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THE INTERSECTION OF A TRIANGULAR BÉZIER PATCH AND A PLANE*

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Abstract

In this paper, the problem of finding the intersection of a triangular Bézier patch and a plane is studied. For the degree that one frequently encounters in practice, i.e. $n = 2, 3$, an efficient and reliable algorithm is obtained, and computational steps are presented.

1. Introduction

In this paper, the problem of finding the intersection of a triangular Bézier patch and a plane is considered. Such a problem quite frequently comes up in practical CAGD computations. Of course, in practice one has to work with pp-surface rather than with a single patch. However, the basic algorithm has to deal with a single patch on its own. Valuable information from the neighbouring patches can be available only in the simplest case, i.e. when a plane intersects the boundary of the patch.

The intersection problem in a particular form arises often when one works with algebraically rather than parametrically represented planar curves.

Similar problems were considered in [1] and [4], where the intersection of a bicubic Bézier patch and a plane was considered.

Let T be a given triangle. The most natural way to express the parametric Bézier surface S on T is to write it in the barycentric form

$$S := S(F) := S^n(F) := \sum_{i+j+k=n} F_{ijk} B^n_{ijk}, \tag{1.1}$$

where

$$F_{ijk} := (x_{ijk}, y_{ijk}, z_{ijk})$$

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are given control points. The Bernstein basic functions B^n_{ijk} are defined as

$$B^n_{ijk}(q) := B^n_{ijk}(u,v,w) := \frac{n!}{i!j!k!}u^i v^j w^k$$

with (u,v,w),

$$0 \le u,v,w \le 1, \quad u+v+w=1$$

being the barycentric coordinates of a point $q \in T$. Let $\wp$ be a plane, given by the equation

$$\wp: \quad ax+by+cz+d=0. \tag{1.2}$$

Let us denote $S_\wp := S \bigcap \wp$. Take (1.1) componentwise into (1.2). This shows that $S^n(F)(q) = S^n(F)(u,v,w) \in S_\wp$ iff the Bernstein polynomial

$$B_n(f) := \sum_{i+j+k=n} f_{ijk} B^n_{ijk}$$

has a zero at $q = (u,v,w)$. Here the coefficients f_{ijk} are given as

$$f_{ijk} := ax_{ijk} + by_{ijk} + cz_{ijk} + d.$$

There is a very natural way of searching for zeros of a Bernstein polynomial defined on a triangle. Let T_1, T_2, T_3 denote the vertices of T, and $T_4 = (1-s)T_2 + sT_3$ for some fixed s, $0 \le s \le 1$. Let us denote

$$Q_s := B_n(f)|_{T_1T_4}.$$

Then by [2] Q_s is a Bernstein polynomial of one variable, with coefficients being polynomials in s. To be precise, recall $B^n_i(t) := \binom{n}{i} t^i (1-t)^{n-i}$. Then

$$Q_s(t) = \sum_{i=0}^{n} a_i(s) B^n_i(t) \tag{1.3}$$

with

$$a_i(s) := \sum_{j=0}^{i} f_{n-i,i-j,j} B^i_j(s). \tag{1.4}$$

The idea of a general algorithm is quite clear. Move s from 0 to 1, and at each step find the zeros of Q_s. The information from the previous step can be taken as good starting approximation. Of course, at each step some zeros might disappear, and some others might be introduced.

If one is looking for an efficient and reliable algorithm, it is of crucial importance to know in advance if $S_\wp$ is actually not empty. If it is not, some information on positions of zeros is also necessary.

It is easy to verify numerically the following sufficient condition: $S_\wp$ is not empty if $B_n(f)$ has a zero on the border of T. This condition is also necessary for $n = 1$. The main result of this paper gives necessary and sufficient conditions also for $n = 2, 3$:

Theorem 1. *Let $B_2(f)$ be positive on the boundary of T. $S_\wp$ is not empty iff the polynomial*

$$A_1(s) \quad := \quad a_1^2(s) - a_0(s)a_2(s)$$

ha as zero $s^* \in (0,1)$, and at this zero $a_1(s^*) < 0$.

As will be pointed out in the next section, the algorithm for this case has to deal with constants rather than with the polynomial itself.

