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MODELS OF ASYNCHRONOUS PARALLEL NONLINEAR MULTISPLITTING RELAXED ITERATIONS*

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Abstract

In the sense of the nonlinear multisplitting and based on the principle of sufficiently using the delayed information, we propose models of asynchronous parallel accelerated overrelaxation iteration methods for solving large scale system of nonlinear equations. Under proper conditions, we set up the local convergence theories of these new method models.

1. Introduction

Consider the large scale system of nonlinear equations

$$F(x) = 0, \qquad F : \mathcal{D} \subset R^n \to R^n. \tag{1.1}$$

Given α ($\alpha \leq n$, an integer) nonempty subsets $J_i (i = 1, 2, \cdots, \alpha)$ of the set $\{1, 2, \cdots, n\}$ with

$$\bigcup_{i=1}^{\alpha} J_i = \{1, 2, \cdots, n\},$$

where $J_1, J_2, \cdots, J_\alpha$ may overlap among them. For $i = 1, 2, \cdots, \alpha$, we assume that

a) $f^{(i)} : \mathcal{D} \times \mathcal{D} \subset R^n \times R^n \to R^n$ satisfies

$$f^{(i)}(x; x) = (f_1^{(i)}(x; x), f_2^{(i)}(x; x), \cdots, f_n^{(i)}(x; x))^T = F(x), \qquad \forall x \in \mathcal{D};$$

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b) $E_i = \mathrm{diag}(e_1^{(i)}, e_2^{(i)}, \cdots, e_n^{(i)}) \in L(R^n)$ satisfies

$$\begin{cases} e_m^{(i)} = \begin{cases} e_m^{(i)} \geq 0, & \text{for } m \in J_i \\ 0, & \text{for } m \notin J_i \end{cases}, \quad m = 1, 2, \cdots, n \\ \sum_{i=1}^{\alpha} E_i = I \quad (I \in L(R^n) \text{ being identity matrix}). \end{cases}$$

Then, the collection of pairs $(f^{(i)}, E_i)(i = 1, 2, \cdots, \alpha)$ is called a nonlinear multisplitting of the mapping $F : \mathcal{D} \subset R^n \to R^n$.

Nowadays, there have been a lot of more deepened research results on both the parallel methods, designed by making use of this concept for solving the system of nonlinear equations (1.1) on the high-speed multiprocessor systems, and their theory analyses[1−4]. Considering the intrinsic shortcomings of the synchronous parallel methods, the parallel iterative methods suitable to the asynchronous computational environments are particularly considerable. It was just in the sense of the nonlinear multisplitting that paper [4] set up a class of asynchronous parallel AOR iterative methods for solving the system of nonlinear equations (1.1).

Based on the already existed results, in this paper we propose a class of method models of asynchronous parallel accelerated overrelaxation iterations for solving the system of nonlinear equations (1.1) by making use of the above concept of nonlinear multisplitting and in light of the principle of sufficiently using the delayed information. These method models give consideration to both the advantages of the nonlinear multiple splittings and the concrete characterizations of the multiprocessor systems, and are of a lot of good behaviours such as convenient computations, flexible and freed communications and so on. Therefore, they can greatly execute the efficiency of practical computations of the multiprocessor systems. Following different choices of the relaxation parameters, not only can the convergence properties of the new asynchronous parallel relaxation method models be improved, but also many applicable and efficient asynchronous parallel nonlinear multisplitting relaxed iteration methods such as the Jacobi, Gauss-Seidel, SOR and so on can be obtained. Meanwhile, the asynchronous parallel nonlinear multisplitting AOR-Newton, -Chord, -Steffensen programs, being of highly practical value, are set up, which makes the new method models become further more convenient, applicable and effective in concrete implementations. Under suitable conditions, we establish local convergence theories for the new models of the asynchronous parallel nonlinear multisplitting relaxed methods, and estimate the asymptotic convergence rates of them, too. At last, the local convergence properties of the asynchronous parallel nonlinear multisplitting AOR-Newton, -Chord, -Steffensen programs are discussed in detail in a unified form.

This work is really developments of the results shown in paper [2], and is also further improvements and generalizations of those in paper [4].

2. Asynchronous Relaxed Method Models

Assume that the considered multiprocessor system is made up of α CPU's, we introduce the following notations:

i) for $\forall i \in \{1, 2, \cdots, \alpha\}, \forall p \in N_0 := \{0, 1, 2, \cdots\}, J^{(i)} = \{J_i(p)\}_{p \in N_0}$ is used to denote a sequence of subset (may be empty set ϕ) of the set J_i;

ii) for $\forall m \in \{1, 2, \cdots, n\}, \forall p \in N_0, N_m(p) := \{i | m \in J_i(p), i = 1, 2, \cdots, \alpha\}$;

iii) for $\forall i \in \{1, 2, \cdots, \alpha\}, S^{(i)} = \{s_1^{(i)}(p), s_2^{(i)}(p), \cdots, s_n^{(i)}(p)\}_{p \in N_0}$ is n infinite sequences.

The sets $J^{(i)}$ and $S^{(i)} (i = 1, 2, \cdots, \alpha)$ have the following properties:

a) for $\forall i \in \{1, 2, \cdots, \alpha\}, \forall m \in \{1, 2, \cdots, n\}$, the set $\{p \in N_0 | m \in J_i(p)\}$ is infinite;

b) for $\forall p \in N_0, \bigcup_{i=1}^{\alpha} J_i(p) \neq \phi$;

c) for $\forall i \in \{1, 2, \cdots, \alpha\}, \forall m \in \{1, 2, \cdots, n\}, \forall p \in N_0, s_m^{(i)}(p) \leq p$;

d) for $\forall i \in \{1, 2, \cdots, \alpha\}, \forall m \in \{1, 2, \cdots, n\}, \lim_{p \to \infty} s_m^{(i)}(p) = \infty$.

For $\forall p \in N_0$, once we define

$$s(p) = \min_{\substack{1 \leq m \leq n \\ 1 \leq i \leq \alpha}} s_m^{(i)}(p),$$

there evidently hold

$$s(p) \leq p, \qquad \lim_{p \to \infty} s(p) = \infty.$$

For the large scale system of nonlinear equations (1.1), we now construct the following asynchronous parallel nonlinear multisplitting relaxed method for solving it numerically:

Method I: Suppose that $x^0 \in \mathcal{D}$ is an approximation of the solution of (1.1), and that we have got the approximate sequence $x^0, x^1, \cdots, x^p$, then the $(p+1)$th approximation $x^{p+1} = (x_1^{p+1}, x_2^{p+1}, \cdots, x_n^{p+1})^T$ of the solution can be calculated by the following three processes:

