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$(0, 1, \cdots, m-2, m)$ INTERPOLATION FOR THE LAGUERRE ABSCISSAS[*1)]

Shi Ying-guang

(Computing Center, Academia Sinica, Beijing, China)

Abstract

A necessary and sufficient condition of regularity of $(0, 1, \cdots, m-2, m)$ interpolation on the zeros of the Laguerre polynomials $L_n^{(\alpha)}(x)$ $(\alpha \geq -1)$ in a manageable form is established. Meanwhile, the explicit representation of the fundamental polynomials, when they exist, is given. Moreover, it is shown that, if the problem of $(0, 1, \cdots, m-2, m)$ interpolation has an infinity of solutions, then the general form of the solutions is $f_0(x) + Cf_1(x)$ with an arbitrary constant C.

1. Introduction

Let us consider a system A of nodes

$$0 \leq x_1 < x_2 < \cdots < x_n, \quad n \geq 2. \tag{1.1}$$

Let $\mathbf{P}_n$ be the set of polynomials of degree at most n and $m \geq 2$ fixed integer. The problem of $(0, 1, \cdots, m-2, m)$ interpolation is, given a set of numbers

$$y_{kj}, \quad k \in N := \{1, 2, \cdots, n\}, \ \ j \in M := \{0, 1, \cdots, m-2, m\}, \tag{1.2}$$

to determine a polynomial $R_{mn-1} \in \mathbf{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k) = y_{kj}, \quad \forall k \in N, \ \ \forall j \in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1} \in \mathbf{P}_{mn-1}$ satisfying (1.3), then we say that the problem of $(0, 1, \cdots, m-2, m)$ interpolation on A is regular (otherwise, singular) and $R_{mn-1}(x)$ can be written uniquely as

$$R_{mn-1}(x) = \sum_{\substack{k \in N \\ j \in M}} y_{kj} r_{kj}(x) \tag{1.4}$$

where $r_{kj} \in \mathbf{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k, \nu \in N, \ \ j, \mu \in M \tag{1.5}$$

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and are called the fundamental polynomials. In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \quad k = 1, 2, \cdots, n. \tag{1.6}$$

In [1] and [2] the exact condition of regularity on the parameter $\alpha \geq -1$ of the Laguerre polynomials $L_n^{(\alpha)}(x)$ is found for $(0,2)$ interpolation based on the zeros of these polynomials. The problem of determining the fundamental polynomials is also discussed. But the latter problem is solved for $\alpha = -1$ only. For $\alpha > -1$ the representation of fundamental polynomials is given only in the case when α is an odd integer and only on $(-\infty, 0)$, while a representation on $[0, \infty)$ would be more important. Following the main idea of [1] and [2], in this paper we attempt to give a necessary and sufficient condition of regularity of $(0, 1, \cdots, m-2, m)$ interpolation for the Laguerre abscissas. Meanwhile, we develop a method of finding the explicit representation of the fundamental polynomials when they exist without exception. Thus, our results improve and extend the ones of [1] and [2]. Finally, when the problem of $(0, 1, \cdots, m-2, m)$ interpolation on A is not regular, then for a given set of numbers y_{kj} either there is no polynomial $R_{mn-1}(x)$ satisfying (1.3) or there is an infinity of polynomials with the property (1.3). The possibility of an infinity of solutions raises the question on the dimensionality of their number. We show that in the case of infinitely many solutions the general form of the solutions is

$$R_{mn-1}(x) = f_0(x) + Cf_1(x),$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials and C is an arbitrary number.

2. An Auxiliary Lemma

We first state a lemma given by the author in [3]. To this end we introduce the fundamental polynomials of $(0, 1, \cdots, m-1)$ interpolation. Let $A_{kj},\ B_k \in \mathbf{P}_{mn-1}$ be defined by

$$A_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k, \nu = 1, 2, \cdots, n, \quad j, \mu = 0, 1, \cdots, m-1 \tag{2.1}$$

and

$$B_k(x) := A_{k,m-1}(x) = \frac{1}{m!}(x - x_k)^{m-1} l_k^m(x), \quad k = 1, 2, \cdots, n, \tag{2.2}$$

where

$$l_k(x) := \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \quad \omega_n(x) = c(x - x_1)(x - x_2)\cdots(x - x_n), \quad c \neq 0. \tag{2.3}$$