Theorem 2. *Let $B_3(f)$ be positive on the boundary of T. $S_\wp$ is not empty iff the polynomial*

$$A_1(s) := a_0^2(s)a_3^2(s) + 4a_0(s)a_2^3(s) + 4a_3(s)a_1^3(s) - 3a_1^2(s)a_2^2(s) \\ -6a_0(s)a_1(s)a_2(s)a_3(s)$$

has at least one zero $s^* \in (0,1)$, and at this zero

$$A_2(s^*) := \min(a_1(s^*), a_2(s^*)) < 0.$$

The properties of A_1, A_2 will be used to produce an efficient and reliable algorithm for this particular n. Restricting T_4 to the line T_2T_3 is quite arbitrary. One can easily find the other two formulations of Theorems 1 and 2, as well as translate them into the case of negative $B_2(f)$ or $B_3(f)$ along the boundary.

It is quite obvious that for $n = 4$ the same approach would work since the values of the zeros can be still exactly formulated in terms of coefficients, but higher degree algebraic equations admit no radical solutions. However, the work of [3] suggests that an algorithm could be found also in the general case and gives a good starting point for the future research work.

The proofs will be given in the next section; the algorithm for the case $n = 3$ and numerical examples will be given in Section 3.

2. The Proofs

In order to prove the theorems, we need the following lemmas.

Lemma 1. *Let*

$$p := \sum_{i=0}^{3} a_i B_i^3$$

be a cubic polynomial with $a_0 > 0, a_3 > 0$, and let

$$\Delta := a_0^2 a_3^2 + 4a_0a_2^3 + 4a_3a_1^3 - 3a_1^2a_2^2 - 6a_0a_1a_2a_3.$$

Then

1) *p has two different zeros in the interval $(0,1)$ iff $\min(a_1, a_2) < 0$ and $\Delta < 0$,*
2) *p has two equal zeros in the interval $(0,1)$ iff $\min(a_1, a_2) < 0$ and $\Delta = 0$,*
3) *p has no zero in the interval $(0,1)$ iff $\min(a_1, a_2) \geq 0$ or $\Delta > 0$.*

Proof. The transformation $x := \frac{t}{1-t}$ carries $(0,1)$ over to $(0,\infty)$. Put

$$f(x) := (1-t)^3 p(t) = a_0 + 3a_1 x + 3a_2 x^2 + a_3 x^3.$$

It is enough to study zeros of f. Since $a_0 > 0, a_3 > 0$, it is obvious that f has two different (equal) zeros in $(0,\infty)$ iff f has three zeros in $(-\infty,\infty)$ and $\min(a_1,a_2) < 0$ (one negative zero and one double zero as well as $\min(a_1,a_2) < 0$).

It is a well known fact how to tackle the cubic equation. Transformation $y := x + \frac{a_2}{a_3}$ reduces it to canonical form

$$a_3 y^3 + 3\Big(a_1 - \frac{a_2^2}{a_3}\Big) y + \frac{1}{a_3^2}(2a_2^3 - 3a_1 a_2 a_3 + a_0 a_3^2) = 0,$$

and the number of zeros depends on the sign of the discriminant

$$\Big(a_1 - \frac{a_2^2}{a_3}\Big)^3 + \Big(\frac{1}{2a_3^2}(2a_2^3 - 3a_1 a_2 a_3 + a_0 a_3^2)\Big)^2 = \frac{\Delta}{4a_3^4},$$

and our conclusions follow.

Lemma 2. *Let $B_3(f)$ have no zero on the boundary of T. Let there exist $s \in (0,1)$ such that Q_s has a zero in $(0,1)$. Then there must exist $\tilde{s} \in (0,1)$ such that $Q_{\tilde{s}}$ has a double zero in $(0,1)$.*

Proof. Since $Q_s(t)$ has no zero at the boundary, without loss of generality, assume that $Q_s(0) > 0$ for any s. If $Q_{\tilde{s}}$ has no zero in $(0,1)$, then there exists $\epsilon > 0$ such that for arbitrary $s \in (\tilde{s}-\epsilon, \tilde{s}+\epsilon)$, $Q_s(t) > 0, \ \ t \in (0,1)$. In fact, $Q_s(t)$ is always positive for $(s,t) \in (\tilde{s}-\epsilon, \tilde{s}+\epsilon) \times [0,1]$.