(I) successively solve systems of nonlinear equations

$$\begin{cases} f_m^{(i)}(x^{s^{(i)}(p)}; \tilde{x}_1^{i,p}, \cdots, \tilde{x}_{m-1}^{i,p}, \hat{x}_m^{i,p}, x_{m+1}^{s_{m+1}^{(i)}(p)}, \cdots, x_n^{s_n^{(i)}(p)}) = 0, \qquad \text{for} \quad m \in J_i(p) \\ m = 1, 2, \cdots, n; \quad i = 1, 2, \cdots, \alpha \end{cases} \tag{2.1}$$

to obtain $\hat{x}_m^{i,p} (m \in J_i(p), i = 1, 2, \cdots, \alpha)$, where

$$x^{s^{(i)}(p)} = (x_1^{s_1^{(i)}(p)}, x_2^{s_2^{(i)}(p)}, \cdots, x_n^{s_n^{(i)}(p)})^T, \qquad i = 1, 2, \cdots, \alpha \tag{2.2}$$

while $\tilde{x}^{i,p} = (\tilde{x}_1^{i,p}, \tilde{x}_2^{i,p}, \cdots, \tilde{x}_n^{i,p})^T$ is given by

$$\begin{cases} \tilde{x}_m^{i,p} = \begin{cases} r\hat{x}_m^{i,p} + (1-r)x_m^{s_m^{(i)}(p)}, & \text{for } m \in J_i(p) \\ x_m^{s_m^{(i)}(p)}, & \text{for } m \notin J_i(p) \end{cases} \\ m = 1, 2, \cdots, n; \quad i = 1, 2, \cdots, \alpha; \end{cases} \tag{2.3}$$

(II) compute $x^{i,p} = (x_1^{i,p}, x_2^{i,p}, \cdots, x_n^{i,p})^T$ by

$$\begin{cases} x_m^{i,p} = \begin{cases} \frac{\omega}{r}\tilde{x}_m^{i,p} + (1-\frac{\omega}{r})x_m^{s_m^{(i)}(p)}, & \text{for } m \in J_i(p) \\ x_m^p, & \text{for } m \notin J_i(p) \end{cases} \\ m = 1,2,\cdots,n; \quad i = 1,2,\cdots,\alpha; \end{cases} \tag{2.4}$$

(III) form the global variable x^{p+1} according to

$$x_m^{p+1} = \sum_{i=1}^{\alpha} e_m^{(i)} x_m^{i,p}, \qquad m = 1,2,\cdots,n. \tag{2.5}$$

Here, $r \in (0,\infty)$ is called as relaxation factor, while $\omega \in (0,\infty)$ is called as acceleration factor.

Clearly, by making use of (2.3), (2.4) can be equivalently written as

$$\begin{cases} x_m^{i,p} = \begin{cases} \omega\hat{x}_m^{i,p} + (1-\omega)x_m^{s_m^{(i)}(p)}, & \text{for } m \in J_i(p) \\ x_m^p, & \text{for } m \notin J_i(p) \end{cases} \\ m = 1,2,\cdots,n; \quad i = 1,2,\cdots,\alpha. \end{cases} \tag{2.6}$$

It is easy to see from (2.1)-(2.3) and (2.5)-(2.6) that corresponding to the special choices $(0,1),(0,\omega),(1,1),(1,\omega)$ and (ω,ω) of the parameter pair (r,ω), the practical and effective asynchronous parallel nonlinear multisplitting Jacobi, extrapolated Jacobi, Gauss-Seidel, extrapolated Gauss-Seidel and SOR methods can be obtained. Additionally, for $\forall m \in \{1,2,\cdots,n\}, \forall i \in \{1,2,\cdots,\alpha\}$, when

$$\begin{cases} J_i = \{1,2,\cdots,n\} \\ \forall p \in N_0, \quad J_i(p) = J_i, \quad s_m^{(i)}(p) = p, \end{cases}$$

Method I reduces to the familiar synchronous parallel nonlinear multisplitting AOR method(see [2]); when

$$\begin{cases} J_i \subset \{1,2,\cdots,n\} \\ \forall p \in N_0, \quad (J_i(p) = J_i) \bigvee (J_i(p) = \phi) = True, \quad s_m^{(i)}(p) = s_i(p) \in R^1, \end{cases}$$

Method I becomes the asynchronous parallel nonlinear multisplitting AOR method proposed in [4].

In Method I, the exact solution of the implicit nonlinear equations (2.1) is usually much difficult to obtain, so in concrete applications, we always make use of known procedures to get an approximate solution of (2.1).

Method II: Given an initial approximation $x^0 \in \mathcal{D}$ of the solution of (1.1), and suppose that we have got the approximate sequence $x^0, x^1, \cdots, x^p$, then the $(p+1)$th approximation x^{p+1} of the solution is determined by

$$\hat{x}_m^{i,p} = x_m^{s_m^{(i)}(p)} - \frac{f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p})}{H_{mm}^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p})}, \qquad m \in J_i(p), \quad i = 1,2,\cdots,\alpha \tag{2.7}$$

as well as (2.2)-(2.5), respectively. Where for $i = 1, 2, \cdots, \alpha; m = 1, 2, \cdots, n$,

$$u_m^{i,p} = (\tilde{x}_1^{i,p}, \cdots, \tilde{x}_{m-1}^{i,p}, x_m^{s_m^{(i)}(p)}, \cdots, x_n^{s_n^{(i)}(p)})^T,$$

$H_{mm}^{(i)}(x;y)$ is the m-th diagonal element of an approximate matrix $H^{(i)}(x;y)$ of the matrix $\partial_2 f^{(i)}(x;y)$, while $\partial_2 f^{(i)}(x;y)$, $\partial_1 f^{(i)}(x;y)$ are the first order derivatives of $f^{(i)}(x;y)$ with respect to its variables y, x, respectively.

Corresponding to different choices of $H^{(i)}(x;y)(i = 1, 2, \cdots, \alpha)$ in Method II, we can derive various practical and effective programs. For example, as

$$\begin{cases} H_{mm}^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p}) = \partial_2^{(m)} f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p}) \\ m \in J_i(p), \quad i = 1, 2, \cdots, \alpha, \end{cases}$$

the asynchronous parallel nonlinear multisplitting AOR-Newton program can be obtained, since the nonlinear equations (2.1) is now solved approximately by the Newton procedure. Where $\partial_2^{(m)} f_m^{(i)}(x;y)$ and $\partial_1^{(m)} f_m^{(i)}(x;y)$ represent the m-th diagonal elements of $\partial_2 f^{(i)}(x;y)$ and $\partial_1 f^{(i)}(x;y)$, individually; as

$$\begin{cases} H_{mm}^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p}) = \dfrac{f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p} + h_m^{i,p} e_m) - f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p})}{h_m^{i,p}} \\ m \in J_i(p), \quad i = 1, 2, \cdots, \alpha, \end{cases}$$

the asynchronous parallel nonlinear multisplitting AOR-Chord program can be got, as the nonlinear equations (2.1) is presently solved approximately by the Chord procedure. Here $h_m^{i,p}(m \in J_i(p), i = 1, 2, \cdots, \alpha, \forall p \in N_0)$ are given difference step sizes, while $e_m \in R^n$ is the m-th unit vector; as

$$\begin{cases} H_{mm}^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p}) = \dfrac{f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p} + f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p}) e_m) - f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p})}{f_m^{(i)}(x^{s^{(i)}(p)}; u_m^{i,p})} \\ m \in J_i(p), \quad i = 1, 2, \cdots, \alpha, \end{cases}$$

the asynchronous parallel nonlinear multisplitting AOR-Steffensen program can be obtained, since the nonlinear equations (2.1) is now solved approximately by the Steffensen procedure.

Analogously, with different choices of the parameter pair (r, ω) in Method II, we can also get an extensive sequence of asynchronous parallel nonlinear multisplitting accelerated overrelaxation methods. For the length of the paper, we will not enumerate them one by one, here.

