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ON AN ESSENTIAL ESTIMATE IN THE ANALYSIS OF DOMAIN DECOMPOSITION METHODS*1)

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Abstract

A class of nonconforming finite elements is considered in this paper, which is continuous only at the nodes of the quasi-uniform mesh. We show that there exists an essential estimate which indicates the equivalence relation, independent of the mesh parameter, between the energies of the nonconforming discrete harmonic extensions in different subdomains. The essential estimate is of great importance in the analysis of the nonoverlapping domain decomposition methods applied to second order partial differential equations discretized by nonconforming finite elements.

1. Main Result

Let Ω be a bounded connected open domain in R^2 with a piecewise smooth boundary $\partial\Omega$, $a_{ij}(x), i,j = 1,2$, piecewise smooth bounded functions in Ω, and $(a_{ij}(x))$ a symmetric, uniformly positive definite matrix in Ω. Ω is divided into two open subdomains Ω_1, Ω_2 by an open smooth curve Γ , which satisfies

$$\Omega_1 \cap \Omega_2 = \phi, \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}, \quad \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma.$$

We make the following definitions:

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \tag{1.1}$$

$$a_k(u,v) = \int_{\Omega_k} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad k = 1,2,$$

$$V_H = \{(v_1,v_2) : v_k \in H^1(\Omega_k), \quad v_k|_{\partial\Omega_k \cap \partial\Omega} = 0, \quad v_1|_\Gamma = v_2|_\Gamma,$$

$$a_k(v_k,\theta) = 0, \quad \forall \theta \in H_0^1(\Omega_k), \quad k = 1,2\}.$$

v_k is called the a_k harmonic extension in $\Omega_k, k = 1,2$, if $(v_1,v_2) \in V_H$. Using the trace theorem and a well-known priori inequality[1,6], we obtain

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Theorem 1[1]. *There exist two positive constants σ_1, τ_1, such that*

$$\tau_1 a_2(v_2, v_2) \leq a_1(v_1, v_1) \leq \sigma_1 a_2(v_2, v_2), \quad \forall (v_1, v_2) \in V_H.$$

In what follows, for simplicity, we assume that Ω is a bounded polygonal domain in R^2 with a quasi-uniform mesh $\Omega_h = \{e\}$, where e , a triangle or a quadrilateral, represents the typical element in Ω_h . Let Ω_h be compatible with the subdomain division, i.e.

$$e \cap \Gamma = \phi, \quad \forall e \in \Omega_h$$

Let $S^h(\Omega)$ be a conforming finite element space, e.g. the space of continuous piecewise linear or bilinear functions defined relative to the mesh Ω_h. We define

$$\begin{aligned} &S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega), \quad \Gamma_h = \Omega_h|_\Gamma, \\ &S^h(\Omega_k) = S_0^h(\Omega)|_{\overline{\Omega}_k}, \quad k = 1, 2, \\ &S_0^h(\Omega_k) = S^h(\Omega_k) \cap H_0^1(\Omega_k), \quad k = 1, 2, \\ &V_C = \{(v_1^h, v_2^h) : v_k^h \in S^h(\Omega_k), \quad v_1^h|_{\Gamma_h} = v_2^h|_{\Gamma_h}, \\ &\qquad\qquad a_k(v_k^h, \theta_h) = 0, \quad \forall \theta_h \in S_0^h(\Omega_k), \quad k = 1, 2\}. \end{aligned}$$

v_k^h is called the conforming discrete a_k harmonic extension in Ω_k, $k = 1, 2$, if $(v_1^h, v_2^h) \in V_C$. Concerning V_C, we have

Theorem 2[2]. *If the mesh Ω_h is quasi-uniform, then there exist two positive constants σ_2, τ_2 , independent of the mesh parameter h, such that*

$$\tau_2 a_2(v_2^h, v_2^h) \leq a_1(v_1^h, v_1^h) \leq \sigma_2 a_2(v_2^h, v_2^h), \quad \forall (v_1^h, v_2^h) \in V_C.$$

Theorems 1 and 2 are the foundations of the analysis of nonoverlapping domain decomposition methods applied to second order partial differential equations in the continuous case and in the conforming discrete case respectively. The aim of this paper is to show that an estimate , similar to *Theorem* 1 and *Theorem* 2, is ture for a class of nonconforming finite elements.

