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PRECONDITIONING OF THE STIFFNESS MATRIX OF LOCAL REFINED TRIANGULATION*[1)]

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Abstract

A preconditioning method for the finite element stiffness matrix is given in this paper. The triangulation is refined in a subregion; the preconditioning process is composed of resolution of two regular subproblems; the condition number of the preconditioned matrix is $O(1+\log\frac{H}{h})$, where H and h are mesh sizes of the unrefined and local refined triangulations respectively.

1. Introduction

In practical computation, the triangulation is often refined in a subregion. In this case, the condition number of the stiffness matrix , determined by the mesh size of the local refined triangulation, will be increased seriously.

Let $\Omega\subset R^2$ be a polygonal region, and

$$Lu=-\sum_{i,j=1}^{2}\frac{\partial u}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})+cu$$

be an elliptic operator defined on it, where $(a_{i,j})_{i,j=1,2}$ is symmetric positive definite and bounded from above and below on Ω, $c\geq 0$.

$$\begin{cases} a(u,v)=(f,v), \quad v\in H_0^1(\Omega), \\ u\in H_0^1(\Omega) \end{cases} \tag{1.1}$$

is the variational form of the boundary value problem, with the bilinear form

$$a(u,v)=\int_\Omega\Big[\sum_{i,j=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+cuv\Big].$$

For convenience we discuss only the homogeneous Dirichlet boundary value problem here. The norm in $H_0^1(\Omega)$ introduced by $a(\cdot,\cdot)$ is equivalent to the original one. $H_0^1(\Omega)$ will be treated as a Hilbert space with inner product $a(\cdot,\cdot)$ in the following.

(1.1) is discretized by the finite element method. Triangulation and the linear continuous element will be discussed.

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$\Omega_0 \subset \Omega$ is a polygonal subregion, and $\mathcal{T}$ is a triangulation on Ω, quasi-uniform and locally regular, the mesh size of which is $O(H)$. The boundary of Ω_0 coincides with this triangulation. The triangulation is refined on Ω_0, and we get a triangulation $\mathcal{T}_0$ on Ω_0, where $\mathcal{T}_0$ is quasi-uniform and locally regular on Ω_0, the mesh size of which is $O(h)$. $\mathcal{T}$ and $\mathcal{T}_0$ compose the finite element triangulation.

$S_0^H(\Omega) \subset H_0^1(\Omega)$ is the finite element space corresponding to $\mathcal{T}$ on Ω, $S_0^h(\Omega_0) \subset H_0^1(\Omega_0)$ is the finite element space corresponding to $\mathcal{T}_0$ on Ω_0, and $S = S_0^H(\Omega) + S_0^h(\Omega_0)$ is the finite element space. $\hat{\Omega}$ is the set of finite element node points in Ω corresponding to $\mathcal{T}$, $\hat{\Omega}_0$ is the set of finite element node points in Ω_0 corresponding to $\mathcal{T}_0$, and $\hat{\Omega} \cup \hat{\Omega}_0$ amounts to the set of finite element node points. $\{\phi_i, i \in \hat{\Omega}\}$ is the usual finite element basis functions of $S_0^H(\Omega)$, and $\{\phi_i^0, i \in \hat{\Omega}_0\}$ is the usual finite element basis functions of $S_0^h(\Omega_0)$. Let

$$\begin{cases} \psi_i = \phi_i, & i \in \hat{\Omega} - \hat{\Omega}_0, \\ \psi_i = \phi_i^0, & i \in \hat{\Omega}_0. \end{cases}$$

The discrete form of (1.1) will be

$$\begin{cases} a(u, \psi_i) = (f, \psi_i), & i \in \hat{\Omega} \cup \hat{\Omega}_0, \\ u \in S. \end{cases} \tag{1.2}$$

The matrix form is

$$Ax = b \tag{1.3}$$

where $A = a(\psi_i, \psi_j)_{i,j \in \hat{\Omega} \cup \hat{\Omega}_0}$. It is well known that Cond $(A) = O(h^{-2})$.

An iterative method is often used to solve (1.3). Preconditioning is an efficient technique to accelerate various iterations. A good preconditioner Q should satisfy the following two conditions: 1) Cond $(Q^{-1}A)$ is small; 2) $Qx = b$ can be solved easily.