A similar argument reveals that, if $Q_{\tilde{s}}$ has two different zeros in $(0,1)$, then there exists $\epsilon > 0$ such that for arbitrary $s \in (\tilde{s}-\epsilon, \tilde{s}+\epsilon)$, Q_s also has two different zeros.

Put

$$\tilde{s} := \sup_{s'} A_{s'} := \sup\{s' | Q_s(t) \text{ has no zero for any } s \in [0,s']\}.$$

We proceed to show that $Q_{\tilde{s}}$ has a double zero in $(0,1)$. Note that $Q_{\tilde{s}}$ has at least one zero in $(-\infty,0)$ since it is a cubic polynomial, positive at 0. Hence $Q_{\tilde{s}}$ has no zero or two zeros in $(0,1)$. If it has none, by the previous argument one can find $\epsilon > 0$ such that $Q_s(t)$ has no zero in $(0,1)$ for $s \in (\tilde{s}-\epsilon, \tilde{s}+\epsilon)$. This implies $\tilde{s} + \frac{\epsilon}{2} \in A_{s'}$, a contradiction. Similar argument works to prove that zeros are not different.

Proof of Theorem 2. By Lemma 1 the sufficiency is obvious. Now if $S_\wp \neq \emptyset$, Q_s has a zero for some $s \in (0,1)$. But then, by Lemma 2, there exists $\tilde{s}$ such that $Q_{\tilde{s}}$ has a double zero. Lemma 1 then implies necessity.

Proof of Theorem 1. Note that Q_s from (1.3) is now a quadratic polynomial, and standard arguments can be used to pin the zero down to $(0,1)$.

Since A_1 and a_1 of Theorem 1 are now polynomials of degree two and one respectively, Theorem 1 can be stated in a computationally simpler form:

$B_2(f)$ being positive on the boundary of T is equivalent to

$$f_{200} > 0, \ f_{020} > 0, \ f_{002} > 0,$$

$$f_{110} + \sqrt{f_{200}f_{020}} > 0,\ f_{101} + \sqrt{f_{200}f_{002}} > 0,\ f_{011} + \sqrt{f_{020}f_{002}} > 0. \tag{2.5}$$

If (2.5) holds, $S_\wp \neq \emptyset$ if and only if

1) Of $f_{110}, f_{101}, f_{011}$ at least two are negative (without loosing generality assume $f_{110}, f_{101} < 0$),

2) $f_{110}f_{101} > f_{200}f_{011}$,

3) $(f_{110}f_{101} - f_{200}f_{011})^2 > (f_{110}^2 - f_{200}f_{020})(f_{101}^2 - f_{200}f_{002})$.

To conclude this section, we add two cubic examples.

Example 1. Let

$$T = \{(x,y)|, 0 \le x, y \le x+y \le 1\},$$

$$\wp : z = 0,$$

$$S : z = -\frac{1}{2}x^3 + 3x^2(1-x-y) + 3x(1-x-y)^2 - \frac{1}{2}(1-x-y)^3 + 3x^2y$$
$$- 18x(1-x-y)y + 3(1-x-y)^2y - 3(1-x-y)y^2 + 2y^3.$$

Then

$$(x_{ijk}) = \begin{pmatrix} x_{300} & & & \\ x_{210} & x_{201} & & \\ x_{120} & x_{111} & x_{102} & \\ x_{030} & x_{021} & x_{012} & x_{003} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(y_{ijk}) = \begin{pmatrix} y_{300} & & & \\ y_{210} & y_{201} & & \\ y_{120} & y_{111} & y_{102} & \\ y_{030} & y_{021} & y_{012} & y_{003} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ \frac{1}{3} & 0 & & \\ \frac{2}{3} & \frac{1}{3} & 0 & \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix},$$

$$(f_{i,j,k}) = (z_{ijk}) = \begin{pmatrix} z_{300} & & & \\ z_{210} & z_{201} & & \\ z_{120} & z_{111} & z_{102} & \\ z_{030} & z_{021} & z_{012} & z_{003} \end{pmatrix} = \begin{pmatrix} 2 & & & \\ -1 & 0 & & \\ 1 & -3 & 1 & \\ -\frac{1}{2} & 1 & 1 & -\frac{1}{2} \end{pmatrix}.$$

