Journal of Computational Mathematics, Vol.13, No.4, 1995, 343–350.

ALGEBRAIC-GEOMETRY FOUNDATION FOR CONSTRUCTING SMOOTH INTERPOLANTS ON CURVED SIDES ELEMENT*[1)]

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Abstract

The aim of this paper is to lay a algebraic geometry foundation for constructing smoothing interpolants on curved side element. Some interpolation theorems in polynomial space are given. The main results effectively for CAGD are presented.

1. Introduction

The algebraic geometry theory plays a very important role in many areas. For example, the fundamental theorem in the multivariate polynomial interpolation theory[8] is established by means of Bezout's theorem, the fundamental theoretical frame of multivariate spline functions is generated by means of compatible co-factor method[8] which is also closely related with Bezout's theorem[9].

The piecewise smooth function skill is used frequently in CAGD, FEC and scattered data fitting, etc. In using the skill, the smoothing interpolation scheme and its explicit representation on a given element are required. For a line grid partition of a given polygonal region, many results and effective skills in the application fields published by many mathematicians recently[1,2,4]. But few papers for the case of curved side element, which is also need to be considered in the application field, are found yet. In this case, the author will prove in section 3 that it is impossible to established Ženišek′s[10] type theorem on any curved sides element in any polynomial space. We consider the problem by using rational function. In fact, the results of this paper are sequel of the author's paper concentrating on the case of line grid partition element in which the generalized wedge function method for rational spline is introduced.

* Received April 24, 1994.
1) The project was supported by National Natural Science Foundation of China.

2. Notations, Definitions and Preliminaries

Let Ω be a simply or multiply connected bounded region with piecewise algebraic curved boundary in the real plane $\mathbf{R^2}$. Denote by $\mathbf{P_k}$ the set of bivariate polynomials of total degree $\leq k$.

A real algebraic curve of order n in $\mathbf{R^2}$ is defined by the set

$$\{(x,y)|p(x,y)=0\},$$

where $p(x,y)\in \mathbf{P_n}$ is a real polynomial of degree n ($n=\deg p(x,y)$). A real polynomial $p(x)$ is called to be irreducible in $\mathbf{R^2}$, if it can not be expressed a multiplication of any two real polynomials $g_1(x,y), g_2(x,y)$, $\deg g_i(x,y)>0$. The divisor of algebraic curve F on G in the complex projective plane is defined by

$$F\circ G=\sum_p (m_p(F)\cdot m_p(G))p$$

where $m_p(F)$ and $m_p(G)$ are the multiplicity of F and G at p, respectively, and the symbolic summation is over all p, including neighbors.

The definition of polypol in R^2, curved sides element, is introduced by Wachspress[7]. Let curve C_m have n distinct irreducible components: $\Gamma_1,\Gamma_2,\cdots,\Gamma_n$. Let point v_1 in $\Gamma_n\circ\Gamma_1$, v_2 in $\Gamma_1\circ\Gamma_2,\cdots v_n$ in $\Gamma_{n-1}\circ\Gamma_n$ be designated as vertices. Let $\bar{\Gamma_i}$ denote a given segment of curve Γ_i between vertices v_i and v_{i+1}(with $v_{n+1}=v_1$) such that the n segments define a simple closed planar figure. This figure is an algebraic element and is said to be *well-set (n-pol of degree m) polypol*[7] iff

(a) the vertices are all ordinary double-point of C_m,

(b) the (open) segments Γ_i contain only simple points of C_m,

(c) the polypol interior contains no point of C_m.

If a polypol is only satisfied (a) and (c), the polypol is called *ill-set polypol*.

The following two theorems are crucial results of the classical algebraic geometry.

Theorem 1. (Bezout's[9]) *The order of $F\circ G$ is $O(F\circ G)=\sum_p m_p(F)\circ m_p(G)$. If F and G have no common component in the complex projective plane, then $O(F\circ G)=t\cdot s$, where t and s are the order of F and G,respectively.*

For the case in the real plane, the following weak form of Bezout's theorem[8] is stated.