Then we have

Lemma. *If there is one index i, $1 \leq i \leq n$, such that $\rho_i \in \mathbf{P}_{mn-1}$ with the properties (1.5) exists uniquely, then the problem of $(0, 1, \cdots, m-2, m)$ interpolation is* *regular and*

$$r_{kj}(x) = A_{kj}(x) - \sum_{\nu=1}^{n} A_{kj}^{(m)}(x_\nu)\rho_\nu(x), \quad k=1,2,\cdots,n, \quad j=0,1,\cdots,m-2. \tag{2.4}$$

Remark. Since the explicit representation for A'_{kj}s is well known, by (2.4) it is sufficient to find the one for ρ'_ks.

3. Main Results

In what follows let n be fixed and (1.1) the zeros of $\omega_n(x) := L_n^{(\alpha)}(x)$. Write

$$\gamma := \frac{1}{2}(m-1)(\alpha+1), \tag{3.1}$$

$$\gamma_k := \frac{(-1)^k \binom{n+\alpha}{n-k}}{k!}, \quad k=0,1,\cdots,n. \tag{3.2}$$

It is well known that [2]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \gamma_k x^k \tag{3.3}$$

satisfies the differential equation

$$xL''^{(\alpha)}_n(x) + (\alpha+1-x)L'^{(\alpha)}_n(x) + nL_n^{(\alpha)}(x) = 0 \tag{3.4}$$

and

$$L_n^{(-1)}(x) = \frac{1}{n}xL'^{(0)}_n(x). \tag{3.5}$$

The first main result in this paper is the following

Theorem 1. *The problem of* $(0,1,\cdots,m-2,m)$ *interpolation on the zeros of* $L_n^{(\alpha)}(x)$ $(\alpha > -1)$ *is regular if and only if*

$$D_n(\alpha) = \sum_{k=0}^{n}\left(-\frac{m-1}{2}\right)^{n-k}\binom{n+\alpha}{n-k}\binom{\gamma}{k} \neq 0. \tag{3.6}$$

When $\alpha = -1$, *the problem is always regular.*

If the problem is regular, then for each i, $1 \le i \le n$, *the fundamental polynomial* $\rho_i(x) := \rho_i(x;\alpha)$ *is given by*

$$\rho_i(x) = [L_n^{(\alpha)}(x)]^{m-1}q_i(x), \tag{3.7}$$

in which $q_i \in \mathbf{P}_{n-1}$ *is of the form*

$$q_i(x) = x^\gamma e^{-\frac{(m-1)x}{2}}\left\{d_i + \int_a^x [Q_i(t) - c_i L_n^{(\alpha)}(t)]t^{-\gamma-1}e^{\frac{(m-1)t}{2}}dt\right\}, \tag{3.8}$$

with certain constants d_i and c_i, where

$$a = \begin{cases} 1, & \alpha > -1, \\ 0, & \alpha = -1, \end{cases} \tag{3.9}$$

$$Q_i(x) = \frac{x_i l_i(x)}{m![L_n'^{(\alpha)}(x_i)]^{m-1}}. \tag{3.10}$$

Proof. By the definition of ρ_i we may set

$$\rho_i(x) = \omega_n^{m-1}(x) q_i(x), \tag{3.11}$$

where $q_i \in \mathbf{P}_{n-1}$ will be determined later. Then the requirement (1.5) yields

$$[\omega_n^{m-1}(x) q_i(x)]_{x=x_k}^{(m)} = \delta_{ik}, \quad k = 1, 2, \cdots, n. \tag{3.12}$$

It is easy to see that

$$[\omega_n^{m-1}(x)]_{x=x_k}^{(m)} = \frac{1}{2}(m-1)m!\omega_n'(x_k)^{m-2}\omega_n''(x_k)$$

and

$$[\omega_n^{m-1}(x)]_{x=x_k}^{(m-1)} = (m-1)!\omega_n'(x_k)^{m-1}.$$

Then (3.12) becomes

$$\frac{1}{2}(m-1)\omega_n''(x_k) q_i(x_k) + \omega_n'(x_k) q_i'(x_k) = \frac{\delta_{ik}}{m!\omega_n'(x_k)^{m-2}}, \quad k = 1, 2, \cdots, n. \tag{3.13}$$

It follows from (3.4) that

$$x_k \omega_n''(x_k) + (\alpha + 1 - x_k)\omega_n'(x_k) = 0, \quad k = 1, 2, \cdots, n. \tag{3.14}$$