In order to set up the convergence theories of the above two asynchronous relaxed method models, we introduce an infinite number sequence $\{m_l\}_{l \in N_0}$ in accordance with the following rule:

m_0 is the least positive integer such that

$$\bigcup_{0 \le s(p) \le p < m_0} J_i(p) = J_i, \qquad i = 1, 2, \cdots, \alpha,$$

in general, m_{l+1} is the least positive integer such that

$$\bigcup_{m_l \le s(p) \le p < m_{l+1}} J_i(p) = J_i, \qquad i = 1, 2, \cdots, \alpha; \qquad l = 0, 1, 2, \cdots.$$

3. Preliminary Knowledge

In the subsequent discussion, we will carry on the notations, concepts and essential conclusions used in [4]-[8]. Particularly, we use $< \cdot >$ and $\rho(\cdot)$ to denote the comparison matrix and spectral radius of the corresponding matrix, respectively, while $|\cdot|$ represents the absolute value of either a vector or a matrix. Additionally, we cite several lemmas set up in [5], which are crucial for the convergence demonstrations of the asynchronous relaxed method models established in last section.

Lemma 1. *Given $\bar{x}^* \in R^n$ and $\{\bar{x}^t\}_{t=0}^p \subset R^n (\forall p \in N_0)$. Assume that for all $t \in \{0, 1, \cdots, p\}$, there exist positive number δ and positive vector $v = (v_1, v_2, \cdots, v_n)^T \in R^n$ such that*

$$|\bar{x}^t - \bar{x}^*| \le \delta v.$$

Then there identically hold

$$|\bar{x}^{s^{(i)}(p)} - \bar{x}^*| \le \delta v, \qquad i = 1, 2, \cdots, \alpha$$

provided $s_m^{(i)}(p) \le p (m = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha)$. Where

$$\bar{x}^{s^{(i)}(p)} = (\bar{x}_1^{s_1^{(i)}(p)}, \bar{x}_2^{s_2^{(i)}(p)}, \cdots, \bar{x}_n^{s_n^{(i)}(p)})^T, \qquad i = 1, 2, \cdots, \alpha.$$

Presently, we introduce nonnegative sequences $\{i_m^p\}_{p \in N_0}$ and $\{j_m^p\}_{p \in N_0}$, where $(m = 1, 2, \cdots, \alpha)$, according to

$$i_m^p = \sum_{i \in N_m(p)} e_m^{(i)}, \qquad j_m^p = \sum_{i \notin N_m(p)} e_m^{(i)}, \qquad p = 0, 1, 2, \cdots; \quad m = 1, 2, \cdots, n.$$

Lemma 2. *Let $\xi_m > 0 (m = 1, 2, \cdots, n)$. Assume that the sequence $\{\varepsilon_m^p\}_{p \in N_0} (m = 1, 2, \cdots, n)$ are defined to satisfy*

$$|\varepsilon_m^{p+1}| \le i_m^p \xi_m + j_m^p |\varepsilon_m^p|, \qquad p = 0, 1, 2, \cdots.$$

Then for any nonnegative integer $q \le p - 1$ there hold

$$|\varepsilon_m^{p+1}| \le (1 - \prod_{k=p-q-1}^{p} j_m^k) \xi_m + \prod_{k=p-q-1}^{p} j_m^k |\varepsilon_m^{p-q-1}|, \qquad m = 1, 2, \cdots, n.$$

Lemma 3. *Let the sequence $\{j_m^{(l)}\}_{l \in N_0} (m = 1, 2, \cdots, n)$ be defined as*

$$j_m^{(0)} = \prod_{p=0}^{m_0 - 1} j_m^p, \quad j_m^{(l+1)} = \prod_{p=m_l}^{m_{l+1}-1} j_m^p, \qquad l = 0, 1, 2, \cdots.$$

Then, there hold $\{j_m^{(l)}\}_{l\in N_0} \subset [0,1)(m=1,2,\cdots,n)$.

4. Convergence Analysis of Method I

Initially, assume that $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in \mathcal{D}$ is a solution of the system of nonlinear equations (1.1), and that for each $i \in \{1,2,\cdots,\alpha\}$, $f^{(i)} : \mathcal{D}\times\mathcal{D} \subset R^n \times R^n \to R^n$ is differentiable in a neighbourhood of $(x^*; x^*)$. We again introduce the following notations:

$$\begin{cases} M_i = (M_{mj}^{(i)}) = \partial_2 f^{(i)}(x^*;x^*) \\ N_i = (N_{mj}^{(i)}) = -\partial_1 f^{(i)}(x^*;x^*) \ , \qquad i=1,2,\cdots,\alpha \\ D_i = diag(M_i) \end{cases} \tag{4.1}$$

while for $i=1,2,\cdots,\alpha$, $L_i = (l_{mj}^{(i)}), U_i = (u_{mj}^{(i)}) \in L(R^n)$ are respectively taken to be

$$\begin{cases} l_{mj}^{(i)} = \begin{cases} -M_{mj}^{(i)}, & \text{for } m,j\in J_i \text{ and } m>j \\ 0, & \text{otherwise} \end{cases} \\ u_{mj}^{(i)} = \begin{cases} -M_{mj}^{(i)}, & \text{for } m,j\in J_i, m<j \text{ or } m,j\notin J_i \text{ and } m\neq j \\ 0, & \text{otherwise} \end{cases} \\ m,j=1,2,\cdots,n. \end{cases} \tag{4.2}$$

Evidently, L_i is a strictly lower triangular matrix, U_i is a strictly zero-diagonal matrix, and there have

$$M_i = D_i - L_i - U_i, \qquad i=1,2,\cdots,\alpha. \tag{4.3}$$

Noticing that $F : \mathcal{D} \subset R^n \to R^n$ is also differentiable in a neighbourhood of $x^* \in \mathcal{D}$ at this time, by the chain rule we know that there hold

$$\begin{aligned} F'(x^*) &= \partial_1 f^{(i)}(x^*;x^*) + \partial_2 f^{(i)}(x^*;x^*) \\ &= M_i - N_i \\ &= D_i - L_i - (U_i + N_i) \\ &= D - B(i=1,2,\cdots,\alpha), \end{aligned} \tag{4.4}$$

where

$$D = \mathrm{diag}(F'(x^*)), \qquad B = D - F'(x^*).$$

Clearly, when $\det(D_i) \neq 0 (i=1,2,\cdots,\alpha)$, $(D_i - L_i, U_i + N_i, E_i)(i=1,2,\cdots,\alpha)$ naturally induces a multisplitting of the matrix $F'(x^*) \in L(R^n)$.

Now, we begin to establish local convergence theory for Method I.