It is straightforward to compute

$$a_0(s) = 2, \quad a_1(s) = s - 1, \quad a_2(s) = (1-s)^2 - 6(1-s)s + s^2,$$

$$a_3(s) = -\frac{1}{2}(1-s)^3 + 3(1-s)^2s + 3(1-s)s^2 - \frac{1}{2}s^3,$$

$$A_1(s) = 2 - 72s + 930s^2 - 5130s^3 + 11481s^4 - 11106s^5 + 3904s^6,$$

$$A_2(s) = \min(a_1(s), a_2(s)) \le a_1(s) = s - 1 < 0.$$

Since $A_1(0) = 2$, $A_1(\frac{1}{2}) = -\frac{45}{4} < 0, A_1(s)$ has at least one zero in $(0,1)$. Theorem 2 implies $S_\wp \neq \emptyset$.

Example 2. Let (x_{ijk}), (y_{ijk}) be the same as in Example 1, and

$$(f_{i,j,k}) = (z_{ijk}) = \begin{pmatrix} 1 & & & \\ 0 & 0 & & \\ \frac{1}{2} & -\frac{1}{4} & 0 & \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$a_0(s) = 1, \;\; a_1(s) = 0,$$

$$a_2(s) = \frac{1}{2}(1-s)(1-2s), \;\; a_3(s) = (1-s)^3 + s^3,$$

$$A_1(s) = ((1-s)^3 + s^3)^2 + \frac{1}{2}(1-s)^3(1-2s)^3,$$

$$A_2(s) = \begin{cases} 0, & 0 \le s < \frac{1}{2}, \\ \frac{1}{2}(1-s)(1-2s), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since, for $0 \le s \le \frac{1}{2}$, $A_2(s) = 0$, Theorem 2 implies $S_\wp = \emptyset$. For $\frac{1}{2} \le s \le 1$, one has

$$0 < (1-s)(2s-1) \le \frac{1}{4}s^2;$$

hence

$$A_1(s) \ge ((1-s)^3 + s^3)^2 - \frac{1}{128}s^6 > 0,$$

and $S_\wp = \emptyset$ also for this part of the interval.

3. The algorithm for finding $S_\wp$

We shall consider only the case $n = 3$. The idea of the algorithm is quite simple. The number of zeros of Q_s can change only if $B_3(f)$ has a zero on the boundary for that s or Q_s has a double zero. Thus one has to determine the zeros at the boundary T_2T_3, i.e. the set

$$M_1 := \{s \mid Q_s(1) = a_3(s) = 0, \; 0 \le s \le 1\}.$$

Since a_3 is a cubic polynomial, Lemma 1 can be applied here also. A necessary condition for Q_s having a double zero is $A_2 < 0$. Put

$$M_2 := \{s \mid A_2(s) < 0, \; 0 \le s \le 1\}. \tag{3.6}$$

and

$$M_3 := \{s \mid A_1(s) = 0, \; s \in M_2\}. \tag{3.7}$$

(3.6) requires finding a minimum of the linear and quadratic functions, and (3.7) the solution of a polynomial equation of degree 6. In order to take care of the zeros at T_1T_2

and T_1T_3 we add also $0, 1$ to produce finally

$$M := \{0 = s_0 < s_1 < \ldots < s_m = 1\} := M_1 \cup M_3 \cup \{0, 1\}.$$

For each i now the number of zeros in (s_i, s_{i+1}) is constant. Also, the change of the number of zeros at s_i is at most 3. Actual computation of the zeros in (s_i, s_{i+1}) is as follows:

1) Determine the number and the positions of zeros at

$$s_{i+\frac{1}{2}} := \frac{s_i + s_{i+1}}{2}$$

by some reliable method such as the Sturm sequence combined with the bisection (or reliable polynomial solver). This gives the t parameter starting values $t_{i+\frac{1}{2},j}$.

2) If positive, follow up each curve, starting at

$$(s_{i+\frac{1}{2}}, t_{i+\frac{1}{2},j})$$

and proceeding in both directions in small steps (h_s, h_t), using Newton iteration.