Domain decomposition is an important approach of the construction of a preconditioner. In the following, Ω_0 will be decomposed from Ω, and the preconditioning process is composed of resolution of two regular subproblems (discrete problems on quasi-uniform triangulation).

2. Construction of the Preconditioner

$$\begin{cases} a(u, \phi_i) = (f, \phi_i), & i \in \hat{\Omega}, \\ u \in S^H \end{cases} \tag{2.1}$$

and

$$\begin{cases} a(u, \phi_i^0) = (f, \phi_i^0), & i \in \hat{\Omega}_0, \\ u \in S^h \end{cases} \tag{2.2}$$

are two discrete problems on Ω and Ω_0 respectively.

We use A^H and A^h to represent the coefficient matrices of (2.1) and (2.2). (2.1) and (2.2) are regular problems on uniform triangulations. If triangulations $\mathcal{T}$ and $\mathcal{T}_0$ are only quasi-uniform, we can use regular problems on uniform triangulations to replace (2.1) and (2.2) in preconditioning.

For any finite element function $u \in S$, we use $\boldsymbol{u}$ to represent the vector in $R^{|\hat{\Omega}\cup\hat{\Omega}_0|}$ corresponding to the restriction of u on the finite element node points. For any finite element function $u \in S^H$ or $u \in S^h$, we use $\boldsymbol{u}$ to represent the vector in $R^{|\hat{\Omega}|}$ or $R^{|\hat{\Omega}_0|}$ corresponding to the restriction of u on $\hat{\Omega}$ or $\hat{\Omega}_0$.

C^H is a restriction operator from $R^{|\hat{\Omega}\cup\hat{\Omega}_0|}$ to $R^{|\hat{\Omega}|}$ defined by

$$(C^H\boldsymbol{u})(i) = \sum_{j\in\hat{\Omega}\cup\hat{\Omega}_0} \phi_i(j)\boldsymbol{u}(j), \quad i \in \hat{\Omega}, \quad u \in S. \tag{2.3}$$

C^h is a restriction operator from $R^{|\hat{\Omega}\cup\hat{\Omega}_0|}$ to $R^{|\hat{\Omega}_0|}$ defined by

$$(C^h\boldsymbol{u})(i) = \boldsymbol{u}(i), \quad i \in \hat{\Omega}_0, \quad u \in S. \tag{2.4}$$

E^H is an extension operator from $R^{|\hat{\Omega}|}$ to $R^{|\hat{\Omega}\cup\hat{\Omega}_0|}$ defined by

$$(E^H\boldsymbol{u})(i) = (\sum_{j\in\hat{\Omega}} \boldsymbol{u}(j)\phi_j)(i) \quad i \in \hat{\Omega}\cup\hat{\Omega}_0, u \in S^H. \tag{2.5}$$

E^h is an extension operator from $R^{|\hat{\Omega}_0|}$ to $R^{|\hat{\Omega}\cup\hat{\Omega}_0|}$ defined by

$$(E^h\boldsymbol{u})(i) = \begin{cases} \boldsymbol{u}(i), & i \in \hat{\Omega}_0 \\ 0, & i \in \hat{\Omega} - \hat{\Omega}_0 \end{cases}, \quad u \in S^h. \tag{2.6}$$

With the above preparations, we give the expression for the inverse of the preconditioner Q:

$$Q^{-1} = E^H(A^H)^{-1}C^H + E^h(A^h)^{-1}C^h. \tag{2.7}$$

Remark 1. Q is symmetric positive definite.

Remark 2. In preconditioning iteration, we need only Q^{-1} but not Q in the operation; the representation of Q is useless.

Remark 3. The action of Q^{-1} is composed of parallelly solving two regular problems (2.1) and (2.2). If (2.1) and (2.2) are not regular, we may use regular problems to replace them, and the condition number will remain unchanged.