Theorem 1. *Let $x^* \in \mathcal{D}$ be a solution of the system of nonlinear equations (1.1), $(f^{(i)}, E_i)(i=1,2,\cdots,\alpha)$ be a nonlinear multisplitting of $F : \mathcal{D} \subset R^n \to R^n$, and $f^{(i)} : \mathcal{D}\times\mathcal{D} \subset R^n \times R^n \to R^n$ be continuously differentiable in a neighbourhood* *of* $(x^*; x^*)$ *for each* $i \in \{1, 2, \cdots, \alpha\}$. *Suppose* $F'(x^*) \in L(R^n)$ *be an* H*-matrix, and* $(D_i - L_i, U_i + N_i, E_i)(i = 1, 2, \cdots, \alpha)$ *be a multisplitting of it with*

$$< F'(x^*) >= |D_i| - |L_i| - |U_i + N_i| = |D| - |B|, \qquad i = 1, 2, \cdots, \alpha. \tag{4.5}$$

Then, there exists a neighbourhood $N(x^*, \delta)$ *of* $x^* \in \mathcal{D}$ *such that the sequence* $\{x^p\}_{p \in N_0}$ *generated by Method I starting from any initial approximation* $x^0 \in N(x^*, \delta)$ *converges to the solution* $x^* \in \mathcal{D}$ *of the system of nonlinear equations* (1.1) *provided the relaxation parameters* r *and* ω *satisfy*

$$0 < r \leq \omega, \qquad 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}. \tag{4.6}$$

Proof. Because of $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in \mathcal{D}$ being a solution of the system of nonlinear equations (1.1), there hold

$$f^{(i)}(x^*; x^*) = F(x^*) = 0, \qquad i = 1, 2, \cdots, \alpha.$$

For each $i \in \{1, 2, \cdots, \alpha\}$, Take arbitrarily a nonempty subset $\hat{J}_i \subseteq J_i$. Noticing $\det(D_i) \neq 0 (i = 1, 2, \cdots, \alpha)$, in accordance with the implicit function theorem, there exist for each $m \in \{1, 2, \cdots, n\}$ an open neighbourhood $N(x^*, \delta^{(i)})$ of x^* and an open neighbourhood $N(x_m^*, \delta_m^{(i)})$ of x_m^* such that for any $x \in N(x^*, \delta^{(i)})$, the function $T_m^{(i)}$: $N(x^*, \delta^{(i)}) \to N(x_m^*, \delta_m^{(i)})$ determined by

$$f_m^{(i)}(x; g_1^{(i)}(x), \cdots, g_{m-1}^{(i)}(x), T_m^{(i)}(x), x_{m+1}, \cdots, x_n) = 0, \qquad m \in \hat{J}_i$$

and

$$T_m^{(i)}(x^*) = x_m^*, \qquad m \in \hat{J}_i$$

is uniquely well-defined and continuously differentiable, where

$$g_m^{(i)}(x) = \begin{cases} rT_m^{(i)}(x) + (1 - r)x_m, & \text{for } m \in \hat{J}_i \\ x_m, & \text{for } m \notin \hat{J}_i. \end{cases} \tag{4.7}$$

Evidently, $g_m^{(i)}(m = 1, 2, \cdots, n)$ are also continuously differentiable in $N(x^*, \delta^{(i)})$. Let

$$\begin{cases} \bar{\delta}^{(i)} = \min\limits_{1 \leq m \leq n} \delta_m^{(i)} \\ N(x^*, \bar{\delta}^{(i)}) = N(x_1^*, \bar{\delta}^{(i)}) \times N(x_2^*, \bar{\delta}^{(i)}) \times \cdots \times N(x_n^*, \bar{\delta}^{(i)}) \\ g^{(i)}(x) = (g_1^{(i)}(x), g_2^{(i)}(x), \cdots, g_n^{(i)}(x))^T. \end{cases} \tag{4.8}$$

Then, $g^{(i)} : N(x^*, \delta^{(i)}) \to N(x^*, \bar{\delta}^{(i)})$ defined by the equations

$$f_m^{(i)}(x; g_1^{(i)}(x), \cdots, g_{m-1}^{(i)}(x), T_m^{(i)}(x), x_{m+1}, \cdots, x_n) = 0, \qquad m \in \hat{J}_i, \quad m = 1, 2, \cdots, n$$

and (4.7) is continuously differentiable and satisfies

$$\begin{cases} T_m^{(i)}(x^*) = x_m^*, \qquad m \in \hat{J}_i, \quad m = 1, 2, \cdots, n \\ g^{(i)}(x^*) = x^*. \end{cases} \tag{4.9}$$

Moreover, by the chain rule, there have

$$\begin{cases} -N_{mj}^{(i)}+\sum_{l=1}^{m-1}M_{ml}^{(i)}\dfrac{\partial g_l^{(i)}(x^*)}{\partial x_j}+M_{mm}^{(i)}\dfrac{\partial T_m^{(i)}(x^*)}{\partial x_j}=0, \\ \qquad\qquad \text{for } j\le m \\ -N_{mj}^{(i)}+\sum_{l=1}^{m-1}M_{ml}^{(i)}\dfrac{\partial g_l^{(i)}(x^*)}{\partial x_j}+M_{mm}^{(i)}\dfrac{\partial T_m^{(i)}(x^*)}{\partial x_j}+M_{mj}^{(i)}=0, \\ \qquad\qquad \text{for } j> m \\ m\in\hat{J}_i, \qquad m,j=1,2,\cdots,n. \end{cases} \tag{4.10}$$

From (4.7) we know that

$$\begin{cases} \dfrac{\partial g_m^{(i)}(x^*)}{\partial x_j}=\begin{cases} r\frac{\partial T_m^{(i)}(x^*)}{\partial x_j}+(1-r), & \text{for } j=m \\ r\frac{\partial T_m^{(i)}(x^*)}{\partial x_j}, & \text{for } j\ne m \end{cases} \\ m\in\hat{J}_i, \qquad m,j=1,2,\cdots,n. \end{cases} \tag{4.11}$$

Substitute (4.11) into (4.10), the following relations can be obtained

$$\begin{cases} -rN_{mj}^{(i)}+r\sum_{l=1}^{m-1}M_{ml}^{(i)}\dfrac{\partial g_l^{(i)}(x^*)}{\partial x_j}+M_{mm}^{(i)}\dfrac{\partial g_m^{(i)}(x^*)}{\partial x_j}=\begin{cases} 0, & j<m \\ (1-r)M_{mm}^{(i)}, & j=m \\ -rM_{mj}^{(i)}, & j>m \end{cases} \\ m\in\hat{J}_i, \qquad m,j=1,2,\cdots,n. \end{cases} \tag{4.12}$$

Now, noticing (4.7) we get

$$e_m^T(D_i-rL_i)\frac{\partial g^{(i)}(x^*)}{\partial x}=e_m^T[(1-r)D_i+r(U_i+N_i)], \qquad m\in\hat{J}_i, \quad m=1,2,\cdots,n.$$

Therefore, there hold

$$\frac{\partial g_m^{(i)}(x^*)}{\partial x}=e_m^T(D_i-rL_i)^{-1}[(1-r)D_i+r(U_i+N_i)], \qquad m\in\hat{J}_i, \quad m=1,2,\cdots,n. \tag{4.13}$$

Write

$$N(x^*,\hat{\delta})=\bigcap_{1\le i\le\alpha}N(x^*,\delta^{(i)}), \qquad N(x^*,\bar{\delta})=\bigcup_{1\le i\le\alpha}N(x^*,\bar{\delta}^{(i)}).$$

Then, for all $i\in\{1,2,\cdots,\alpha\}, g^{(i)}:N(x^*,\hat{\delta})\to N(x^*,\bar{\delta})$ are continuously differentiable and obey (4.7), (4.9) and (4.13).