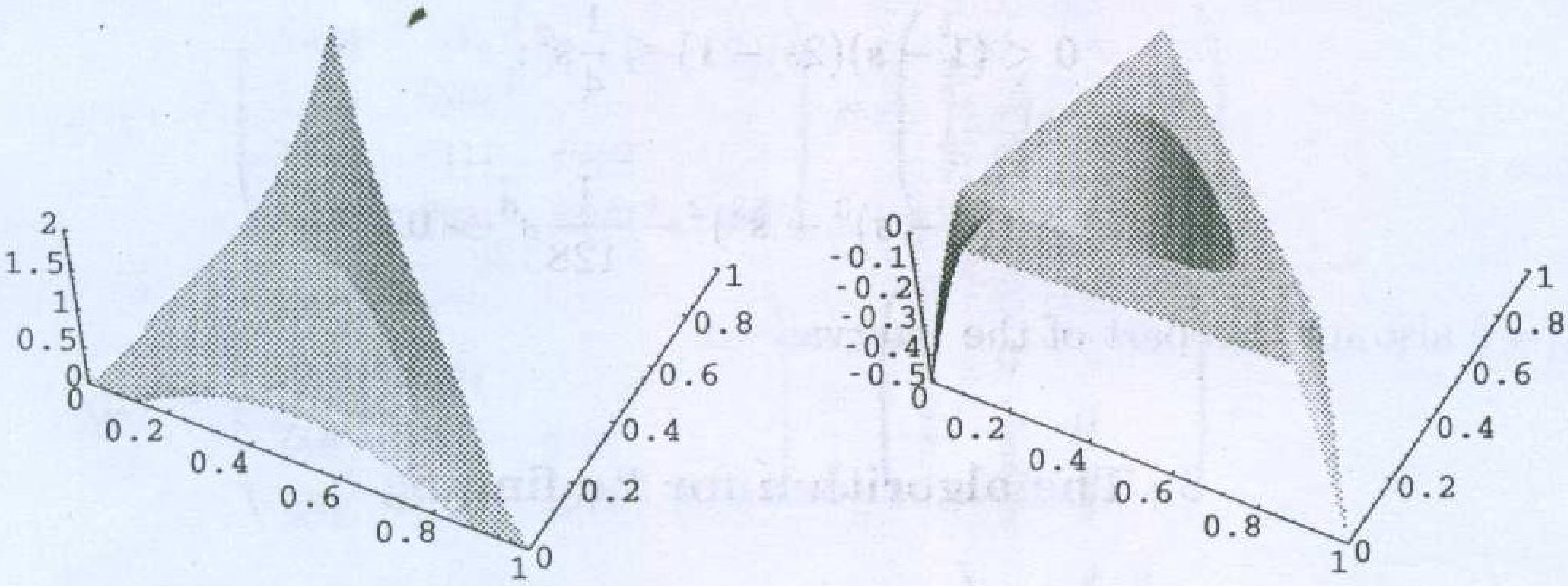

Fig.1. Positive and negative part of $B_n(f)$

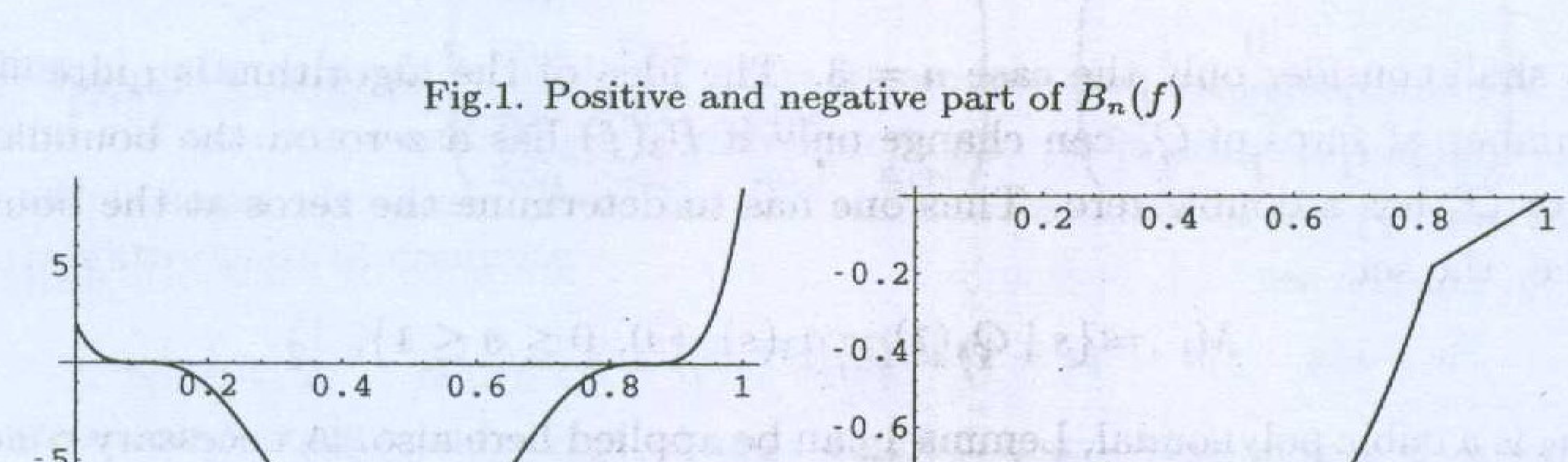

Fig.2. Plot of $A_1(s), A_2(s)$

Recall Example 1 of the previous section. Plot of the positive and negative part of $B_n(f)$ (Fig. 1) clearly indicates the intersection curves.

The (s,t) form is given as

$$Q_s(t) = 2(1-t)^3 + 3(s-1)(1-t)^2 t + 3((1-s)^2 - 6(1-s)s + s^2)(1-t)t^2$$
$$+ \Big(-\frac{1}{2}(1-s)^3 + 3(1-s)^2 s + 3(1-s)s^2 - \frac{1}{2}s^3\Big)t^3.$$

The first step of the algorithm reveals two boundary zeros

$$M_1 = \{0.12732200375003,\ 0.87267799624996\}.$$

Since $M_2 = (0,1)$ one has to search for zeros of $A_1(s)$ on the whole interval (Fig. 2).

This produces four double zeros

$$M_3 = \{\ 0.081174734178281,\ 0.083790254377921,\ 0.121559890757154,\ 0.823024808975617\ \},$$

and finally $|M| = 8$. The computation of zeros of Q_s for middle points $s = s_{i+\frac{1}{2}}$,

$$\{\ 0.0405873670891405,\ 0.082482494278101,\ 0.102675072567537,\ 0.1244409472535945,\ 0.475173406362826,\ 0.847851402612791,\ 0.9363389981249820\ \},$$

gives the following intersection point values of the parameter t (Table 1).

Table 1. The starting intersection values $t_{i+\frac{1}{2},j}$

	# of z.	1	2	3
1	1	0.826789814115618		
2	3	0.597944879897707	0.70188818752868	0.78176733854285
3	1	0.502083652673187		
4	3	0.465220805263716	0.88354066065651	0.96898115127681
5	2	0.38476130622908 7	0.83784538244413	
6	0			
7	1	0.872379230723569		

The final computation is shown in Fig. 3. The dashed lines represent the coordinate lines $s \in M$. Note that the first two are actually very close. Of course, putting T_1 at $\{0,0\}$ would be in this example numerically better. But a robust algorithm must make its way out also in chosen circumstances.

Note that the spacing steps h_s, h_t for each of the regions has to take account of how fast the curves are changing in t, not only in s. A difference between two neighbouring elements in M can be taken as a measure for h_s, and the Hausdorf distance between $\{t_{i+\frac{1}{2},j}\}$ as a measure of h_t.