Let $T^h(\Omega)$ be the set $\{v^h : v^h = w^h + u^h, w^h \in \hat{T}^h(\Omega),\ u^h|_e$ is a finite order polynomial, $\forall e \in \Omega_h, u^h(x) = 0, \forall$ node $x \in \Omega_h\}$, where $\hat{T}^h(\Omega) = \{v^h \in C(\Omega) : v^h|_e$ is a linear function if e is a triangle, or a bilinear function if e is a quadrilateral, $\forall e \in \Omega_h\}$. Here, a node $x \in \Omega_h$ is defined to be the vertex of some $e \in \Omega_h$.

$T^h(\Omega)$ is a class of nonconforming finite elements, which is continuous only at the nodes of the mesh Ω_h, and many practical nonconforming elements possess this property (e.g. [3], [4], [5]). In what follows, we assume that the nonconforming approximate solution of a given problem exists uniquely and converges to the exact solution of the problem.

Now, we make the following definitions:

$$T_0^h(\Omega) = \{v^h \in T^h(\Omega) : v^h(x) = 0, \forall \text{ node } x \text{ on } \partial\Omega\},$$

$$T^h(\Omega_k) = T_0^h(\Omega)|_{\overline{\Omega}_k},$$

$$T_0^h(\Omega_k) = \{v^h \in T^h(\Omega_k) : v^h(x) = 0, \forall \text{ node } x \text{ on } \Gamma_h\},$$

$$A_k(u^h, v^h) = \sum_{e \subset \Omega_k} \int_e \sum_{i,j=1}^{2} a_{ij} \frac{\partial u^h}{\partial x_i} \frac{\partial v^h}{\partial x_j}, \quad k = 1, 2,$$

$$V_N = \{(v_1^h, v_2^h) : v_k^h \in T^h(\Omega_k), \quad v_1^h(x) = v_2^h(x), \forall \text{ node } x \text{ on } \Gamma_h, \\ A_k(v_k^h, \theta_h) = 0, \quad \forall \theta_h \in T_0^h(\Omega_k), \quad k = 1, 2\}.$$

v_k^h is called the nonconforming discrete A_k harmonic extension in $\Omega_k, k = 1, 2$, if $(v_1^h, v_2^h) \in V_N$. The property of V_N is stated as follows:

Theorem 3. *If the mesh Ω_h is quasi-uniform, then there exist two positive constants σ_3, τ_3, independent of the mesh parameter h, such that*

$$\tau_3 A_2(v_2^h, v_2^h) \le A_1(v_1^h, v_1^h) \le \sigma_3 A_2(v_2^h, v_2^h), \quad \forall (v_1^h, v_2^h) \in V_N. \tag{1.2}$$

The complete proof of *Theorem* 3 is given in §2. In what follows, c will denote a generic positive constant, which is possibly different in different places, but is independent of h.

2. The Proof of Theorem 3

Denote

$$\|v\|_{0,\Omega_k,h}^2 = \sum_{e \subset \Omega_k} \|v\|_{0,e}^2; \quad |v|_{1,\Omega_k,h}^2 = \sum_{e \subset \Omega_k} |v|_{1,e}^2, \quad k = 1, 2$$

and define the interpolation operator $I_h : T^h(\Omega) \longrightarrow C(\overline{\Omega})$ as follows:

$$\forall v^h \in T^h(\Omega), \quad I_h v^h \in \hat{T}^h(\Omega), \quad (I_h v^h)(x) = v^h(x), \quad \forall \text{ node } x \in \Omega_h,$$

with regard to I_h, we have

Lemma 4. *There exists a positive constant c, such that for arbitrary $v^h \in T^h(\Omega_k)$ and $k = 1, 2$,*

$$\|I_h v^h\|_{0,\Omega_k,h} \le c\|v^h\|_{0,\Omega_k,h}, \quad |I_h v^h|_{1,\Omega_k,h} \le c|v^h|_{1,\Omega_k,h}.$$

Proof. It follows from the interpolation theorem and an "inverse property" implied by the quasi-uniformness of the mesh Ω_h that

$$\|I_h v^h\|_{0,e} \le \|v^h\|_{0,e} + \|v^h - I_h v^h\|_{0,e} \le \|v^h\|_{0,e} + ch|v^h|_{1,e} \le c\|v^h\|_{0,e} \quad \forall e \subset \Omega_k.$$

Summing up over all the elements $e \subset \Omega_k$ gives the first inequality. The second one can be done similarly.