Bezout's Theorem (weak form). *If two real algebraic curves $C_1:p_1(x,y)=0$ and $C_2:p_2(x,y)=0$, of orders m resp. n, have $N>mn$ intersection points in the complex plane, then there is a real polynomial $Q(x,y)$ of degree $<min\{m,n\}$ such that*

$$p_1(x,y)=Q(x,y)R_1(x,y),$$

$$p_2(x,y)=Q(x,y)R_2(x,y),$$

where $R_1(x,y) \in \mathbf{P_{m_1}}, R_2(x,y) \in \mathbf{P_{n_1}}, m_1 < m, n_1 < n.$

It is well known that let C and C' be two cubics which meet in exactly 9 different intersection points in the complex projective plane, and if $p_1, p_2, \cdots, p_8$ are any eight of the nine intersection points, then the remainder 9-th point lies on a curve through the eight point. For general case, the crucial result on algebraic curves is connected with the linear series of curves and is utmost important to our work. That is

Theorem 2[7]**.** *Let the curves* F_k *and* G_t *be prime,* $k \geq t$*. Then there exist* $kt-(t-1)(t-2)/2$ *linear independent elements in* $F_k \circ G_t$ *determining the* $(t-1)(t-2)/2$ *remainder elements in* $F_k \circ G_t$ *uniquely.*

Theorem 2 implies that for a polynomial of degree k, the number of degree of freedom on a curve of order t is $d(k,t) = kt - (t-1)(t-2)/2 + 1 = kt - \frac{t^2-3t}{2}$. It is easy to verify that, for $k \geq t$, there are $kt - (t-1)(t-2)/2$ linear independent points (including neighbors) on a curve of order t. Those points are called $k-th$ linear independent points on the curve of order t.

3. Some Conclusions on Polynomial Interpolation

Let Γ be an irreducible real algebraic curve of order t, and let k, μ be positive integers such that $k \geq (\mu+1)\cdot t$. Denote $\hat{d}(k,\mu) = \begin{pmatrix} k-(\mu+1)\cdot t+2 \\ 2 \end{pmatrix}$.

The following conclusion is held.

Theorem 3. *Let* $\Gamma : f(x,y) = 0$ *be an irreducible real algebraic curve of order* $t \geq 1, k \geq (\mu+1)\cdot t$*. Let* $\{v_i\}_{i=1}^{\hat{d}(k,\mu)}$ *be a suitable knot-set for* $\mathbf{P_{k-(\mu+1)t}}$ *and none of the knots lies on* Γ*.* $\{v_j^{(|r|)}\}_{j=1,2,\cdots,d(|r|)}, d(|r|) = (k-|r|t)t - (t^2-3t)/2$ *be chosen from* Γ *such that all elements except one of the divisor* $\sum_{j=1}^{d(|r|)} v_j^{(|r|)}$*, are* $(k-|r|t)-th$ *linear independent on* $\Gamma, |r| = 0, 1, \cdots, \mu$*. Then there exists uniquely a polynomial* $p_k(x,y) \in \mathbf{P_k}$ *determined by the following conditions: for a given real values* $\{z_i\}$ *and* $\{z_j^{(|r|)}\}$

$$p_k(v_i) = z_i, \quad \frac{\partial^{|}r|}{\partial n^{|r|}} p_k(v_j^{(|r|)}) = z_j^{(|r|)};$$

$$|r| = 0, 1, \cdots, \mu;\ j = 1, 2, \cdots, d(|r|);\ i = 1, 2, \cdots, \hat{d}(k,\mu).$$

Proof. It is easy to calculate the total number of the conditions

$$\begin{aligned} N &= \begin{pmatrix} k-(\mu+1)t+2 \\ 2 \end{pmatrix} + \sum_{r=0}^{\mu} d(|r|) \\ &= \begin{pmatrix} k-(\mu+1)t+2 \\ 2 \end{pmatrix} + \sum_{|r|=0}^{\mu} (k-|r|t)t-(\mu+1)\begin{pmatrix} t-1 \\ 2 \end{pmatrix} + (\mu+1) \\ &= \begin{pmatrix} k+2 \\ 2 \end{pmatrix} = \dim \mathbf{P_k}. \end{aligned}$$