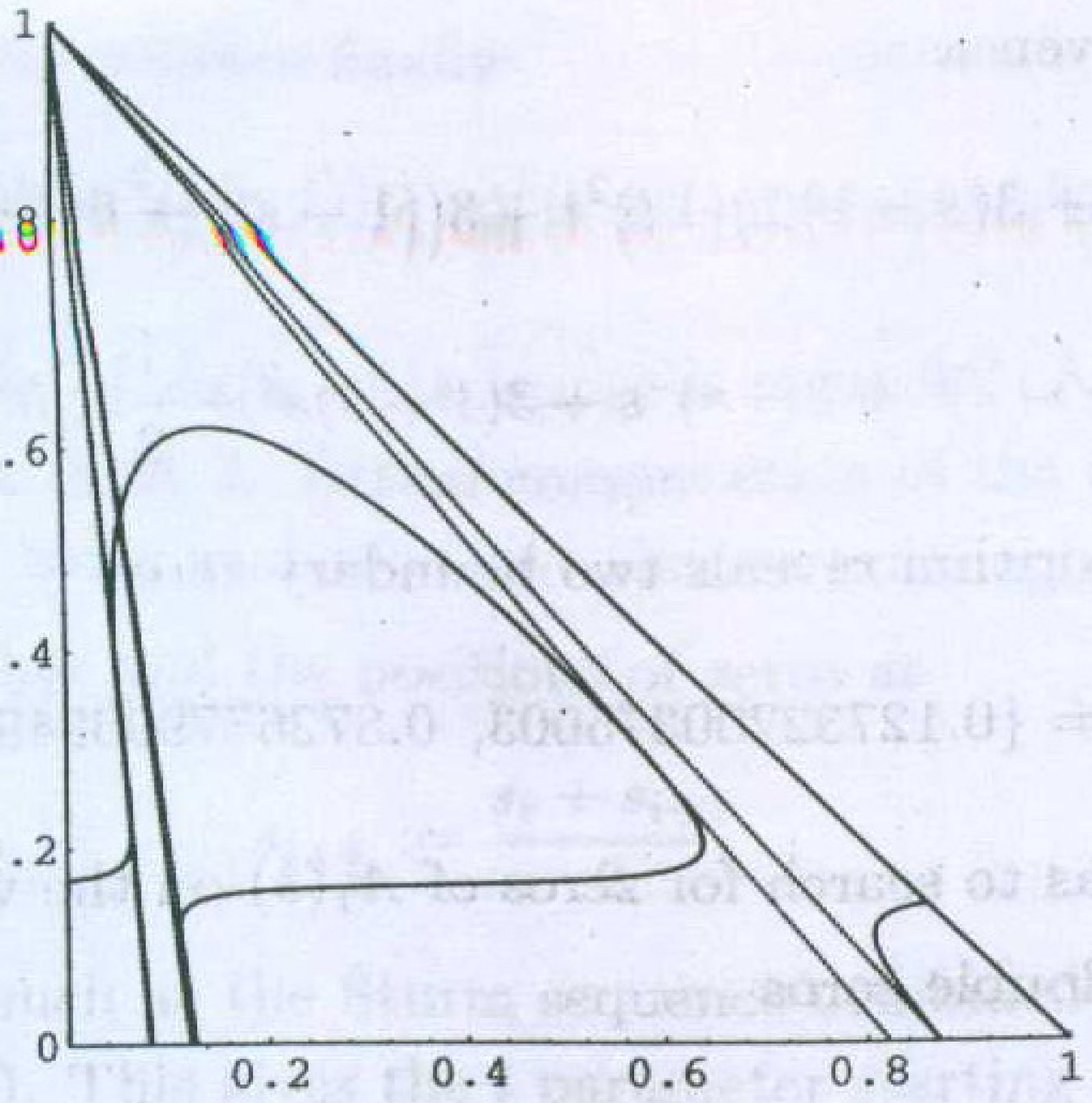

Fig.3. The intersection curves

References

[1] V. Chandru and B.S.Kochar, Analytic techniques for geometric intersection problems, in Geometric Modelling, ed. G. Farin, SIAM, 1987, Philadelphia, 316–317.

[2] G. Z. Chang and P. J. Davis, The convexity of Bernstein polynomials over triangles, *J. of Approx. Theory,* 40 (1984), 11–28.

[3] R. N. Goldman, T. W. Sederberg and D. C. Anderson, Vector elimination: A technique for implicitization, inversion, and intersection of planar parametric rational polynomial curves, *CAGD,* **1**(1984), 327–356.

[4] F. Qing-xiang, The intersection of a bicubic Bézier patch and a plane, *CAGD,* **7** (1990), 475–488.