This, coupled with (3.13), gives

$$x_k q_i'(x_k) + \left(\frac{m-1}{2} x_k - \gamma\right) q_i(x_k) = \frac{x_k \delta_{ik}}{m!\omega_n'(x_k)^{m-1}}, \quad k = 1, 2, \cdots, n$$

or

$$x_k q_i'(x_k) + \left(\frac{m-1}{2} x_k - \gamma\right) q_i(x_k) = \frac{x_i \delta_{ik}}{m!\omega_n'(x_i)^{m-1}}, \quad k = 1, 2, \cdots, n. \tag{3.15}$$

Denote by $\mathbf{D}$ the differential operator

$$\mathbf{D}y := xy' + \left(\frac{m-1}{2} x - \gamma\right) y.$$

Then (3.15) implies

$$\mathbf{D}q_i(x) = Q_i(x) - c_i \omega_n(x), \tag{3.16}$$

where c_i is a constant to be determined and $Q_i(x)$ is given by (3.10). Solving this differential equation we get (3.8) with a constant d_i to be determined.

Now let us determine c_i and d_i. To this end, put

$$q_i(x) = \sum_{k=0}^{n-1} \alpha_k x^k. \tag{3.17}$$

Meanwhile we write

$$Q_i(x) = \sum_{k=0}^{n-1} \beta_k x^k. \tag{3.18}$$

We distinguish two cases.

Case I. $\alpha > -1$. Using (3.17), (3.18), and (3.3), and comparing the coefficients of x^k on both sides of (3.16) we obtain the system of equations with the unknowns $\alpha_0, \cdots, \alpha_{n-1}, c_i$:

$$\begin{cases} \frac{m-1}{2}\alpha_{k-1} + (k-\gamma)\alpha_k + \gamma_k c_i = \beta_k, \quad k = 0, 1, \cdots, n, \\ \alpha_{-1} = \alpha_n = \beta_n = 0. \end{cases} \tag{3.19}$$

The coefficient determinant of this system is

$$D_n(\alpha) = \begin{vmatrix} -\gamma & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_0 \\ \frac{m-1}{2} & 1-\gamma & 0 & 0 & \cdots & 0 & 0 & \gamma_1 \\ 0 & \frac{m-1}{2} & 2-\gamma & 0 & \cdots & 0 & 0 & \gamma_2 \\ 0 & 0 & \frac{m-1}{2} & 3-\gamma & \cdots & 0 & 0 & \gamma_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{m-1}{2} & n-1-\gamma & \gamma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{m-1}{2} & \gamma_n \end{vmatrix}.$$

Expanding the coefficient determinant of this system in terms of the elements of the last column we get

$$\begin{aligned} D_n(\alpha) &= \sum_{k=0}^{n} (-1)^{n+2+k} \gamma_k \left(\frac{m-1}{2}\right)^{n-k} (-1)^k k! \binom{\gamma}{k} \\ &= \sum_{k=0}^{n} \left(-\frac{m-1}{2}\right)^{n-k} \binom{n+\alpha}{n-k} \binom{\gamma}{k}. \end{aligned}$$

We know that the system (3.19) has a unique solution if and only if (3.6) is true. By the lemma this is equivalent to the regularity of $(0,1,\cdots,m-2,m)$ interpolation.

Solving (3.19) by Cramer's rule we get c_i. As for d_i we note that by (3.8) and (3.17)

$$d_i = e^{\frac{m-1}{2}} q_i(1) = e^{\frac{m-1}{2}} \sum_{k=0}^{n-1} \alpha_k.$$

To calculate α_k we use (3.19) to get

$$\alpha_{k-1} = \frac{2}{m-1}\{\beta_k - \gamma_k c_i + (\gamma - k)\alpha_k\}, \quad k = n, n-1, \cdots, 1. \tag{3.20}$$