3. Estimation of the Condition Number

Convergence rate of the preconditioned iteration is determined by the condition number of $Q^{-1}A$, Cond $(Q^{-1}A)$, which is in turn determined by the ratio of the upper and lower bounds of the generalized Rayleigh quotient

$$\frac{(AQ^{-1}A\boldsymbol{u},\boldsymbol{u})}{(A\boldsymbol{u},\boldsymbol{u})}, \quad \boldsymbol{u} \in R^{|\hat{\Omega}\cup\hat{\Omega}_0|}. \tag{3.1}$$

We use P^H and P^h to represent the orthogonal projections from S to S^H and S^h under the inner product $a(\cdot,\cdot)$ respectively. The following can be proved[1]:

Lemma 3.1.

$$\frac{(AQ^{-1}Au,u)}{(Au,u)}=\frac{a(P^Hu+P^hu,u)}{a(u,u)}. \tag{3.2}$$

$\mathrm{Cond}(Q^{-1}A)$ can be estimated through the estimation of the lower and upper bounds of the quotient (3.2). Since P^H and P^h are orthogonal projections,

$$a(P^Hu+P^hu,u)=a(P^Hu,P^Hu)+a(P^hu,P^hu)\le 2a(u,u) \tag{3.3}$$

from which we get the upper bound of (3.2). To estimate the lower bound, we need the following lemma.

Lemma 3.2. *If there exists a constant C_0, so that for any $u\in S$, there exist $u^H\in S^H, u^h\in S^h, u=u^h+u^H$ and*

$$\|u^H\|^2+\|u^h\|^2\le C_0\|u\|^2,$$

we have

$$a(u,u)\le C_0a(P^Hu+P^hu,u).$$

This result is a special case of Lions lemma[2]. We will get the lower bound estimate of (3.2) by application of lemma 3.2. We need to find a decomposition of the function in S and the corresponding constant C_0.

For any $u\in S$, we use u^H to represent its interpolation in S^H. It is obvious that $u^h=u-u^H\in S^h$; hence, $u=u^H+u^h$. This is the decomposition we need. In the following, C will always be a constant independent of H and h.

Since u^H is zero on $\partial\Omega$, by the Poincare inequality,

$$\|u^H\|^2\le C|u^H|^2_{1,\Omega}=C\sum_{T\in\mathcal{T}}|u^H|^2_{1,T}. \tag{3.4}$$

On a fixed element T of the unrefined triangulation $\mathcal{T}$, if $T\subset\Omega-\Omega_0$, then $u^H=u$, and

$$|u^H|^2_{1,T}=|u|^2_{1,T}; \tag{3.5}$$

if $T\subset\Omega_0$, u^H is a linear function on T and we have

$$|u^H|^2_{1,T}\le C|u^H|^2_{0,\infty,T}\le C|u|^2_{0,\infty,T} \tag{3.6}$$

by the discrete Sobolev inequality[3],[4] we get

$$|u|^2_{0,\infty,T}\le C\Big(\frac{1}{H^2}\|u\|^2_{L^2(T)}+\log\frac{H}{h}|u|^2_{1,T}\Big). \tag{3.7}$$

Hence

$$|u^H|^2_{1,T}\le C\Big(\frac{1}{H^2}\|u\|^2_{L^2(T)}+\log\frac{H}{h}|u|^2_{1,T}\Big). \tag{3.8}$$

By adding an arbitrary constant on both sides of (3.7) and using Poincare inequality we get

$$|u^H|^2_{1,T}\le C\Big(1+\log\frac{H}{h}\Big)|u|^2_{1,T}. \tag{3.9}$$

Summing (3.5) or (3.9) up with respect to all elements of the triangulation $\mathcal{T}$, we get

$$|u^H|_{1,\Omega}^2 \le C\Big(1+\log\frac{H}{h}\Big)|u|_{1,\Omega}^2.$$

Hence

$$\|u^H\|_{1,\Omega}^2 \le C\Big(1+\log\frac{H}{h}\Big)\|u\|_{1,\Omega}^2.$$

Therefore,

$$\|u^H\|^2+\|u^h\|^2 = \|u^H\|^2+\|u-u^H\|^2 \le 2(\|u^H\|^2+\|u\|^2) \le C\Big(1+\log\frac{H}{h}\Big)\|u\|^2. \quad (3.10)$$

From (3.10) and Lemma 3.2 we get

$$a(u,u) \le C(1+log\frac{H}{h})a(P^H u+P^h u, u). \quad (3.11)$$

By (3.3) and (3.11), we have proved

Theorem 3.1. *There exists a constant C independent of H and h, so that*

$$\mathrm{Cond}(Q^{-1}A) \le C\Big(1+\log\frac{H}{h}\Big).$$

This result shows that the preconditioner constructed in the last section for the local refined triangulation stiffness matrix is almost optimal.

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