It suffice to show that only zero polynomial exists for the corresponding homogeneous interpolation problem. Suppose that $p_k \in \mathbf{P_k}$ is a polynomial of degree k satisfing the homogeneous interpolation conditions. From the hypothesis of theorem, we know that the curve $p_k(x,y)=0$ and $f(x,y)=0$ have commom zero points $\{v_j^{(0)}\}_{j=1,2,\cdots,d(0)}$. Because all points except one of them are $k-th$ independent points on $f(x,y)=0$, thus it follows from theorem 2 that the total common zero points of $p_k(x,y)=0$ and $f(x,y)=0$ is $d(0)+(t-1)(t-2)/2 = kt+1$. But $f(x,y)=0$ is irreducible, it follows from Bezout's theorem(weak form) that $f(x,y)=0$ is a component of $p_k(x,y)=0, p_k(x,y)=f(x,y)p_{k-t}(x,y)$. We claim that $p_k(x,y)$ can be of the form $p_k(x,y)=[f(x,y)]^{\mu+1}\cdot p_{k-(\mu+1)t}(x,y)$. Suppose the statement for an integer r is true, that is $p_k(x,y)=[f(x,y)]^{|r|+1}\cdot p_{k-(|r|+1)t}(x,y)$. Then from the hypothesis we know that $\partial^{|r|+1}/\partial\tau^{|r|+1}p_k(x,y)=0$ at every point on $f(x,y)=0$ and that $p_{k-(|r|+1)t}(v_j^{(|r|+1)})=0, j=1,2,\cdots,d(|r|+1)$. Since except one, all the points of $v_j^{(|r|+1)}, j=1,2,\cdots,d(|r|+1)$ are $(k-(|r|+1)t)-th$ linear independent, thus the total number of the common zero points of $p_{k-(|r|+1)t}(x,y)=0$ and $f(x,y)=0$ is $(k-(|r|+1)t)t+1$. Using Bezout's theorem (weak form) again, we get $p_k(x,y)=[f(x,y)]^{|r|+2}\cdot p_{k-(|r|+2)t}(x,y)$, which implies that the statement is valid. Because the set $\{v_i\}_{i=1}^{\hat{d}(k,\mu)}$ is a suitable knot-set for $\mathbf{P_{k-(\mu+1)t}}$ and none of them lies on Γ. Hence $p_k(x,y)$ must be zero polynomial. The theorem is proved.

Let $d=\begin{pmatrix} k-2t+2 \\ 2 \end{pmatrix}$, $e=kt-\frac{t^2-3t}{2}-2$, $g=(k-t)t-\frac{t^2-3t}{2}-2$, $h=(k-2t)t-\frac{t^2-3t}{2}-2$.

Corollary 4. *Let $\Gamma: f(x,y)=0$ be an irreducible algebraic curve of order $t\geq 1, k\geq(\mu+1)\cdot t$. Let $\{v_i\}_{i=1}^{\hat{d}(k,1)}$ be a suitable knot-set for $\mathbf{P_{k-2t}}$ and none of the knots lies on Γ, $\{\hat{v}_j\}_{j=1}^{e}, \{\tilde{v}_l\}_{l=1}^{g}$ be chosen on Γ such that all elements except one of the divisor $2\hat{v}_1+2\hat{v}_e+\sum_{j=2}^{e-1}\hat{v}_j$ and $\hat{v}_1+\hat{v}_e+\sum_{l=1}^{g}\tilde{v}_l$ are $k-th$ and $(k-t)-th$ linear independent on Γ, respectively. Then there exists uniquely a polynomial $p_k(x,y)\in\mathbf{P_k}$*

determined by the following information:

$$\{D^r p_k(\hat{v}_1), \quad D^r p_k(\hat{v}_e), \quad |r| \le 1; \quad p_k(\hat{v}_j), \quad j = 2,3,\cdots,e-1;$$
$$\frac{\partial}{\partial n} p_k(\tilde{v}_l), \quad l = 1,2,\cdots,g; \quad p_k(v_i), \quad i = 1,2,\cdots,\hat{d}(k,1)\} \ .$$

where $\frac{\partial}{\partial n} p_k(\tilde{v}_l)$ *means the normal derivative of* $p_k(x,y)$ *along curve* Γ *at* $\tilde{v}_l$*.*

