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THE CONVERGENCE OF MULTIGRID METHODS FOR SOLVING FINITE ELEMENT EQUATIONS IN THE PRESENCE OF SINGULARITIES*1)

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Abstract

We analyze the convergence of multigrid methods applied to finite element equations of second order with singularities caused by reentrant angles and abrupt changes in the boundary conditions. Provided much more weaker demand of classical multigrid proofs, it is shown in this paper that, for symmetric and positive definite problems in the presence of singularities, multigrid algorithms with even one smoothing step converge at a rate which is independent of the number of levels or unknowns. Furthermore, we extend this result to the nonsymmetric and indefinite problems.

1. Introduction

Multigrid Methods provide optimal order solvers for linear systems of finite element equations arising from elliptic boundary value problems. The convergence of multigrid methods was proved by many authors[2−6,9−12]. All these proofs, require strong regularities and quasi-uniformity of grids[3,10]. For example, assuming $H^{1+\alpha}$ regularity and quasi-uniform triangulations, Bank & Dupont[3] showed a convergence rate of $O(m^{\frac{-\alpha}{2}})$, for a growing number m of smoothing steps per level. In the optimal case $\alpha = 1$, the problem has to be H^2–regular. When the region has reentrant angles or abrupt changes in the boundary condition, H^2–regularity is violated, and in addition, the approximation properties of the finite element space deteriorate because of the presence of singularities not captured by the quasi-uniform grids.

Yserentant[11] proved the convergence of multigrid methods for symmetric and definite problems with singularities. However, a sufficiently large number of smoothing steps m was required. Shangyou Zhang[12] got the similar result using nonnested multigrid methods, but it also assumed that m is larger than a certain constant. In this

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work, we prove the convergence of multigrid methods for symmetric and definite problems with singularities of even one smoothing step. Furthermore, it is shown in this paper that, multigrid methods applied to indefinite and nonsymmetric problems also converge on nonquasiuniform grids.

The outline of the remainder of the paper is as follows.

In section 2, we define a weighted function $\phi_r(x)$ and a family of triangulations governed by $\phi_r(x)$, and describe a j-level multigrid iterative procedure. An important lemma is given in section 3. In section 4, we prove our multigrid convergence theorems. We provide some results for nonsymmetric and indefinite problems with singularities in section 5.

Throughout this paper, c and C will denote generic positive constants which may take on different values in different places. These constants will always be independent of the mesh parameters.

2. Notation and Multigrid Scheme

For simplicity, we consider the model problem

$$\begin{aligned} -\Delta u + u &= f, && \text{in } \Omega, \\ u &= 0, && \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where Ω is an open bounded polygonal domain in R^2 with the boundary subdivided into two parts Γ_D and Γ_N. Let x_i, $1 \le i \le M$, denote the vertices of Ω with θ_i, where θ_i is the interior angle of Ω at x_i. Because of possible changes in the boundary conditions, the case $\theta_i = \pi$ is permitted. Let $0 < \theta_i < 2\pi$. For each vertex x_i, we define $k_i = 1$ if the two sides of x_i belong either both to Γ_D or both to Γ_N, and $k_i = 1/2$ otherwise. Let $\alpha_i = \min(1, (k_i\pi)/\theta_i)$, then $1/4 \le \alpha_i \le 1$ holds. If we have pure Dirichlet or Neumann boundary conditions, $\alpha_i < 1$ only holds for reentrant angles. We choose r_i with $1 - \alpha_i \le r_i < 1$ if $\alpha_i < 1$, and $r_i = 0$ if $\alpha_i = 1$. Define

$$\phi_r(x) = \prod_{i=1}^{M} \mid x - x_i \mid^{r_i} \tag{2}$$

for $r = (r_1, r_2, \ldots, r_M)$, where $\mid x \mid$ denotes the Euclidean norm. We assume that the family $T_0, T_i \ldots$of triangulations has the following two properties[1]: Let $\tau \in T_j$ be a triangle, then

$$ch_j\phi_r(x) \le d(\tau) \le ch_j\phi_r(x), \quad \text{if } \phi_r(x) \ne 0, \quad \forall x \in \tau, \tag{3}$$

$$ch_j \max_{x\in\tau} \phi_r(x) \le d(\tau) \le ch_j \max_{x\in\tau} \phi_r(x), \quad \text{if } \phi_r(x) \ne 0 \quad \text{for some } x \in \tau. \tag{4}$$