Since $F'(x^*)\in L(R^n)$ is an H-matrix, $\rho(|D|^{-1}|B|)<1$. For any $\varepsilon>0$, denote

$$J_\varepsilon=|D|^{-1}|B|+\varepsilon ee^T, \qquad e=(1,1,\cdots,1)^T\in R^n. \tag{4.14}$$

By continuity of the spectral radius of matrix and (4.6) we see that

$$\rho_\varepsilon=\rho(J_\varepsilon)<1, \qquad \sigma_\varepsilon=\frac{\omega}{r}\varepsilon+|1-\omega|+\omega\rho_\varepsilon<1 \tag{4.15}$$

provided ε is taken to be small enough. Recalling the Perron-Frobinuis theorem in nonnegative matrix theory, there exists a positive vector

$$v^{(\varepsilon)} = (v_1^{(\varepsilon)}, v_2^{(\varepsilon)}, \cdots, v_n^{(\varepsilon)})^T \in R^n$$

such that

$$J_\varepsilon v^{(\varepsilon)} = \rho_\varepsilon v^{(\varepsilon)}. \tag{4.16}$$

For this ε, in light of the continuous differentiability of $g^{(i)} : N(x^*, \hat{\delta}) \to N(x^*, \bar{\delta})$, we can take $\delta \in (0, \hat{\delta})$ properly small such that

$$|g^{(i)}(x) - g^{(i)}(x^*) - \frac{\partial g^{(i)}(x^*)}{\partial x}(x - x^*)| \le \varepsilon |x - x^*|, \qquad i = 1, 2, \cdots, \alpha \tag{4.17}$$

hold as long as

$$x \in N(x^*, \delta) := \{x| \quad |x - x^*| \le \delta v^{(\varepsilon)}\} \subset N(x^*, \hat{\delta}).$$

Up to now, the proof can be proceeded in three parts.

Part I. Suppose $x^0 \in N(x^*, \delta)$, then

$$x^p \in N(x^*, \delta), \qquad \forall p \in N_0. \tag{4.18}$$

In fact, when $p = 0$, (4.18) is obviously true. Assume that for all $p \le t$, (4.18) hold. By making use of Lemma 1 we know that there have

$$x^{s^{(i)}(t)} \in N(x^*, \delta), \qquad i = 1, 2, \cdots, \alpha \tag{4.19}$$

at this time, too. Using (4.7) we can equivalently express Method I as

$$\begin{cases} x_m^{t+1} = \sum\limits_{i \in N_m(t)} e_m^{(i)} \left[\frac{\omega}{r} g_m^{(i)}(x^{s^{(i)}(t)}) + (1 - \frac{\omega}{r}) x_m^{s_m^{(i)}(t)} \right] + \sum\limits_{i \notin N_m(t)} e_m^{(i)} x_m^t \\ m = 1, 2, \cdots, n. \end{cases} \tag{4.20}$$

Considering (4.13), there immediately hold for $m = 1, 2, \cdots, n$ that

$$\begin{aligned} x_m^{t+1} - x_m^* = &\sum_{i \in N_m(t)} e_m^T E_i \mathcal{L}^{(i)}(r, \omega)(x^{s^{(i)}(t)} - x^*) + \frac{\omega}{r} \sum_{i \in N_m(t)} e_m^T E_i R^{(i)}(x^{s^{(i)}(t)}) \\ &+ \sum_{i \notin N_m(t)} e_m^T E_i (x^t - x^*), \end{aligned} \tag{4.21}$$

where

$$\begin{cases} \mathcal{L}^{(i)}(r, \omega) = (D_i - rL_i)^{-1}[(1 - \omega)D_i + (\omega - r)L_i + \omega(U_i + N_i)] \\ i = 1, 2, \cdots, \alpha, \end{cases} \tag{4.22}$$

$$\begin{cases} R^{(i)}(x) = g^{(i)}(x) - g^{(i)}(x^*) - \dfrac{\partial g^{(i)}(x^*)}{\partial x}(x - x^*) \\ \forall x \in N(x^*, \delta), \quad i = 1, 2, \cdots, \alpha. \end{cases} \tag{4.23}$$

Because of $(D_i - rL_i)(i = 1, 2, \cdots, \alpha)$ being all H-matrices, we can get the following inequalities

$$|(D_i - rL_i)^{-1}| \leq < D_i - rL_i >^{-1} = (|D_i| - r|L_i|)^{-1}, \qquad i = 1, 2, \cdots, \alpha.$$

Noticing (4.14), the following estimations can be obtained by direct calculations

$$\begin{aligned} |\mathcal{L}^{(i)}(r,\omega)| &\leq |(D_i - rL_i)^{-1}|[|1-\omega||D_i| + (\omega - r)|L_i| + \omega|U_i + N_i|] \\ &\leq (|D_i| - r|L_i|)^{-1}[(|D_i| - r|L_i|) + (|1-\omega| - 1)|D_i| + \omega(|L_i| + |U_i + N_i|)] \\ &\leq I + (|D_i| - r|L_i|)^{-1}|D_i|[(|1-\omega| - 1)I + \omega|D_i|^{-1}|B|] \\ &\leq I + (|D_i| - r|L_i|)^{-1}|D_i|[(|1-\omega| - 1)I + \omega J_\varepsilon](i = 1, 2, \cdots, \alpha), \end{aligned}$$

where in the third inequality we have applied the condition (4.5). Presently, by making use of (4.16) as well as inequalities

$$|D_i| - r|L_i| \leq |D_i|, \qquad i = 1, 2, \cdots, \alpha,$$

we can immediately get the following relations

$$|\mathcal{L}^{(i)}(r,\omega)|v^{(\varepsilon)} \leq (|1-\omega| + \omega\rho_\varepsilon)v^{(\varepsilon)}, \qquad i = 1, 2, \cdots, \alpha. \tag{4.24}$$

Using (4.17), (4.23), (4.19) and (4.24), from (4.21) we know that

$$\begin{aligned} |x_m^{t+1} - x_m^*| &\leq i_m^t \sigma_\varepsilon \delta v_m^{(\varepsilon)} + j_m^t |x_m^t - x_m^*| \\ &\leq (i_m^t + j_m^t)\delta v_m^{(\varepsilon)} = \delta v_m^{(\varepsilon)}. \end{aligned} \tag{4.25}$$

According to the induction, the correctness of (4.18) is confirmed.

Part II. Suppose $x^0 \in N(x^*, \delta)$, then

$$x^p \in N(x^*, \Delta_l), \qquad \forall p \geq m_l, \tag{4.26}$$

where

$$\begin{cases} \Delta_0 = (\sigma_\varepsilon + (1-\sigma_\varepsilon)\gamma^{(0)})\delta \\ \Delta_{l+1} = (\sigma_\varepsilon + (1-\sigma_\varepsilon)\gamma^{(l+1)})\Delta_l \\ \gamma^{(l)} = \max\limits_{1\leq m\leq n} j_m^{(l)} \in [0,1) \\ l = 0, 1, 2, \cdots. \end{cases}$$

As $l = 0$, by (4.25) and Lemma 2 we can get for $m = 1, 2, \cdots, n$ that

$$\begin{aligned} |x_m^{p+1} - x_m^*| &\leq i_m^p \sigma_\varepsilon \delta v_m^{(\varepsilon)} + j_m^p |x_m^p - x_m^*| \\ &\leq (1 - \prod_{k=0}^p j_m^k)\sigma_\varepsilon \delta v_m^{(\varepsilon)} + \prod_{k=0}^p j_m^k |x_m^0 - x_m^*| \\ &\leq (1 - \prod_{k=0}^p j_m^k)\sigma_\varepsilon \delta v_m^{(\varepsilon)} + \prod_{k=0}^p j_m^k \delta v_m^{(\varepsilon)} \\ &= (\sigma_\varepsilon + (1-\sigma_\varepsilon)\prod_{k=0}^p j_m^k)\delta v_m^{(\varepsilon)} \\ &\leq (\sigma_\varepsilon + (1-\sigma_\varepsilon) j_m^{(0)})\delta v_m^{(\varepsilon)} \\ &\leq \Delta_0 v_m^{(\varepsilon)}. \end{aligned}$$

Hence, (4.26) is valid.

