac{\alpha^2(1-\underline{\mu}^2) + 4\alpha(1-\overline{\mu}^2) - 4(1-\overline{\mu}^2)}{2(2-\underline{\mu}^2-\overline{\mu}^2)^2} \\ &\quad + \frac{\sqrt{(\alpha-2)^2\left[(1-\underline{\mu}^2)^2\alpha^2 - 4(1-\overline{\mu}^2)^2\alpha + 4(1-\overline{\mu}^2)^2\right]}}{2(2-\underline{\mu}^2-\overline{\mu}^2)^2}. \end{aligned} \tag{2.30}$$

Moreover, we have $\rho_2(\alpha) > 0$ and $\rho_2(\alpha)$ belongs to the interval for γ given in (2.26). Since $M(\alpha,\gamma)$ is increasing for γ, whereas $N(\alpha,\gamma)$ is decreasing for γ we obtain

$$\begin{aligned} \min_{\alpha,\gamma} \rho(\Lambda_{\alpha,\gamma}) &= \min_{\alpha}\min_{\gamma} \rho(\Lambda_{\alpha,\gamma}) = \min_{\alpha} N(\alpha,\rho_2(\alpha)) = \min_{\alpha} M(\alpha,\rho_2(\alpha)) \\ &= \min_{\alpha} \sqrt{1-\alpha+\rho_2(\alpha)(1-\underline{\mu}^2)}. \end{aligned} \tag{2.31}$$

Setting

$$R(\alpha) \equiv 1-\alpha+\rho_2(\alpha)(1-\underline{\mu}^2), \quad \lambda = \frac{1-\overline{\mu}^2}{1-\underline{\mu}^2} \;\; (0 < \lambda \le 1), \tag{2.32}$$

we find

$$R(\alpha) = \frac{1}{2(1+\lambda)^2}\left[\alpha^2 - 2\alpha(1+\lambda^2) + 2(1+\lambda^2) + \sqrt{(\alpha-2)^2(\alpha^2-4\lambda^2\alpha+4\lambda^2)}\right]. \tag{2.33}$$

It can be proved that $R'(\alpha) < 0$ if $\alpha < 2$, while $R'(\alpha) > 0$ if $\alpha > 2$, which implies that

$$\min_{\alpha} \sqrt{R(\alpha)} = \sqrt{R(2)} = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{2-\underline{\mu}^2-\overline{\mu}^2}}. \tag{2.34}$$

This means that in this case the optimal values of α, γ are

$$\alpha^* = 2, \quad \gamma^* = \rho_2(2) = \frac{2}{2-\underline{\mu}^2-\overline{\mu}^2}, \tag{2.35}$$

and

$$\rho(\Lambda_{\alpha^*,\gamma^*}) = \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{2-\underline{\mu}^2-\overline{\mu}^2}}. \tag{2.36}$$

Taking into consideration the obtained optimal results for the three cases and since

$$\sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{2-\underline{\mu}^2-\overline{\mu}^2}} \le \sqrt{\frac{\overline{\mu}^2-\underline{\mu}^2}{1-\underline{\mu}^2}},$$

we conclude that if $0 \le \underline{\mu} \le \overline{\mu} < 1$, then the optimal values of α and γ are given by (2.35). Consequently, ω_1^* and ω_2^* are the two roots of

$$z^2 - \alpha^* z + \gamma^* = 0, \tag{2.37}$$

that is, the following complex numbers

$$1 \pm i\sqrt{\frac{\underline{\mu}^2 + \overline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2}} \tag{2.38}$$

appearing in case 1 of table 1 (Note that the values of ω_1^*, ω_2^* can be interchanged).

For the remaining case 2 of table 1, that is, if statement (ii) of Theorem 2.1 holds, the corresponding proof is omitted, since the work is analogous to that for case 1 of table 1. It is worth noticing that ω_1^*, ω_2^* are real now.

4. Comparison with Other Methods and Concluding Remarks

In this section some results are obtained from the comparison of the optimum spectral radius $\rho(L_{\omega_1^*,\omega_2^*}) = \rho(\text{MEJ})$ of the MEJ method with the corresponding ones of the following iterative methods: Jacobi (J), Extrapolated Jacobi (EJ), Gauss-Seidel (GS), Extrapolated Gauss-Seidel (EGS), Extrapolated Double Jacobi (EDJ), Successive Overrelaxation (SOR) and Modified SOR (MSOR).

Under the hypothesis of Theorem 3.1 we restrict our consideration to the case $0 \le \underline{\mu} \le \overline{\mu} < 1$.

Since $\rho(\text{J}) = \rho(\text{EJ}) = \overline{\mu} < 1$, it is clear that

$$\rho(\text{MEJ}) = \sqrt{\frac{\overline{\mu}^2 - \underline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2}} \le \overline{\mu}, \tag{4.1}$$

with equality holding if and only if $\overline{\mu} = 0$.