Corollary 5. *Let* $\Gamma : f(x,y) = 0$ *be an irreducible real algebraic curve of order* $t \ge 1, k \ge (\mu+1)\cdot t, \tilde{d}(|r|) = d(|r|) - 2(\mu - |r| + 1)$*. Let* $\{v_i\}_{i=1}^{\hat{d}(k,\mu)}$ *be a suitable knot-set for* $\mathbf{P_{k-(\mu+1)t}}$ *and none of the knots lies on* Γ*, and let* v_0*,* v_1 *and* $v_{|r|j}$ *be chosen on* Γ *such that except one, all elements of the divisor* $(\mu+1)v_0 + (\mu+1)v_1 + \sum_{j=1}^{\tilde{d}(|r|)} v_{|r|j}, |r| = 1,2,\cdots,\mu$ *are* $(k-|r|t)-th$ *linear independent on* Γ*, respectively. Then there uniquely exists a polynomial* $p_k(x,y) \in \mathbf{P_k}$ *determined by the following information:*

$$\frac{\partial^{|r|}}{\partial x^{r_1} \partial y^{r_2}} p_k(v_s), \quad s = 0,1; \quad \frac{\partial^{|r|}}{\partial n^{|r|}} p_k(v^{(|r|)}_{j_{|r|}}), \quad |r| = 1,2,\cdots,\mu, \quad j_{|r|} = 1,2,\cdots,d(|r|);$$
$$p_k(v_i), \qquad i = 1,2,\cdots,\hat{d}(k,\mu).$$

From the process of the proof of the preceding theorem, we can obtain the following conclusion.

Proposition 6. *Suppose that the conditions of theorem 3 are satisfied. If* $p_1(x,y)$*,* $p_2(x,y) \in \mathbf{P_k}$ *satisfy the same interpolation conditions on* $\Gamma : f(x,y) = 0$*, then there exists a polynomial* $Q(x,y) \in \mathbf{P_{k-(\mu+1)t}}$ *such that*

$$p_1(x,y) - p_2(x,y) = [f(x,y)]^{\mu+1} \cdot Q(x,y).$$

4. Constructing Approach for Smooth Interpolants on Polypol

We propose first to show that the following important statement.

Theorem 7. *It is impossible to establish Ženišek's type theorem on any polypol in any polynomial space.*

Proof. Suppose that our claim is false. Without loss of generality, we take the polypol, curved sides element, composed of a curved sides triangle. Let $v_i (i = 1,2,3)$ be the vertices of the element, t be the order of the curve segment between v_2 and v_3 and the derivative conditions up to order μ_1 be given at $v_i (i = 1,2,3)$. To assure μ-times smoothness, it is sufficient from theorem 3 to introduce the following nodes on the polypol: for $r = 1,2,\cdots,\mu$ and a positive integer k,

a). $(k-|r|t)t - 2(\mu_1 - |r| + 1) - \frac{t^2-3t}{2}$ $(k-|r|t)$–th linear independent points on interior of the curve boundary,

b). $k - |r| + 1 - 2(\mu_1 - |r| + 1)$ points on interior of the every line boundaries, c). $\begin{pmatrix} k-(\mu+1)t-2\mu \\ 2 \end{pmatrix}$ suitable knots for $\mathbf{P_{k-(\mu+1)t-2\mu-2}}$ in interior of the element. This condition will lose sense when $k < (\mu+1)t+2\mu+2$.