Here, $d(\tau)$ denotes the diameter of τ. For convenience, we simply let $h_j = 2^{-j}$ in this paper.

We now state finite element approximation problems of (1) as follows: Assume u is the exact solution of (1), find $P_j u \in S_j$, such that

$$A(P_j u, v) = F(v), \quad \forall v \in S_j, \quad j = 1, 2, \ldots \tag{5}$$

where $A(u,v) = \int_\Omega (\nabla u \nabla v + uv)dx, F(v) = \int_\Omega f v dx$, and S_j is the space of continuous, piecewise linear functions (on a grid T_j defined above)which vanish on Γ_D . Define $P_j : H_D^1(\Omega) \longrightarrow S_j$ is a finite element projection operator, where $H_D^1(\Omega)$ be a space of H^1–functions which vanish on Γ_D in the sense of traces. Of course, $S_j \subset H_D^1(\Omega)$. We define some weighted Soblev norms associated with the function $\phi_r(x)$ given by (2). Let

$$\| u \|_{o,r} = \sqrt{\int_\Omega \phi_r^2(x) \mid u(x) \mid^2 dx} \tag{6}$$

be a weighted L_2–norm and $H^{o,r}$ be the space of all measurable functions having finite $\| \, . \, \|_{o,r}$ norm. Replace ϕ_r^2 by ϕ_r^{-2} in (6), we define $\| u \|_{o,-r}$ and $H^{0,-r}$ similarly. Let $\| u \|_{2,r} = (\| u \|_1^2 + \sum_{|\alpha|=2} \| D^\alpha u \|_{0,r}^2)^{\frac{1}{2}}$ and $H^{2,r}$ be the corresponding space. Babuška, Kellogg and Pitkaranta [1] proved some results based on triangulations defined above. We state as follows.

Al: (Regularity[1,8]) *If the coefficients* r_i *are chosen as above and* $f \in H^{0,r}(\Omega)$, *then the solution of (1) belongs to* $H^{2,r}(\Omega)$ *and* $\| u \|_{2,r} \leq C \| f \|_{0,r}$ *holds.*

A2: (Approximation Properties) *For all functions* $u \in H^{2,r}(\Omega)$, *there holds*

$$\| u - P_j u \|_1 \leq C h_j \| u \|_{2,r} \; .$$

The solution of (5) is equivalent to the solution of linear system $A_{k_j} x^{k_j} = b^{k_j}$, where A_{k_j} is a matrix with the entries $A_{k_j}(i,k) = A(\psi_i^{k_j}, \psi_k^{k_j})$, $b^{k_j} = (f, \psi_i^{k_j})$ and $\{\psi_i^{k_j}, i = 1, 2 \ldots, \dim(S_j)\}$ constitutes a basis of S_j.

Now we describe a j-level multigrid iterative algorithm.

Let $u^0 \in S_j$ be an approximation of $P_j u$, a full multigrid iterative steps from u^0 to u^{2m+1} are as follows.

a . Pre-smoothing steps. $u^1, u^2, \ldots, u^m$ are computed by using the recursion

$$u^i = G_j u^{i-1}, \tag{7}$$

$$G_j x = x + \omega(b^{k_j} - A_{k_j} x), \quad \forall x \in S_j. \tag{8}$$

$P_j u$ satisfies (8), i.e.

$$G_j P_j u = P_j u. \tag{9}$$

For simplicity we assume $\omega = 1/(\text{largest eigenvalue of } A_{k_j})$.