Case II. $\alpha = -1$. In this case $x_1 = 0$ and $\gamma = \gamma_0 = \beta_0 = 0$. Then the equation corresponding to $k = 0$ in (3.19) becomes an identity. So we must find another equation. To this end, using (3.2) and (3.3) we get

$$L_n'^{(-1)}(0) = \gamma_1 = -1, \quad L_n''^{(-1)}(0) = 2\gamma_2 = n-1.$$

Substituting these values into (3.13) yields

$$\frac{(m-1)(n-1)}{2} q_i(0) - q_i'(0) = \frac{(-1)^m \delta_{i1}}{m!}.$$

Meanwhile, by means of (3.17) we obtain

$$\frac{(m-1)(n-1)}{2}\alpha_0 - \alpha_1 = \frac{(-1)^m \delta_{i1}}{m!}.$$

Adding this equation to the equation corresponding to $k = 1$ in (3.19) we get

$$\frac{n(m-1)}{2}\alpha_0 + \gamma_1 c_i = \beta_1 + \frac{(-1)^m \delta_{i1}}{m!}. \tag{3.21}$$

At last we obtain the system of equations for this case:

$$\begin{cases} \dfrac{n(m-1)}{2}\alpha_0 + \gamma_1 c_i = \beta_1 + \dfrac{(-1)^m \delta_{i1}}{m!}, \\ \dfrac{m-1}{2}\alpha_{k-1} + k\alpha_k + \gamma_k c_i = \beta_k, \quad k = 1, \cdots, n, \\ \alpha_n = \beta_n = 0. \end{cases} \tag{3.22}$$

Expanding the coefficient determinant of this system in terms of the elements of the last column we get

$$\begin{aligned} D_n(-1) &= (-1)^{n+2}\left\{\frac{n(m-1)}{2}\sum_{k=1}^{n}(-1)^k (k-1)!\left(\frac{m-1}{2}\right)^{n-k}\gamma_k + \left(\frac{m-1}{2}\right)^n \gamma_1\right\} \\ &= (-1)^n \frac{m-1}{2}\left\{\sum_{k=1}^{n}\left(\frac{m-1}{2}\right)^{n-k}\frac{n!}{k!(n-k)!} - \left(\frac{m-1}{2}\right)^{n-1}\right\} \\ &= (-1)^n \frac{m-1}{2}\left\{\sum_{k=0}^{n}\left(\frac{m-1}{2}\right)^{n-k}\binom{n}{k} - \left(\frac{m-1}{2}\right)^n - \left(\frac{m-1}{2}\right)^{n-1}\right\} \\ &= (-1)^n \frac{m-1}{2}\left\{\left(1 + \frac{m-1}{2}\right)^n - \frac{m+1}{2}\left(\frac{m-1}{2}\right)^{n-1}\right\} \\ &= (-1)^n \frac{m^2-1}{4}\left\{\left(\frac{m+1}{2}\right)^{n-1} - \left(\frac{m-1}{2}\right)^{n-1}\right\} \neq 0. \end{aligned}$$

Similarly, solving (3.22) we can determine c_i. Meanwhile, (3.8) and (3.17) imply

$$d_i = eq_i(0) = e\alpha_0 = \frac{2e}{n(m-1)}\left\{\beta_1 + c_i + \frac{(-1)^m\delta_{i1}}{m!}\right\}.$$

This completes the proof.

Although Theorem 1 gives a necessary and sufficient condition of regularity in a manageable form, it does not provide practical information of regularity on n and α. The next theorem provides a sufficient condition of this type in which $[\gamma]$ stands for the largest integer not larger than γ.

Theorem 2. *Let $\alpha > -1$. If for $\gamma = [\gamma]$*

$$n > \frac{1}{2}[(m+1)\gamma^2 + (3-m)\gamma - 2] \tag{3.23}$$

and if for $\gamma \neq [\gamma]$

$$n > \frac{(m-3)[\gamma]^2 + (4\gamma + m - 1)[\gamma] + 2\gamma}{2(\gamma - [\gamma])}, \tag{3.24}$$

then the problem is regular.

Proof. Let

$$a_k = \left(\frac{m-1}{2}\right)^{n-k}\binom{n+\alpha}{n-k}\binom{\gamma}{k}, \quad k = 0, 1, \cdots, n. \tag{3.25}$$

Then

$$a_k = \frac{2(n+1-k)(\gamma+1-k)}{(m-1)k(k+\alpha)}a_{k-1} := \tau_k a_{k-1}, \quad k = 1, \cdots, n. \tag{3.26}$$

Obviously, (3.23) and (3.24) imply $n \geq [\gamma]+1$. Thus $a_k > 0,\ 0 \leq k < \gamma+1$.