It is also know that

$$\rho(\text{GS}) = \overline{\mu}^2 \text{ and } \rho(\text{EGS}) = \frac{\overline{\mu}^2}{2 - \overline{\mu}^2}, \tag{4.2}$$

implying that

$$\rho(\text{EGS}) \le \rho(\text{GS}) \le \rho(\text{J}), \tag{4.3}$$

with equality holding if and only if $\overline{\mu} = 0$.

The Double Jacobi (DJ) method [4] for (1.1) is defined by

$$x^{(m+1)} = T^2 x^{(m)} + (I + T)D^{-1}b, \quad m = 0, 1, 2, \dots \tag{4.4}$$

and consequently for the EDJ we have

$$\rho(\text{EDJ}) = \frac{\overline{\mu}^2 - \underline{\mu}^2}{2 - \underline{\mu}^2 - \overline{\mu}^2} = \rho^2(\text{MEJ}). \tag{4.5}$$

Hence we obtain

$$\rho(\text{EDJ}) \le \rho(\text{MEJ}) \quad \text{(equality holds if and only if } \underline{\mu} = \overline{\mu}) \tag{4.6}$$

and

$$\rho(\mathrm{EDJ}) \leq \rho(\mathrm{EGS}) \quad (\text{equality holds if and only if } \underline{\mu} = 0). \tag{4.7}$$

For the SOR and MSOR methods we have the known results (see [8], [7], [1])

$$\rho(\mathrm{SOR}) = \frac{1-\sqrt{1-\overline{\mu}^2}}{1+\sqrt{1-\overline{\mu}^2}}, \tag{4.8}$$

$$\rho(\mathrm{MSOR}) = \frac{\sqrt{1-\underline{\mu}^2}-\sqrt{1-\overline{\mu}^2}}{\sqrt{1-\underline{\mu}^2}+\sqrt{1-\overline{\mu}^2}} \tag{4.9}$$

and

$$\rho(\mathrm{MSOR}) \leq \rho(\mathrm{SOR}) \leq \rho(\mathrm{EGS}), \tag{4.10}$$

with equality holding, in the left inequality if and only if $\underline{\mu} = 0$, while in the right inequality if and only if $\overline{\mu} = 0$.

Using the above given comparison results we can obtain some others, which are summarized in the following theorem:

Theorem 4.1. *(i) Under the hypothesis of Theorem 3.1, let* $0 = \underline{\mu} < \overline{\mu} < 1$. *Then we have*

$$\rho(\mathrm{MSOR}) = \rho(\mathrm{SOR}) < \rho(\mathrm{EDJ}) = \rho(\mathrm{EGS}) < \rho(\mathrm{GS}) < \rho(\mathrm{MEJ}) < \rho(\mathrm{EJ}) = \rho(\mathrm{J}).$$

(ii) *Under the hypothesis of Theorem* 3.1, *let* $0 < \underline{\mu} \leq \overline{\mu} < 1$. *Then:*

(1) $\rho(\mathrm{MEJ}) < \rho(\mathrm{GS})$ *if and only if* $\overline{\mu}\sqrt{\dfrac{1-\overline{\mu}^2}{1+\overline{\mu}^2}} < \underline{\mu}$,

(2) $\rho(\mathrm{MEJ}) < \rho(\mathrm{EGS})$ *if and only if* $\overline{\mu}\sqrt{\dfrac{2-\overline{\mu}^2}{2}} < \underline{\mu}$,

(3) $\rho(\mathrm{MEJ}) < \rho(\mathrm{SOR})$ *if and only if* $\sqrt{1-\dfrac{(2-\overline{\mu}^2)\sqrt{1-\overline{\mu}^2}}{2}} < \underline{\mu}$,

(4) $\rho(\mathrm{MSOR}) \leq \rho(\mathrm{EDJ})$ *(equality holds if and only if* $\underline{\mu} = \overline{\mu}$. *If* $\underline{\mu} = \overline{\mu}$, *then* $\rho(\mathrm{MSOR}) = \rho(\mathrm{EDJ}) = \rho(\mathrm{MEJ}) = 0$).

Proof. It follows after some manipulation by comparing the corresponding spectral radii and using the results given previously.

As Theorem 4.1 shows, the optimum MEJ method can be under some assumptions concerning $\underline{\mu}$ and $\overline{\mu}$ asymptotically faster even than the optimum SOR method, but is always slower than the MSOR one for $\underline{\mu} \neq \overline{\mu}$. Moreover, it has as the MSOR method the property of converging if $\underline{\mu} > 1$ (see (2.8)).

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