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ON THE SPLITTINGS FOR RECTANGULAR SYSTEMS*

H.J. Tian
(*Department of Mathematics, Shanghai Normal University, Shanghai, China*)

Abstract

Recently , M. Hanke and M. Neumann[4] have derived a necessary and sufficient condition on a splitting of $A = U-V$, which leads to a fixed point system , such that the iterative sequence converges to the least squares solution of minimum 2-norm of the system $Ax = b$. In this paper, we give a necessary and sufficient condition on the splitting such that the iterative sequence converges to the weighted Moore-Penrose solution of the system $Ax = b$ for every $x_0 \in C^n$ and every $b \in C^m$. We also provide a necessary and sufficient condition such that the iterative sequence is convergent for every $x_0 \in C^n$.

1. Introduction

It is well-known that the most prevalent approach for obtaining a fixed point system of the following system

$$Ax = b, \qquad A \in C^{m\times n} \tag{1.1}$$

is via a splitting of the coefficient matrix A into

$$A = U - V. \tag{1.2}$$

If $m = n$ and U is nonsingular, we present the equivalent formulation of (1.1) by

$$x = U^{-1}Vx + U^{-1}b. \tag{1.3}$$

If $m \neq n$ or if U is not invertible, we can, by taking a generalized inverse U^- of U (instead of U^{-1}), extend (1.3) by considering the fixed point system

$$x = U^-Vx + U^-b. \tag{1.4}$$

Generalized inverses of matrices play a key role in our present work. It is instructive for our purposes to think of reflexive inverses as weighted Moore-Penrose inverses and to call the corresponding solution which induce weighted Moore-Penrose solution. In section 2, we summarize preliminary results from the literature on generalized inverses

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which are most relevant to this paper briefly. In section 3 , we derive a necessary and sufficient condition for a splitting (1.2) to yield a fixed point iterative scheme such that the limit point $\bar{x}$ is a weighted Moore-Penrose solution to (1.1). In section 4, we provide a necessary and sufficient condition such that the iterative sequence is convergent for every $x_0 \in C^n$ and every $b \in C^m$. In section 5, a numerical experiment is presented to illustrate the performance of the splitting.

2. Preliminary and Background Results

Let $A \in C^{m\times n}$ and suppose $X \in C^{n\times m}$. Then X is called a reflexive inverse of A if

$$AXA = A \quad \text{and} \quad XAX = X. \tag{2.1}$$

Given a subspace $R \subseteq C^n$ which is complementary to $N(A)$ and a subspace $N \subseteq C^m$ which is complementary to $R(A)$, then there exists a unique reflexive inverse X of A such that

$$R(X) = R \quad \text{and} \quad N(X) = N \tag{2.2}$$

and conversely, if X is a reflexive inverse of A, then $R(X)$ and $N(X)$ are complementary subspace of $N(A)$ and $R(A)$. In the following we shall use $R(\mathrm{A})$ and $N(\mathrm{A})$ to denote the range and the nullspace of a matrix A. Accordingly we write $A^-_{R,N} := X$. It is known that

$$A^-_{R,N}A = P_{R,N(A)} \quad \text{and} \quad AA^-_{R,N}, = P_{R(A),N}, \tag{2.3}$$

where $P_{R,N(A)}$ and $P_{R(A),N}$ denote the projectors on R along $N(A)$ and on $R(A)$ along N, respectively.

With any reflexive inverse X of A one can associate two vector norms, one in $C^n$and one in C^m, as follows:

$$\|x\|_{R,N(A)} := (\|P_{R,N(A)}x\|_2^2 + \|(I - P_{R,N(A)})x\|_2^2)^{1/2}, \forall x \in C^n$$

and

$$\|y\|_{R(A),N} := (\|P_{R(A),N}y\|_2^2 + \|(I - P_{R(A),N})y\|_2^2)^{1/2}, \forall y \in C^m.$$

Due to the finite dimensional setting which we work in, for any vector $b \in C^m$ the set

$$\delta_b := \{\bar{x} \in C^n : \|b - A\bar{x}\|_{R(A),N} = \inf_{x\in C^n} \|b - Ax\|_{R(A),N}\} \neq \phi \tag{2.4}$$

and the vector $\bar{z} := A^-_{R,N}b$ has the following properties:

$$\bar{z} \in \delta_b \text{ and } \|\bar{z}\|_{R,N(A)} = \min_{\bar{x}\in\delta_b} \|\bar{x}\|_{R,N(A)}. \tag{2.5}$$

Therefore we can interpret any reflexive inverse as a weighted Moore-Penrose inverse and vice versa $\bar{z}$ as a weighted Moore-Penrose solution to the system $Ax = b$.