b . Coarse grid correction. Let $d \in S_{j-1}$ be the solution of

$$A(d, v) = F(v) - A(u^m, v), \quad \forall v \in S_{j-1}. \tag{10}$$

If $j = 1$, let $\tilde{d} = d$; if $j > 1$, compute an approximate solution $\tilde{d}$ to d by applying v–iteration steps at level $j-1$ to the problem (10) with 0 as initial value, compute

$$u^{m+1} = u^m + \tilde{d}. \tag{11}$$

Note that

$$d = P_{j-1}(P_j u - u^m). \tag{12}$$

c . Post-smoothing steps. For $i = m+2, m+3, \ldots, 2m+1$, compute

$$u^i = G_j u^{i-1}. \tag{13}$$

3. Preliminary Lemmas

We state the eigenvalues and eigenvectors in a weak form

$$A(\Phi_i, v) = \lambda_i(\Phi_i, v), \quad \forall v \in S_j. \tag{14}$$

Since $A(.,.)$ is a symmetric bilinear form we may choose the Φ_i such that $(\Phi_i, \Phi_j) = \delta_{ij}$ (Kronecker's symbol). Define

$$||| \ u \ |||_s^2 = \sum_i \lambda_i^s \mid c_i \mid^2, \tag{15}$$

where $u = \sum_i c_i \Phi_i$. Note that $||| \ u \ |||_0 = || \ u \ ||_{l_2}$ and $||| \ u \ |||_1 = || \ u \ ||, (|| \ u \ ||$ is energy norm). It is easily proved by Cauchy-Schwarz inequality that

$$A(u, v) \leq C \ ||| \ u \ |||_0 ||| \ v \ |||_2 \ . \tag{16}$$

Now we make the following assumption

A3: $$\lambda_{max} \leq C h_j^{-2}. \tag{17}$$

We note that A3 can be verified on many partitions satisfying (3), (4), e.g., the partitions proposed in [7,8]. Define an operator $Q_{j-1} : H_D^1(\Omega) \longrightarrow S_{j-1}, Q_{j-1} = I - P_{j-1}$, then Q_{j-1} is an orthogonal projection operator on S_{j-1} in corresponding of inner product $A(.,.)$, i.e.

$$A(Q_{j-1}u, v) = 0, \quad \forall v \in S_{j-1} \tag{18}$$

and Q_j satisfy

$$Q_j^2 = Q_j. \tag{19}$$

Now we can state and prove an important lemma.

Lemma 3.1. *Given $u \in S_j$, then*

$$\| Q_{j-1}u \| \leq Ch_j \||| u \|||_2 \tag{20}$$

Proof. From (18), (16) we conclude

$$\begin{aligned} \| Q_{j-1}u \|^2 &= A(Q_{j-1}u, Q_{j-1}u) = A(Q_{j-1}u, u) \\ &\leq \||| Q_{j-1}u \|||_0 \||| u \|||_2 \leq \| Q_{j-1}u \|_0 \||| u \|||_2 \,. \end{aligned} \tag{21}$$

Next we proceed by the standard duality argument. Let $f = \phi_r^{-1} Q_{j-1}u$, then $f \in H^{0,r}$ (cf. (3.2) in [1]), by A1 there is a $z \in H^{2,r}$ satisfy $(f, Q_{j-1}u) = A(z, Q_{j-1}u)$ with estimation

$$\| z \|_{2,r} \leq C \| f \|_{0,r} \,. \tag{22}$$

Using A2 we have $\| z - P_{j-1}z \| \leq Ch_j \| z \|_{2,r}$. Hence, we obtain

$$\begin{aligned} \| Q_{j-1}u \|_{0,-r}^2 &= (f, Q_{j-1}u) = A(z, Q_{j-1}u) \\ &= A(z - P_{j-1}z, Q_{j-1}u) \\ &\leq Ch_j \| z \|_{2,r} \| Q_{j-1}u \| \\ &\leq Ch_j \| f \|_{0,r} \| Q_{j-1}u \| \\ &\leq Ch_j \| Q_{j-1}u \|_{0,r} \| Q_{j-1}u \|, \end{aligned} \tag{23}$$

since the uniform boundness of $\phi_r(x)$ we obtain

$$\| Q_{j-1}u \|_0^2 \leq C \| Q_{j-1}u \|_{0,-r}^2 \,. \tag{24}$$

The lemma then follows by (21), (23) and (24) immediately.