The total number of conditions a), b) and c) is

$$\begin{aligned} N_1 &= \sum_{|r|=0}^{\mu}((k-|r|t)t-2(\mu_1-|r|+1)-\frac{t^2-3t}{2}) \\ &+ 2\sum_{|r|=0}^{\mu}((k-|r|+1)-2(\mu_1-|r|+1)) \\ &+ \begin{pmatrix} k-(\mu+1)t-2\mu \\ 2 \end{pmatrix} + 3\begin{pmatrix} \mu_1+2 \\ 2 \end{pmatrix} \end{aligned}$$

we shall show that the inequality $N_1 > \dim\mathbf{P_k}$ for any integer $\mu_1, \mu > 0$ and $t > 1$. This statement implies the conclusion of theorem is true. In fact,

$$N_1 - \dim\mathbf{P_k} = \frac{3}{2}\mu_1^2 + [\frac{9}{2} - 6(\mu+1)]\mu_1 + (4\mu^2 - \mu - 2) + 2(\mu+1)^2 t\ .$$

To prove $N_1 > \dim\mathbf{P_k}$, we consider the discriminant of (1), which is

$$\begin{aligned} \Delta(\mu,t) &= \frac{9}{4}(4\mu+1)^2 - 6[(4\mu^2-\mu-2)+2(\mu+1)^2 t] \\ &= \frac{3}{4}[3(4\mu+1)^2 - 8(4\mu^2-\mu-2+2(\mu+1)^2 t)] \\ &= \frac{3}{4}[(16(1-t)\mu^2 + 32(1-t)\mu + 19 - 16t]\ . \end{aligned}$$

It is clear that $\Delta(\mu,t) < 0$ for integer $\mu > 0, t > 1$. Which implies that the inequality $N_1 > \dim\mathbf{P_k}$ is held. This complete the proof of theorem.

Let π be a well-set polypol with vertices $v_i, i = 1, 2, \cdots, n$ and boundaries Γ_i : $S_{t_i}(x,y) = 0$.

To get μ-times smoothness interpolants on the well-set polypol, we need the following crucial conclusion.

Theorem 8[6]**.** *Let π be a well-set polypol with boundaries $\Gamma_j : S_{t_j}(x,y) = 0, j = 1, 2, \cdots, n, \mu > 0$ be an given integer and $f^i(x,y) \in \mathbf{C}^{\mu}(\mathbf{\Omega}), i = 1, 2, \cdots, n$ be given functions. Let*

$$\alpha_i^{(\mu)}(x,y) = \frac{c_i S_{t_1}^{\mu+1} \cdots S_{t_{i-1}}^{\mu+1} \cdot S_{t_{i+1}}^{\mu+1} \cdots S_{t_n}^{\mu+1}}{\sum_{j=1}^{n} c_j S_{t_1}^{\mu+1} \cdots S_{t_{j-1}}^{\mu+1} \cdot S_{t_{j+1}}^{\mu+1} \cdots S_{t_n}^{\mu+1}}, \quad (S_{t_{i+n}} = S_{t_i}), \tag{1}$$

where $c_i > 0, i = 1, 2, \cdots, n$ are constants. Then the rational function

$$R(x,y) = \sum_{i=1}^{n} \alpha_i^{(\mu)}(x,y) f^i(x,y), \qquad (x,y) \in \pi \tag{2}$$

has the interpolating properties:

$$D^r R(x,y)|_{\Gamma_j} = D^r f^j(x,y)|_{\Gamma_j}, \qquad |r| \le \mu.$$

The functions $f^i(x,y)$ and $\alpha_i^{(\mu)}(x,y)$ in (2) are called boundary function of π and weight function of $f^i(x,y)$, respectively.