First we note

$$\tau_k < \tau_{k-1}, \quad 2 \leq k < \gamma + 1. \tag{3.27}$$

Meanwhile, it is easy to check that (3.23) and (3.24) are equivalent to $\tau_\gamma > 1$ for $\gamma = [\gamma]$ and to $\tau_{[\gamma]+1} > 1$ for $\gamma \neq [\gamma]$, respectively. Thus we have

$$a_k > a_{k-1} > 0, \quad 1 \leq k < \gamma + 1. \tag{3.28}$$

For the proof of the theorem, if γ is an integer, then

$$|D_n(\alpha)| = \left|\sum_{k=0}^{\gamma}(-1)^k a_k\right| \geq a_\gamma - a_{\gamma-1} > 0.$$

If γ is not an integer, then

$$\text{sgn } a_k = (-1)^{k+[\gamma]+1}, \quad k = [\gamma]+1, [\gamma]+2, \cdots, n$$

and hence

$$|D_n(\alpha)| = \left|\sum_{k=0}^{n}(-1)^k a_k\right| \geq \sum_{k=[\gamma]+1}^{n}|a_k| - \left|\sum_{k=0}^{[\gamma]}(-1)^k a_k\right| \geq a_{[\gamma]+1} - a_{[\gamma]} > 0.$$

This completes the proof.

The last result deals with the dimensionality of the set of the solutions when the problem is singular.

Theorem 3. *If the problem of $(0,1,\cdots,m-2,m)$ interpolation on the zeros of $L_n^{(\alpha)}(x)$ $(\alpha>-1)$ is not regular, i.e., $D_n(\alpha)=0$, then the general form of the solutions of (1.3) with $y_{kj}\equiv 0$ is*

$$R_{mn-1}(x)=C[L_n^{(\alpha)}(x)]^{m-1}q(x) \tag{3.29}$$

in which C is an arbitrary number and $q\in\mathbf{P}_{n-1}$ is of the form

$$q(x)=x^{\gamma}e^{-\frac{(m-1)x}{2}}\left\{d+\int_1^x L_n^{(\alpha)}(t)t^{-\gamma-1}e^{\frac{(m-1)t}{2}}\,dt\right\} \tag{3.30}$$

with a certain constant d.

Proof. Obviously, $R_{mn-1}\in\mathbf{P}_{mn-1}$ is a solution of (1.3) with $y_{kj}\equiv 0$ if and only if $R_{mn-1}(x)$ is of the form

$$R_{mn-1}(x)=\omega_n^{m-1}(x)q(x),\quad q\in\mathbf{P}_{n-1} \tag{3.31}$$

and satisfies

$$[\omega_n^{m-1}(x)q(x)]^{(m)}_{x=x_k}=0,\quad k=1,2,\cdots,n. \tag{3.32}$$

Comparing (3.32) with (3.12) and following the line of the proof of Theorem 1 we can show that $q=\sum_{k=0}^{n-1}\alpha_k x^k$ satisfies the differential equation with an arbitrary number C

$$\mathbf{D}q(x)=C\omega_n(x) \tag{3.33}$$

and the system of equations

$$\begin{cases}\frac{m-1}{2}\alpha_{k-1}+(k-\gamma)\alpha_k=C\gamma_k, & k=0,1,\cdots,n,\\ \alpha_{-1}=\alpha_n=0,\end{cases} \tag{3.34}$$

which are analogues of (3.16) and (3.19), respectively. Moreover, if we can show that equation (3.33) with $C=1$ has a unique solution (3.30), then the proof of the theorem is complete. Solving (3.33) with $C=1$ we get (3.30) with a constant d to be determined.

To determine d we note

$$d=e^{\frac{m-1}{2}}\sum_{k=0}^{n-1}\alpha_k.$$

Obviously, $D_n(\alpha)=0$ means that the system (3.34) has a nontrivial solution $\alpha_0,\cdots,\alpha_{n-1},C$. In this case we see that $C\neq 0$, for otherwise $\alpha_0=\cdots=\alpha_{n-1}=C=0$ would occur, which is impossible. This shows that the system (3.34) with $C=1$ must have a solution. On the other hand, from this system we can uniquely solve

$$\alpha_{k-1}=\frac{2}{m-1}\{\gamma_k+(\gamma-k)\alpha_k\},\quad k=n,n-1,\cdots,1$$

and hence uniquely determine d.

This completes the proof.

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