4. Proof of Convergence

The equivalent form of (8) is

$$G_j x = \sum_i c_i \left(1 - \frac{\lambda_i}{\lambda_{max}}\right) \Phi_i, \tag{25}$$

where $x = \sum_i c_i \Phi_i$.

We introduce a weaker seminorm[4]

$$| x |^2 = \sum_i \lambda_i \left(1 - \frac{\lambda_i}{\lambda_{max}}\right) | c_i |^2, \tag{26}$$

$$| x |^2 = A(x, G_j x) = \| G_j^{\frac{1}{2}} x \|^2, \tag{27}$$

we denote

$$\rho = \rho(x) = \begin{cases} \frac{|x|^2}{||x||^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases} \quad (28)$$

For $i = 0, 1, \ldots, 2m+1$, let

$$e^i = P_j u - u^i, \quad (29)$$

then form (7), (9) we have

$$e^m = G_j e^{m-1} = G_j^m e^0. \quad (30)$$

Let $\mu_i = 1 - \lambda_i/\lambda_{max}$, $e^0 = \sum_i c_i \Phi_i$, by applying Holder's inequality we obtain

$$\begin{aligned} || e^m ||^2 &= || G_j^m e^0 || = \textstyle\sum_i \lambda_i \mu_i^{2m} \mid c_i \mid^2 \\ &\leq (\textstyle\sum_i \lambda_i \mu_i^{2m+1} \mid c_i \mid^2)^{\frac{2m}{2m+1}} (\textstyle\sum_i \lambda_i \mid c_i \mid^2)^{\frac{1}{2m+1}} \\ &= \mid e^m \mid^{\frac{4m}{2m+1}} || e^0 ||^{\frac{2}{2m+1}} . \end{aligned} \quad (31)$$

The error estimate of smoothing step now is clear:

Lemma 4.1.

$$|| e^m || \leq \rho^m || e^0 ||$$

holds.

Next we estimate the error of coarse grid correction.

Let $e^m = \sum_i c_i \Phi_i$, from (26) and (15), we obtain

$$|| e^m ||^2 - \mid e^m \mid^2 = \lambda_{max}^{-1} ||| e^m |||_2^2 . \quad (32)$$

We may conclude from Lemma 3.1, (32) and A3 that

$$|| Q_{j-1} e^m ||^2 \leq C h_j^2 ||| e^m |||_2^2 \leq C(|| e^m ||^2 - \mid e^m \mid^2) = C(1-\rho) || e^m ||^2 . \quad (33)$$

together with (19), we see that

$$|| Q_{j-1} e^m ||^2 \leq \mu || e^m ||^2, \quad (34)$$

where $\mu = \min(1, c(1-\rho))$

We usually declare the convergence rate of multigrid method with j auxiliary grids by a contraction number δ_j with respect of the energy norm such that

$$|| e^i || \leq \delta_j || e^0 || \quad (35)$$

If δ_{j-1} for the level j-1 in the sense of (35) is already known, then by the coarse grid correction, we see that

$$|| \tilde{d} - d || \leq \varepsilon || d ||, \quad (36)$$

where $\varepsilon = \delta_{j-1}^{\nu}$. Now we are able to prove the error estimate of coarse grid correction.