a. *Node selection.* every vertex of π is taken as $\mu-$times vertex node which is related the derivative information up to μ. For every Γ_i, if $t_i = 1$, then for $|r| = 0, 1, \cdots, \mu, k + |r| - (2\mu + 1)$ distinct $r - th$ linear independent points are introduced on the open segment $\bar{\Gamma}_i$; if $t_i > 1$, then for $|r| = 0, 1, \cdots, \mu, (k - |r|t)t - 2(\mu - |r| + 1)$, distinct points $v_{|r|j}^{(i)}$ are introduced in the interior of $\bar{\Gamma}_i$ such that all elements except one of the divisor $(\mu - |r| + 1)v_{i-1} + (\mu - |r| + 1)v_i + \sum_{j=1}^{d(|r|)} v_{|r|j}^{(i)}$ are $(k - |r|t)$-th linear independent on Γ_i. For a given r, we call $\{v_{i_j}^{(|r|)}\}$ the $|r| - th$ side nodes on Γ_i.

b. *Constructive approach of boundary interpolation functions.* For the selected side nodes on every side of well-set polypol π as in A, the boundary interpolation polynomial is constructed by using corollary 5 as follows: say for $\bar{\Gamma}_i = \widehat{v_{i-1}v_i} : f_{t_i}(x,y) = 0$, the boundary interpolation polynomial $p^{(i)}(x,y)$ is uniquely determined by the following information:

$t = 1$, $p^{(i)}(x,y) \in \mathbf{P_{2\mu+1}}$

$$\begin{cases} \dfrac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}} p^{(i)}(v_t), \quad 0 \le r_1 + r_2 \le \mu, \quad t = i-1, i, i+1; \\[2ex] \dfrac{\partial^{|r|}}{\partial n^{|r|}} p^{(i)}(v_{|r|j}^{(i)}), \qquad |r| = 1, 2, \cdots, \mu, \quad j = 1, 2, \cdots, |r|; \end{cases}$$

$t > 1, p^{(i)}(x,y) \in \mathbf{P_{(\mu+1)t}}$

$$\begin{cases} \dfrac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}} p^{(i)}(v_i), \quad \dfrac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}} p^{(i)}(v_{i+1}), \quad 0 \le r_1 + r_2 = |r| \le \mu; \\[2ex] \dfrac{\partial^{|r|}}{\partial n^{|r|}} p^{(i)}(v_{|r|j}^{(i)}), \qquad 0 \le |r| \le \mu, \quad j = 1, 2, \cdots, d'(|r|); \\[2ex] p^{(i)}(v_{i-1}) \quad \text{or} \quad p^{(i)}(v_{i+2}) \end{cases} \tag{3}$$

where $d'(|r|) = (\mu - |r| + 1)(t^2 - 2) - (t^2 - 3t)/2$. The detail explicit representation of the boundary interpolation functions is investigated in author's other paper else.

c. *μ-times smoothness rational function.* Using the constructed boundary interpolation polynomials of polypol π, one construct the real rational function $R(x,y)$ by theorem 8, that is

$$R^{\pi}(x,y)=\sum_{i=1}^{n}\alpha_i^{(\mu)}(x,y)p^i(x,y), \qquad (x,y)\in\pi \tag{4}$$

where the meaning of $\alpha_i^{(\mu)}(x,y)$ is same to the one in theorem 8.

From the conclusions above, it is easy to prove from proposition 6 the following theorem.

Theorem 9. *Let Ω be a region in $\mathbf{R^2}$ with piecewise algebraic curve boundary, and Ω_p be a well-set polypol partition of Ω such that every piecewise point on the boundary is a vertex of some well-set polypol. Let $\mu > 0$ be a given integer. Then the piecewise function*

$$R(x,y)=R^{\pi}(x,y), \qquad (x,y)\in\pi, \qquad ,\pi\in\Omega_p$$

is μ-times smoothness differentiable on the set Ω.

For the case of ill-set polypol, if the side nodes are selected as in well-set polypol and none of them is not singular point of the boundary curve, then our deduction and conclusions are also held.

Finally, the following conclusions are stated: for a given well-set polypol π, let $t=\min\{t_i : i=1,2,\cdots,n\}$. If $t=1$, then the smooth interpolant (5) has $2\mu+1$ polynomial precision; if $t>1$, then the smooth interpolant (4) has $(\mu+1)t$ polynomial precision.

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