Assume that for $p \geq m_l$, (4.26) have been proved. Then, when $p \geq m_{l+1}$, by making use of (4.17), (4.23), (4.19) and (4.24) as well as the induction assumption, from (4.21) we have for $m = 1, 2, \cdots, n$ that

$$|x_m^{p+1} - x_m^*| \leq i_m^p \sigma_\varepsilon \Delta_l v_m^{(\varepsilon)} + j_m^p |x_m^p - x_m^*|$$

hold. Similarly, in light of Lemma 2, there hold for $m = 1, 2, \cdots, n$ that

$$\begin{aligned} |x_m^{p+1} - x_m^*| &\leq (1 - \prod_{k=m_l}^{p} j_m^k)\sigma_\varepsilon \Delta_l v_m^{(\varepsilon)} + \prod_{k=m_l}^{p} j_m^k |x_m^{m_l} - x_m^*| \\ &\leq (1 - \prod_{k=m_l}^{p} j_m^k)\sigma_\varepsilon \Delta_l v_m^{(\varepsilon)} + \prod_{k=m_l}^{p} j_m^k \Delta_l v_m^{(\varepsilon)} \\ &\leq (\sigma_\varepsilon + (1-\sigma_\varepsilon) j_m^{(l+1)})\Delta_l v_m^{(\varepsilon)} \\ &\leq \Delta_{l+1} v_m^{(\varepsilon)}. \end{aligned}$$

Therefore, (4.26) is also valid for this case. In accordance with the induction, we can conclude the validity of (4.26).

Part III. Suppose $x^0 \in N(x^*, \delta)$, then $x^p \to x^*(p \to \infty)$.

Let

$$\beta^{(l)} = \sigma_\varepsilon + (1-\sigma_\varepsilon)\gamma^{(l)}, \qquad l = 0, 1, 2, \cdots.$$

Then $\{\beta^{(l)}\}_{l \in N_0} \subset [0, 1)$. Additionally, as

$$\begin{aligned} \Delta_{l+1} &= \beta^{(l+1)} \Delta_l = \cdots \\ &= \prod_{k=0}^{l+1} \beta^{(k)} \delta \to 0 (l \to \infty), \end{aligned}$$

By (4.26) we know that

$$\lim_{p \to \infty} x^p = x^*.$$

5. Convergence Analysis of Method II

For $i = 1, 2, \cdots, \alpha$, let $\hat{J}_i \subseteq J_i$ and define

$$g_m^{(i)}(x) = \begin{cases} x_m - r\frac{f_m^{(i)}(x;\gamma^{m,i}(x))}{H_{mm}^{(i)}(x;\gamma^{m,i}(x))}, & m \in \hat{J}_i \\ x_m, & m \notin \hat{J}_i \end{cases}, \quad m = 1, 2, \cdots, n, \tag{5.1}$$

where

$$\begin{cases} \gamma^{1,i}(x) = x \\ \gamma^{m,i}(x) = (g_1^{(i)}(x), \cdots, g_{m-1}^{(i)}(x), x_m, \cdots, x_n)^T, \quad m = 2, 3, \cdots, n. \end{cases} \tag{5.2}$$

Then, by (2.2)-(2.5), (2.7) and (5.1)-(5.2) we see that Method II can be equivalently represented as

$$\begin{cases} x_m^{p+1} = \sum_{i\in N_m(p)} e_m^{(i)}\left[\frac{\omega}{r} g_m^{(i)}(x^{s^{(i)}(p)}) + (1-\frac{\omega}{r}) x_m^{s_m^{(i)}(p)}\right] + \sum_{i\notin N_m(p)} e_m^{(i)} x_m^p \\ m = 1,2,\cdots,n. \end{cases} \tag{5.3}$$

Based on these identities, we can set up the local convergence theorem of Method II.

Theorem 2. *Under the conditions of Theorem 1, we additionally suppose that $H^{(i)}(x;y)$ is continuously differentiable in a neighbourhood of $(x^*;x^*)$ and satisfies*

$$\lim_{(x;y)\to(x^*;x^*)} H_{mm}^{(i)}(x;y) = \partial_2^{(m)} f_m^{(i)}(x^*;x^*), \qquad m=1,2,\cdots,n \tag{5.4}$$

for each $i\in\{1,2,\cdots,\alpha\}$. Then, there exists a neighbourhood $N(x^*,\delta)$ of $x^*\in\mathcal{D}$ such that the sequence $\{x^p\}_{p\in N_0}$ generated by Method II starting from any initial approximation $x^0\in N(x^*,\delta)$ converges to the solution $x^*\in\mathcal{D}$ of the system of nonlinear equations (1.1) *provided the relaxation parameters r and ω satisfy* (4.6)*.*

Proof. Define $\rho_\varepsilon, \sigma_\varepsilon, J_\varepsilon$ and $v^{(\varepsilon)}$ as (4.14)-(4.16), and take

$$N(x^*,\tilde{\delta}) := \{x| \quad |x-x^*| \le \tilde{\delta} v^{(\varepsilon)}\} \subset \mathcal{D}$$

such that $f^{(i)}$ and $H^{(i)}$ are continuously differentiable on $N(x^*,\tilde{\delta})\times N(x^*,\tilde{\delta})$ for each $i\in\{1,2,\cdots,\alpha\}$. Let

$$\begin{cases} r_m^{(i)}(x;y) = f_m^{(i)}(x;y) - f_m^{(i)}(x^*;x^*) - [\partial_1 f_m^{(i)}(x^*;x^*)(x-x^*) + \partial_2 f_m^{(i)}(x^*;x^*)(y-x^*)] \\ m=1,2,\cdots,n, \end{cases} \tag{5.5}$$

by making use of the induction, we can prove that there exist $\delta^{(i)}\in(0,\tilde{\delta})$ and positive constants $a_m^{(i)}, b_m^{(i)}, c_m^{(i)} (m=1,2,\cdots,n; i=1,2,\cdots,\alpha)$ such that

$$\begin{cases} |r_m^{(i)}(x;\gamma^{m,i}(x))| \le a_m^{(i)}\|x-x^*\| \\ |g_m^{(i)}(x) - g_m^{(i)}(x^*)| \le b_m^{(i)}\|x-x^*\| \\ \|\gamma^{m,i}(x) - \gamma^{m,i}(x^*)\| \le c_m^{(i)}\|x-x^*\| \end{cases}, \qquad \forall x\in N(x^*,\delta^{(i)}) \tag{5.6}$$

hold for $m=1,2,\cdots,n; i=1,2,\cdots,\alpha$.