The proof of Theorem 3. Suppose $(u_1^h, u_2^h) \in V_N$. We denote $\{x_i\}_{i=1}^m$ as the set of nodes on Γ_h. λ is a piecewise linear continuous function on Γ_h with $\lambda(x_i) = u_1^h(x_i) =$

$u_2^h(x_i)$, $i=1,2,\cdots,m$. Now, we construct $u_k \in H^1(\Omega_k)$, the a_k harmonic extension of λ, as follows:

$$\begin{cases} a_k(u_k,\theta)=0, & \forall\theta\in H_0^1(\Omega_k),\\ u_k=\lambda, & \text{on } \Gamma,\\ u_k=0, & \text{on } \partial\Omega_k\backslash\Gamma. \end{cases} \tag{2.1}$$

Correspondingly, u_k^h is the nonconforming approximation of u_k.

For conciseness, we make the following definitions:

$$W^h(\Omega_k)=\hat{T}^h(\Omega)|_{\overline{\Omega}_k},$$

$$W_k^h(\Omega_k)=\{v^h\in W^h(\Omega_k): v^h(x)=u_k^h(x), \forall \text{ node } x\in\partial\Omega_k\},$$

$$T_k^h(\Omega_k)=\{v^h\in T^h(\Omega_k): v^h(x)=u_k^h(x), \forall \text{ node } x\in\partial\Omega_k\}.$$

For arbitrary $w^h\in T_k^h(\Omega_k)$, we have

$$\begin{aligned}0\le A_k(w^h-u_k^h-u_k,w^h-u_k^h-u_k)&=A_k(w^h-u_k^h,w^h-u_k^h)\\&+A_k(u_k,u_k)-2A_k(u_k,w^h-u_k^h)\end{aligned}$$

and then

$$\begin{aligned}A_k(w^h-u_k,w^h-u_k)&=A_k(w^h-u_k^h,w^h-u_k^h)+A_k(u_k^h-u_k,u_k^h-u_k)\\&+2A_k(w^h-u_k^h,u_k^h-u_k)=A_k(w^h-u_k^h,w^h-u_k^h)+A_k(u_k^h-u_k,u_k^h-u_k)\\&-2A_k(w^h-u_k^h,u_k)\ge A_k(u_k^h-u_k,u_k^h-u_k)-A_k(u_k,u_k).\end{aligned}$$

Thus,

$$\begin{aligned}A_k(u_k^h,u_k^h)&\le 2[A_k(u_k,u_k)+A_k(u_k^h-u_k,u_k^h-u_k)]\\&\le 4[A_k(u_k,u_k)+A_k(w^h-u_k,w^h-u_k)].\end{aligned}$$

It follows from the arbitrariness of w^h in $T_k^h(\Omega_k)$ and the last inequality that

$$A_k(u_k^h,u_k^h)\le 4[A_k(u_k,u_k)+\inf_{w^h\in T_k^h(\Omega_k)}A_k(w^h-u_k,w^h-u_k)]. \tag{2.2}$$

Applying a well-known a priori inequality[1,6] to (2.1) gives

$$A_k(u_k,u_k)=a_k(u_k,u_k)\le c|u_k|^2_{\frac{1}{2},\partial\Omega_k}\le c|\lambda|^2_{\frac{1}{2},\Gamma}, \tag{2.3}$$

where $|\cdot|_{\frac{1}{2},\Gamma}$ is the norm of Sobolev space $H_{00}^{\frac{1}{2}}(\Gamma)$[1].