We next mention some choices of R and N which correspond to reflexive inverses that are frequently used in applications and in the literature. First,suppose that $N =$

$N(A^*) = R(A)^\perp$ and $R = R(A^*) = N(A)^\perp$,then $\bar{z} = A^-_{R(A^*),N(A^*)}b$ is the least-squares solution of minimum 2-norm of the system $Ax = b$ and $A^-_{R(A^*),N(A^*)}$ is the familiar Moore-Penrose inverse of A which is usually denoted by A^+. Let P, Q are definite matrices of order m and order n , respectively. If $N = P^{-1}N(A^*)$ and $R = Q^{-1}R(A^*)$, then $\bar{z} = A^-_{Q^{-1}R(A^*),P^{-1}N(A^*)}b$ is the least-squares (P) solution of minimum-norm (Q) of the system $Ax = b$ and $A^-_{Q^{-1}R(A^*),P^{-1}N(A^*)}$ is always denoted as $A^+_{P,Q}$.

A more specialized generalized inverse for a matrix can be defined when the matrix is square.

Let $A \in C^{n\times n}$ and let index(A) be the smallest nonnegative integer l such that $N(A^l) = N(A^{l+1})$. Then there exists a unique matrix $X \in C^{n\times n}$, called the Drazin inverse of A and represented as A^D, that satisfies the following matrix equations

$$XAX = X, \quad AX = XA \quad \text{and} \quad XA^{j+1} = A, \quad \forall j \geq \text{index}(A). \tag{2.6}$$

When index$(A) \leq 1$, or, equivalently, when $R(A) \bigoplus N(A) = C^n$, then A^D is a reflexive inverse of A. This reflexive inverse is called in the literature the group inverse of A and denoted by A_g. It should be noted that A_g is simply $A^-_{R(A),N(A)}$.

Definition 2.1. *Let A have a splitting (1.2). Given subspaces $T, \widetilde{T} \subseteq C^n$ and subspaces ,$\widetilde{S} \subseteq C^m$, such that $T \bigoplus R(A) = C^m, \widetilde{T} \bigoplus R(U) = C^m, S \bigoplus N(A) = C^n$ and $\widetilde{S} \bigoplus N(U) = C^n$. Then the splitting (1.2) is called subproper if*

$$T \subseteq \widetilde{T}, \widetilde{S} \subseteq S \tag{2.7}$$

and it is called proper if equalities hold in (2.7).

3. Proper Splitting

In this section we are interested in semiiterative methods which converge to the weighted Moore-Penrose solution to the system (1.1). The following theorem forms the main result of this section.

Theorem 3.1. *Let $A \in C^{m\times n}$have a splitting (1.2). Given subspaces $T, \widetilde{T} \subseteq C^n$ and subspaces $S, \widetilde{S} \subseteq C^m$ such that $T \bigoplus R(A) = C^m$, $\widetilde{T} \bigoplus R(U) = C^m$, $S \bigoplus N(A) = C^n$ and $\widetilde{S} \bigoplus N(U) = C^n$. Then the sequence of iterates*

$$x_k = U^-_{\widetilde{T},\widetilde{S}}Vx_{k-1} + U^-_{\widetilde{T},\widetilde{S}}b \tag{3.1}$$

converges to the weighted Moore-Penrose solution $A^-_{T,S}b \in C^n$ for every $x_0 \in C^n$ and every $b \in C^m$ if and only if

$$\rho(U^-_{\widetilde{T},\widetilde{S}}V) < 1 \tag{3.2}$$

holds and the splitting (1.2) is proper.