Lemma 4.2. *There holds*

$$\| e^{m+1} \| \le (\varepsilon + (1-\varepsilon)\mu) \| e^m \| .$$

Proof. From (36), we know that, there is a $w \in S_{j-1}$ with $\| w \| \le \| d \|$ such that $\tilde{d} = d + \varepsilon w$. From (12) and the energy orthogonality of Q_{j-1}, we obtain

$$\| Q_{j-1} e^m + w \|^2 \le \| e^m \|^2 . \tag{37}$$

From (19)and (37), we obtain for any $v \in S_j$

$$\begin{aligned} A(v, e^{m+1}) &= A(v, Q_{j-1} e^m + \varepsilon w)) \\ &= A(v, (1-\varepsilon) Q_{j-1}^2 e^m + \varepsilon (Q_{j-1} e^m + w)) \\ &\le (1-\varepsilon) \| Q_{j-1} v \| \| Q_{j-1} e^m \| + \varepsilon \| Q_{j-1} e^m + w \| \| v \| \\ &= (1-\varepsilon) \| Q_{j-1} v \| \| Q_{j-1} e^m \| + \varepsilon \| e^m \| \| v \| \end{aligned} \tag{38}$$

By schwarz inequality and (34), (38) implies

$$A(v, e^{m+1}) \le (\varepsilon + (1-\varepsilon)\mu) \| e^m \| \| v \| \tag{39}$$

$$\| e^{m+1} \| \le (\varepsilon + (1-\varepsilon)\mu) \| e^m \| \tag{40}$$

which completes the proof of Lemma.

The convergence of multigrid method is now clearly from Lemma 4.1 and Lemma 4.2.

Theorem 4.1. *One step of the full multi-level procedure satisfy*

$$\| e^{2m+1} \| \le \delta_j \| e^0 \|, \tag{41}$$

where

$$\delta_j = \max_{0 \le \rho \le 1} \rho^{2m} (\varepsilon + (1-\varepsilon)\mu). \tag{42}$$

The rest of convergence proof is devoted to verify the next Theorem.

Theorem 4.2. *The contraction number δ_j of multigrid method satisfy*

$$\delta_j \le \bar{\delta} = \frac{c}{2m+c} < 1, \quad \forall m \ge 1.$$

Proof. We use mathematical induction. First, $\delta_0 = 0$.
Assume

$$\delta_{j-1} \le \bar{\delta} = \frac{c}{2m+c} < 1$$

has been verified.
Then

$$\varepsilon = \delta_{j-1}^{\nu} \leq \bar{\delta}^{\nu} \leq \frac{c}{2m+c}.$$

We note that δ_j is an increase function for ε and ρ respectively on [0,1], thus,

$$\delta_j \leq \max_{0\leq\rho\leq 1} \rho^{2m}\{\frac{c}{2m+c} + (1-\frac{c}{2m+c})\min(1, c(1-\rho)\} \leq \frac{c}{2m+c}.$$

5. Nonsymmetric and Indefinite Problems with Singularities

To make further research, we study nonsymmetric and indefinite problems with singularities. Remember that the notations cited before will be still available in this section except another definition.

We take the following boundary problem as our prototype.

$$\begin{aligned} -a\Delta u + b\nabla u + cu &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma. \end{aligned} \tag{43}$$

We assume that $a \in L^{\infty}(\Omega)$, $a \geq c_0 < 0$, $b \in C^1(\Omega)$. The weak formulation of (43) is

$$A(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{44}$$

$$A(u,v) = \int_{\Omega} (a\nabla u\nabla v + b\nabla uv + cuv)dx. \tag{45}$$

We note that $A(u,v)$ could be nonsymmetric and indefinite bilinear form because of the presence of ∇u in (43).