Since $W_k^h(\Omega_k)\subset H^1(\Omega_k)\cap T_k^h(\Omega_k)$, $W^h(\Omega_k)\subset H^1(\Omega_k)$, it follows from the approximation properties of $W^h(\Omega_k)$ that, for $0<\varepsilon<\frac{1}{2}$,

$$\begin{aligned}\inf_{w^h\in T_k^h(\Omega_k)}A_k(w^h-u_k,w^h-u_k)&\le\inf_{w^h\in W_k^h(\Omega_k)}A_k(w^h-u_k,w^h-u_k)\\&\le\inf_{w^h\in W_k^h(\Omega_k)}a_k(w^h-u_k,w^h-u_k)\le ch^{2\varepsilon}\|u_k\|^2_{H^{1+\varepsilon}(\Omega_k)}\end{aligned} \tag{2.4}$$

by the a priori inequality[1,6] and an "inverse property" implied by the quasi-uniformness of the mesh Ω_h, we see that

$$h^{2\varepsilon}\|u_k\|^2_{H^{1+\varepsilon}(\Omega_k)} \le ch^{2\varepsilon}|u_k|^2_{H^{\frac{1}{2}+\varepsilon}(\partial\Omega_k)} \le c|u_k|^2_{\frac{1}{2},\partial\Omega_k} \le c|\lambda|^2_{\frac{1}{2},\Gamma}. \tag{2.5}$$

It follows from the substitution of (2.3), (2.4) and (2.5) into (2.2) that

$$A_k(u_k^h, u_k^h) \le c|\lambda|^2_{\frac{1}{2},\Gamma}. \tag{2.6}$$

On the other hand, Lemma 4 implies that

$$|I_h u_k^h|^2_{1,\Omega_k,h} \le c|u_k^h|^2_{1,\Omega_k,h}. \tag{2.7}$$

Since $I_h u_k^h \in H^1(\Omega_k)$, $I_h u_k^h\Big|_{\partial\Omega_k\setminus\Gamma} = 0$, $I_h u_k^h\Big|_{\Gamma} = \lambda$, using the Poincarè inequality yields

$$\|I_h u_k^h\|^2_{1,\Omega_k} \le c|I_h u_k^h|^2_{1,\Omega_k} = c|I_h u_k^h|^2_{1,\Omega_k,h}. \tag{2.8}$$

It follows from the trace theorem that

$$\|I_h u_k^h\|_{1,\Omega_k} \ge c|I_h u_k^h|_{\frac{1}{2},\partial\Omega_k} \ge c|I_h u_k^h|_{\frac{1}{2},\Gamma} = c|\lambda|_{\frac{1}{2},\Gamma}. \tag{2.9}$$

By the uniform positive definiteness of the matrix $\big(a_{ij}(x)\big)$ in Ω, we see that

$$A_k(u_k^h, u_k^h) \ge c|u_k^h|^2_{1,\Omega_k,h}. \tag{2.10}$$

Combination of (2.7), (2.8), (2.9), and (2.10) gives

$$A_k(u_k^h, u_k^h) \ge c|\lambda|^2_{\frac{1}{2},\Gamma}. \tag{2.11}$$

(2.6) and (2.11) complete the proof.

Remark 1. If we change (1.1) into

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + d(x)uv \tag{2.12}$$

where $d(x) \ge 0$ in Ω, Theorem 3 holds in this case also.

Remark 2. From the proof of Theorem 3 we know that $A_k(u_k^h, u_k^h)$ is equivalent to $|\lambda|^2_{\frac{1}{2},\Gamma}$, where λ is the restriction to Γ of the conforming part of u_k^h, if $(u_1^h, u_2^h) \in V_N$. Therefore, $T^h(\Omega)$ can be viewed as

$$T^h(\Omega) = \{v^h : v^h \text{ is continuous at the nodes of mesh } \Omega_h,$$
$$v^h\Big|_e \text{ is a finite order polynomial}, \forall\, e \in \Omega_h\}$$

Remark 3. By Lemma 4 and the uniform positive definiteness of the matrix $\big(a_{ij}(x)\big)$ in Ω, there exists a positive constant c, such that

$$cA_k(I_h v_k^h, I_h v_k^h) \le A_k(v_k^h, v_k^h) \le A_k(I_h v_k^h, I_h v_k^h), \quad \forall (v_1^h, v_2^h) \in V_N,$$

which means that the energies of nonconforming discrete extension v_k^h and its conforming part $I_h v_k^h$ are equivalent.

3. Conclusion

Based on Theorem 3, we can prove that all existing nonoverlapping domain decomposition methods valid for the conforming finite element discretization of second order partial differential equations (e.g. [2], [7], [8], [9], [10], [11]) can be generalized to the nonconforming discrete case, which has been done by the authors and will be given in the forthcoming papers.

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