Proof. Assume (3.2) holds. Since $\rho(U^{-}_{\widetilde{T},\widetilde{S}}V) < 1$, then $I - U^{-}_{\widetilde{T},\widetilde{S}}V$ is nonsingular. Since $T = \widetilde{T}, S = \widetilde{S}$, then

$$\begin{aligned}(I - U^{-}_{\widetilde{T}\widetilde{S}}V)A^{-}_{T,S} &= A^{-}_{T,S} - U^{-}_{\widetilde{T}\widetilde{S}}VA^{-}_{T,S}\\ &= A^{-}_{T,S} - U^{-}_{\widetilde{T}\widetilde{S}}UA^{-}_{T,S} + U^{-}_{\widetilde{T}\widetilde{S}}AA_{T,S}\\ &= A^{-}_{T,S} - A^{-}_{T,S} + U^{-}_{\widetilde{T}\widetilde{S}}\\ &= U^{-}_{\widetilde{T}\widetilde{S}}\end{aligned}$$

Thus

$$(I - U^{-}_{\widetilde{T}\widetilde{S}}V)^{-1}U^{-}_{\widetilde{T}\widetilde{S}} = A^{-}_{T,S} \tag{3.3}$$

From (3.1)it is easily proven by induction that

$$x_k = (U^{-}_{\widetilde{T}\widetilde{S}}V)^k x_0 + \sum_{j=0}^{k-1}(U^{-}_{\widetilde{T},\widetilde{S}}V)^j U^{-}_{\widetilde{T},\widetilde{S}}b. \tag{3.4}$$

From $\rho(U^{-}_{\widetilde{T},\widetilde{S}}V) < 1$ and (3.4), it follows that $(U^{-}_{\widetilde{T},\widetilde{S}}V)^k \to 0$ and

$$\lim_{k\to\infty} x_k = (I - U^{-}_{\widetilde{T},\widetilde{S}}V)^{-1}U^{-}_{\widetilde{T},\widetilde{S}}b = A^{-}_{T,S}b.$$

Assume the sequence $\{x_k\}_0^\infty$ with respect to (3.1) converges to the weighted Moore-Penrose solution $A^{-}_{T,S}b$ independently of the initial vector x_0 and b, we must have $\rho(U^{-}_{\widetilde{T},\widetilde{S}}V) < 1$ and

$$A^{-}_{T,S} = (I - U^{-}_{\widetilde{T},\widetilde{S}}V)^{-1}U^{-}_{\widetilde{T},\widetilde{S}}. \tag{3.5}$$

Since

$$(I - U^{-}_{\widetilde{T},\widetilde{S}}V)U^{-}_{\widetilde{T},\widetilde{S}} = U^{-}_{\widetilde{T},\widetilde{S}}(I - VU^{-}_{\widetilde{T},\widetilde{S}})$$

then

$$A^{-}_{T,S} = (I - U^{-}_{\widetilde{T}\widetilde{S}}V)^{-1}U^{-}_{\widetilde{T}\widetilde{S}} = U^{-}_{\widetilde{T}\widetilde{S}}(I - VU^{-}_{\widetilde{T}\widetilde{S}})^{-1}\ . \tag{3.6}$$

From (3.6), it immediately follows that

$$T = R(A^{-}_{T,S}) = R(U^{-}_{\widetilde{T}\widetilde{S}}) = \widetilde{T},$$

$$S = N(A^{-}_{T,S}) = N(U^{-}_{\widetilde{T}\widetilde{S}}) = \widetilde{S}.$$

This completes the proof of this theorem.

Corollary 3.2.[4] *Let $A = U - V$ be a splitting of $A \in C^{m\times n}$. Then the sequence of iterates*

$$x_k = U^+Vx_{k-1} + U^+b \tag{3.7}$$

converges to A^+b for every $b \in C^m$ and from every $x_o \in C^n$ if and only if

$$\rho(U^+V) < 1, \quad N(A) = N(U) \quad \text{and} \quad R(A) = R(U). \tag{3.8}$$

Corollary 3.3. *Let $A = U - V$ be a splitting for $A \in C^{m\times n}$. Let P and Q be definite matrices with order m and order n, respectively. Then the following sequence of iterates*

$$x_k = U^{+}_{P,Q} V x_{k-1} + U^{+}_{P,Q} b \tag{3.9}$$

converges to $A^{+}_{P,Q} b$ for every $b \in C^m$ and from every $x_o \in C^n$ if and only if

$$\rho(U^{+}_{P,Q} V) < 1, \quad R(A) = R(U) \quad \text{and} \quad N(A) = N(U). \tag{3.10}$$