Multigrid methods for nonsymmetric and indefinite problems was researched by many authors[2,6,9], but all these proofs were based on the quasi-uniform grids. Therefore, we can not apply directly the known results on the strong non-quasi-uniform grid considered in this paper. But this problem can be solved with the same results as A1 and A2. For simplicity, we still denote them as A1, A2, except replacing (1) by (43). Let $A(u,v) = A_s(u,v) + B(u,v)$ where $A_s(u,v) = \int_{\Omega}(a\nabla u\nabla v)dx$ and $B(u,v) = \int_{\Omega}(b\nabla uv + cuv)dx$. without loss of generality we assume

$$\sup_{<u,u>=1} A_s(u,u) = 1, \tag{46}$$

where $< .,. >$ is an inner product, satisfing

$$c^{-1} \| u \|_0^2 \leq h_j^2 < u,u > \leq \| u \|_0^2, \quad \forall u \in S_j. \tag{47}$$

Define a series of norms

$$\| u \|_k^* = \sqrt{A_s^k(u,u)}. \tag{48}$$

Obviously,

$$\| u \|_0^* = \sqrt{< u, u >}, \tag{49}$$

$$\| u \|_1^* \sim \| u \| \, . \tag{50}$$

Define a subspace X of S_j,

$$X = \{u \in S_j, \mid A(u,v) = 0, \forall v \in S_{j-1}\}, \tag{51}$$

Mandel[9] provided three convergence criterion of multigrid methods for nonsymmetric and indefinite problems. In particular, when $k = 1$ we have:

$$\begin{array}{lll} \mathbf{C1}: & \forall u \in S_j, \exists v \in S_{j-1}, \text{st} & \| u - v \|_1^* \le \sqrt{\delta} \, \| u \|_2^* \; . \\ \mathbf{C2}: & \forall u \in X, \forall v \in S_j, & | B(u,v) | \le \tau \, \| u \|_1^* \| v \|_1^* \; . \\ \mathbf{C3}: & \forall u \in S_j, \forall v \in S_j, & | B(u,v) | \le \beta \, \| u \|_1^* \| v \|_0^* \; . \end{array} \tag{52}$$

We note that the criterion were arranged for variational problems, which do not need any background of finite element equations. At this point, what we have to do next is only to verify the criterion for this problems with singularities, and therefore, we obtain the same results as in [9]. We now do the verification.

From (47), (48) and (49), we obtain

$$c^{-1} \| u \|_k \le h_j^{1-k} \| u \|_k^* \le C \; \| u \|_k, \quad k = 0, 1, \tag{53}$$

hence, C3 holds with $\beta = Ch_j = C2^{-j}$.

Theorem 5.1. *On the triangulations described in Section 1,* C2 *holds with*

$$\tau = Ch_j = C2^{-j}.$$

Proof. Consider the adjoint problem of (44)

$$A(w,v) = (w,f), \quad \forall w \in H_0^1(\Omega), \tag{54}$$

then choose $w = u \in X$. we have

$$A(u,v) = (u,f), \quad u \in X. \tag{55}$$

By A1 and A2, there is a function $\xi \in S_{j-1}$, such that

$$\| v - \xi \|_1 \le Ch_j \| v \|_{2,r} \le Ch_j \| f \|_{0,r}, \tag{56}$$

and by A1, we know that

$$\begin{aligned} (u,f) &= A(u,v) = A(u, v-\xi) \\ &\le C \| u \|_1 \| v - \xi \|_1 \\ &\le Ch_j \| u \|_1 \| f \|_{0,r} \; . \end{aligned} \tag{57}$$

From the integration by parts and (57) we obtain that C2 holds with $\tau = Ch_j$. It remains to verify C1. The proof of C1 is almost the same as the verification of A1 in the section 4 of [9] except using Theorem 5.1. We describe as follows for completeness. Let $u \in S_j$, define $v \in S_{j-1}$, such that $A(u-v,z)=0, \forall z \in S_{j-1}$. Then by the Holder inequality and Theorem 4.1 we obtain

$$|| u-v ||_1^{*2} = A_s(u-v,u) + B(u-v,v) \leq || u-v ||_0^* || u ||_2^* + Ch_j || u-v ||_1^* . \quad (58)$$

It is easy to verify that $|| u-v ||_0^* \leq || u-v ||_1^*$. Hence, C1 holds with bounded δ for sufficiently large j .

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