As a matter of fact, by (5.1)—(5.2) we see that

$$g_m^{(i)}(x^*) = x_m^*, \qquad m=1,2,\cdots,n; \quad i=1,2,\cdots,\alpha \tag{5.7}$$

and

$$\begin{cases} g_m^{(i)}(x) - g_m^{(i)}(x^*) \\ \quad = \begin{cases} x_m - x_m^* - r\frac{r_m^{(i)}(x;\gamma^{m,i}(x)) + \partial_1 f_m^{(i)}(x^*;x^*)(x-x^*) + \partial_2 f_m^{(i)}(x^*;x^*)(\gamma^{m,i}(x)-x^*)}{H_{mm}^{(i)}(x;\gamma^{m,i}(x))}, & m\in\hat{J}_i \\ x_m - x_m^*, & m\notin\hat{J}_i \end{cases} \\ m=1,2,\cdots,n; \qquad i=1,2,\cdots,\alpha. \end{cases} \tag{5.8}$$

Additionally, noticing the continuous differentiability of $f_m^{(i)}$ and $H_{mm}^{(i)}$ $(m = 1, 2, \cdots, n;$ $i = 1, 2, \cdots, \alpha)$ on $N(x^*, \tilde{\delta}) \times N(x^*, \tilde{\delta})$, by concrete derivation we can obtain (5.6).

Corresponding to each $i \in \{1, 2, \cdots, \alpha\}$, we now define the sets

$$\mathcal{D}_m^{(i)} = \{x \in N(x^*, \delta^{(i)})| \quad \partial_2^{(m)} f_m^{(i)}(x, \gamma^{m,i}(x)) \neq 0\}, \quad m = 1, 2, \cdots, \alpha. \tag{5.9}$$

By the continuity of $\partial_2^{(m)} f_m^{(i)} (m = 1, 2, \cdots, n)$ in $N(x^*, \delta^{(i)}) \times N(x^*, \delta^{(i)})$ and $\gamma^{m,i} (m = 1, 2, \cdots, n)$ in $N(x^*, \delta^{(i)})$ as well as

$$\gamma^{m,i}(x^*) = x^*, \quad \partial_2^{(m)} f_m^{(i)}(x^*; x^*) \neq 0, \quad m = 1, 2, \cdots, n,$$

we know that each $\mathcal{D}_m^{(i)}$ is open. Again, according to the continuous differentiability of $H^{(i)}$, there exists, corresponding to each $\mathcal{D}_m^{(i)}$, a neighbourhood $\mathcal{D}_m^{'(i)} \subset \mathcal{D}_m^{(i)}$ of x^* such that

$$H_{mm}^{(i)}(x; \gamma^{m,i}(x)) \neq 0, \qquad \forall x \in \mathcal{D}_m^{'(i)}, \quad m = 1, 2, \cdots, n.$$

Write

$$S_0^{(i)} = \mathcal{D}_1^{'(i)}, \quad S_m^{(i)} = S_{m-1}^{(i)} \cap \mathcal{D}_m^{'(i)}, \qquad m = 1, 2, \cdots, n.$$

Clearly,

$$S_1^{(i)} = \mathcal{D}_1^{'(i)}, \quad S_m^{(i)} \subseteq S_{m-1}^{(i)}, \quad S_m^{(i)} \subseteq \mathcal{D}_m^{'(i)}, \qquad m = 1, 2, \cdots, n.$$

By (5.1)—(5.2), for $m = 1, 2, \cdots, n$, $g_m^{(i)}$ is well-defined in $S_m^{(i)}$. As each $\mathcal{D}_m^{'(i)}$ is open, each $S_m^{(i)}$ is open, too. Take $\delta \in (0, \min_{1 \le i \le \alpha} \delta^{(i)})$, a neighbourhood

$$N(x^*, \delta) := \{x| \quad |x - x^*| \le \delta v^{(\varepsilon)}\} \subseteq \bigcap_{i=1}^{\alpha} S_n^{(i)}$$

of x^* is therefore determined. Evidently, $g^{(i)} (i = 1, 2, \cdots, \alpha)$, and hence Method II, is well-defined in $N(x^*, \delta)$.

Denote

$$\begin{cases} q_m^{(i)}(x) = \begin{cases} [H_{mm}^{(i)}(x; \gamma^{m,i}(x)) - \partial_2^{(m)} f_m^{(i)}(x^*; x^*)][g_m^{(i)}(x) - g_m^{(i)}(x^*) - (x_m - x_m^*)] \\ \qquad + r r_m^{(i)}(x; \gamma^{m,i}(x)), \\ \qquad \text{for } m \in \hat{J}_i \\ -r e_m^T F'(x^*)(x - x^*), \\ \qquad \text{for } m \notin \hat{J}_i \end{cases} \\ m = 1, 2, \cdots, n; \qquad i = 1, 2, \cdots, \alpha. \end{cases} \tag{5.10}$$

Then by (5.8) we have for $m \in \hat{J}_i (m = 1, 2, \cdots, n; i = 1, 2, \cdots, \alpha)$ that

$$\begin{aligned} \partial_2^{(m)} f_m^{(i)}(x^*; x^*)(g_m^{(i)}(x) - g_m^{(i)}(x^*)) = \partial_2^{(m)} f_m^{(i)}(x^*; x^*)(x_m - x_m^*) \\ - r[\partial_1 f_m^{(i)}(x^*; x^*)(x - x^*) + \partial_2 f_m^{(i)}(x^*; x^*)(\gamma^{m,i}(x) - x^*)] \\ - q_m^{(i)}(x), \end{aligned}$$

i.e.,

$$M_{mm}^{(i)}(g_m^{(i)}(x)-g_m^{(i)}(x^*))+r\sum_{j=1}^{m-1}M_{mj}^{(i)}(g_j^{(i)}(x)-g_j^{(i)}(x^*))$$
$$=M_{mm}^{(i)}(x_m-x_m^*)-r[\partial_1 f_m^{(i)}(x^*;x^*)(x-x^*)+\sum_{j=m}^{n}M_{mj}^{(i)}(x_j-x_j^*)]$$
$$-q_m^{(i)}(x),$$

or equivalently,

$$e_m^T(D_i-rL_i)(g^{(i)}(x)-g^{(i)}(x^*))=e_m^T[(1-r)D_i+r(U_i+N_i)](x-x^*)-q_m^{(i)}(x), \quad (5.11)$$

where

$$g^{(i)}(x)=(g_1^{(i)}(x),g_2^{(i)}(x),\cdots,g_n^{(i)}(x))^T, \qquad i=1,2,\cdots,\alpha.$$

Let

$$\begin{cases} q^{(i)}(x)=(q_1^{(i)}(x),q_2^{(i)}(x),\cdots,q_n^{(i)}(x))^T \\ R^{(i)}(x)=-(D_i-rL_i)^{-1}q^{(i)}(x) \end{cases}, \qquad i=1,2,\cdots,\alpha. \quad (5.12)$$

By making use of (5.10) and (5.11)—(5.12) we can obtain that

$$\begin{cases} g^{(i)}(x)-g^{(i)}(x^*)=e_m^T(D_i-rL_i)^{-1}[(1-r)D_i+r(U_i+N_i)](x-x^*)+e_m^TR^{(i)}(x) \\ m\in\hat{J}_i, \qquad m=1,2,\cdots,n \end{cases} \quad (5.13)$$

hold for all $i\in\{1,2,\cdots,\alpha\}$.