Corollary 3.4. *Let $A = U - V$ be a splitting for $A \in C^{n\times n}$ with index$(A) = 1$. Then the sequence of iterates*

$$x_k = U_g V x_{k-1} + U_g b \tag{3.11}$$

converges to $A_g b$ for every $b \in C^n$ and from every $x \in C^n$ if and only if

$$\rho(U_g V) < 1, \quad R(A) = R(U) \quad \text{and} \quad N(A) = N(U). \tag{3.12}$$

4. Subproper Splitting

In this section we will discuss the convergence of (3.1) in the case when (1.2) is a subsplitting.

Theoremm 4.1. *Let (1.2) be subproper as Definition 2.1. Then the iterative sequence $\{x_k\}_0^\infty$ generated by (3.1) is convergent for every $x_0 \in C^n$ if and only if the iteration matrix $U^{-}_{\widetilde{T}\widetilde{S}} V$ is semiconvergent, i.e.*

$$\rho(U^{-}_{\widetilde{T},\widetilde{S}} V) \le 1, \tag{i}$$

$$\lambda \in \delta(U^{-}_{\widetilde{T},\widetilde{S}} V) \quad \text{and} \quad |\lambda| = 1 \Longrightarrow \lambda = 1 \quad \text{and} \tag{ii}$$

$$\text{index}(I - U^{-}_{\widetilde{T},\widetilde{S}} V) \le 1. \tag{iii}$$

Proof. From (3.1), we have

$$x_k = (U^{-}_{\widetilde{T},\widetilde{S}} V)^k x_0 + \sum_{j=0}^{k-1} (U^{-}_{\widetilde{T},\widetilde{S}} V)^j U^{-}_{\widetilde{T},\widetilde{S}} b.$$

Since $T \subseteq \widetilde{T}$ and $\widetilde{S} \subseteq S$, then $(I - U^{-}_{\widetilde{T},\widetilde{S}} V) A^{-}_{T,S} = U^{-}_{\widetilde{T},\widetilde{S}}$. Thus

$$\begin{aligned}\sum_{j=0}^{k-1} (U^{-}_{\widetilde{T},\widetilde{S}} V)^j U^{-}_{\widetilde{T},\widetilde{S}} b &= \sum_{j=0}^{k-1} (U^{-}_{\widetilde{T},\widetilde{S}} V)^j (I - U^{-}_{\widetilde{T},\widetilde{S}} V) A^{-}_{T,S} b \\ &= (I - (U^{-}_{\widetilde{T},\widetilde{S}} V)^k) A^{-}_{T,S} b. \end{aligned} \tag{4.1}$$

Therefore

$$x_k - A^{-}_{T,S} b = (U^{-}_{\widetilde{T},\widetilde{S}} V)^k (x_0 - A^{-}_{T,S} b). \tag{4.2}$$

It follows that, from (4.2), a necessary and sufficient condition for the scheme (3.1) to converge from any initial vector x_0 is that the iteration matrix $U^-_{\widetilde{T},\widetilde{S}}V$ is semiconvergent.

If the splitting (1.2) is subproper and the iteration matrix is semiconvergent , then the iteration (3.1) will converge to

$$(I - U^-_{\widetilde{T},\widetilde{S}}V)_g U^-_{\widetilde{T},\widetilde{S}}b + [I - (I - U^-_{\widetilde{T},\widetilde{S}}V)(I - U^-_{\widetilde{T},\widetilde{S}}V)_g]x_0.$$

5. Numerical Experiment

We illustrate the performance of the splitting with respect to the minimum-norm (Q) least-squares (P) solution to (1.1). Let

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

In this case the minimum-norm (Q) least-squares (P) solution of the system $Ax = b$ is

$$x = \begin{bmatrix} \frac{10}{3} \\ \frac{26}{3} \end{bmatrix}$$

Let

$$A = U - V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

be a splitting for A. Using this splitting, after $k = 150$ iterative computations, we obtain the computed result

$$x_k = \begin{bmatrix} 0.33333332E + 01 \\ 0.66666667E + 01 \end{bmatrix}.$$

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