On the other hand, by (5.6) there hold

$$|R^{(i)}(x)|\le\varepsilon|x-x^*|, \qquad \forall x\in N(x^*,\delta), \quad i=1,2,\cdots,\alpha \quad (5.14)$$

for δ sufficiently small.

Now, based on (5.3) and (5.12)—(5.13), the following relations can be concluded,

$$\begin{cases} x_m^{p+1}-x_m^*=\sum_{i\in N_m(p)}e_m^TE_i\mathcal{L}^{(i)}(r,\omega)(x^{s^{(i)}(p)}-x^*)+\sum_{i\notin N_m(p)}e_m^TE_i(x^p-x^*) \\ \qquad\qquad +\sum_{i\in N_m(p)}e_m^T\frac{\omega}{r}E_iR^{(i)}(x^{s^{(i)}(p)}) \\ m=1,2,\cdots,n, \end{cases}$$

where $\mathcal{L}^{(i)}(r,\omega)(i=1,2,\cdots,\alpha)$ are defined in the same way as (4.22).

Up to now, the proof of Theorem 2 can be fulfilled analogous to that of Theorem 1.

We end this section with the following two remarks.

Remark I: The convergence theories of the asynchronous parallel nonlinear multisplitting AOR-Newton, AOR-Chord and AOR-Steffensen methods can be got as special cases of Theorem 2.

Remark II: The varying intervals of the relaxation parameters r and ω in Theorems 1 and 2 can be enlarged to

$$0 \le r \le \omega, \quad 0 < \omega < \frac{2}{1+\rho(|D|^{-1}|B|)}.$$

The proofs are thorough analogies of those of Theorems 1 and 2, for the length of the paper, we will not demonstrate them here in detail.

6. Numerical Results

We adopt the problem used in [4], i.e., the system of nonlinear equations

$$F(x) = 0, \qquad F = (f_1, f_2, \cdots, f_n)^T : R^n \to R^n \tag{6.1}$$

defined by

$$\begin{cases} f_j(x) = a_j x_{j-1} - b_j x_j + c_j x_{j+1} - h^2 g_j e^{x_j}, \quad j = 1, 2, \cdots, n \\ x_0 = 0, \quad x_{n+1} = 1 \end{cases} \tag{6.2}$$

with

$$\begin{cases} a_j = 1 + (j - 1/2)^2 h^2, & b_j = 2 + (2j^2 + 1/2)h^2 \\ c_j = 1 + (j + 1/2)^2 h^2, & g_j = 2(1 + 3j^2 h^2) e^{-j^2 h^2} \\ h = 1/(n+1), & j = 1, 2, \cdots, n, \end{cases} \tag{6.3}$$

and use the asynchronous parallel nonlinear multisplitting AOR-Newton method (ANMAOR(r,ω)-Newton method) as well as its special cases, that is, the asynchronous parallel nonlinear multisplitting SOR-Newton method(ANMSOR(ω)-Newton method), the asynchronous parallel nonlinear multisplitting Gauss-Seidel-Newton method (ANMGS-Newton method) and the asynchronous parallel nonlinear multisplitting Jacobi-Newton method(ANMJ-Newton method), as representatives to imitate the numerical behaviours of our new asynchronous parallel nonlinear multisplitting relaxed method models by solving the system of nonlinear equations (6.1)—(6.3) according to various n, or a fixed n but different choices of the relaxation parameter(s).

We take $\alpha = 2$ and two subsets

$$J_1 = \{1, 2, \cdots, m_1\}, \qquad J_2 = \{m_2, m_2 + 1, \cdots, n\},$$

m_1 and m_2 being positive integers satisfying $1 \le m_2 \le m_1 \le n$, of the number set $\{1, 2, \cdots, n\}$, as well as weighting matrices

$$\begin{cases} E_i = diag(e_1^{(i)}, e_2^{(i)}, \cdots, e_n^{(i)})(i = 1, 2) \\ e_j^{(1)} = \begin{cases} 1, & \text{for } 1 \le j < m_2 \\ 1/2, & \text{for } m_2 \le j \le m_1 \\ 0, & \text{for } m_1 < j \le n \end{cases}, \quad e_j^{(2)} = \begin{cases} 0, & \text{for } 1 \le j < m_2 \\ 1/2, & \text{for } m_2 \le j \le m_1 \\ 1, & \text{for } m_1 < j \le n \end{cases} \\ j = 1, 2, \cdots, n \end{cases}$$

corresponding to two kinds of block-multisplittings resulted from the following two choices of the positive integer pairs (m_1, m_2):

a) $m_1 = [2n/3], \qquad m_2 = [n/3]$;

b) $m_1 = [4n/5], \qquad m_2 = [n/5]$,

here, $[a]$ is used to denote the integer part of a positive number "a".

All our iterations are started from an initial guess having all elements equal to 10.0 and terminated once the current iteration x^p satisfies both

$$\|x^p - x^{p-1}\|_\infty \leq 2 \times 10^{-4}$$

and

$$\|F(x^p)\|_\infty \leq 1.2 \times 10^{-3}.$$

This kind of iteration indexes is written as p_t and we list it in the following numerical tables to show the feasibility and efficiency of the above tested methods. For the length of this paper, we just write several typical data among our numerous numerical results which can describe the numerical characterizations of these methods.

Table I ANMJ-Newton method

n	8	10	12	15	20	30	40	50	60	80
a)	234	334	426	616	1026	2036	3436	5098	6997	11806
b)	222	306	421	596	976	1986	3313	4942	6858	11511

Table II ANMGS-Newton method

n	8	10	12	15	20	30	40	50	60	80
a)	137	185	234	329	537	1048	1753	2588	3544	5958
b)	130	174	229	314	505	1013	1680	2496	3457	5788

Table III ANMSOR(ω)-Newton method(n=30)

ω	0.5	0.8	0.9	1.2	1.5	1.7	1.8	1.85	1.9	1.95	2.0
a)	3080	1557	1274	709	382	230	157	119$*$	135	247	∞
b)	3001	1511	1234	681	369	214	131	111$*$	151	249	794

Table IV ANMAOR(1.85,ω)-Newton method(n=30)

ω	0.5	0.8	0.9	1.2	1.5	[1.7, 1.84]	1.86	1.9	1.95	[1.96, 2.0]
a)	401	257	229	170	134	117$*$	120	130	152	
b)	356	224	199	152	131		109	109	108	104$*$

Table V ANMAOR(r,1.85)-Newton method(n=30)

r	0.9	1.2	1.5	1.7	1.8	1.84	1.86	1.9	1.95	2.0
a)	∞	514	319	212	152	124	117$*$	140	207	427
b)	∞	455	309	199	129	102$*$	114	156	256	799

In the above tables, "∞" is used to denote the case that the stopping criterion is not satisfied after the iteration is continued over 15000 times while the value p_t listed with " $*$ " shows that it is the best among all the choices of the corresponding relaxation parameter(s) in our numerical experiment and, hence, in that table. Evidently, the numerical results listed in the above tables are self-explanatory, so it is no need for us to do further analyses and illustrations.

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