Journal of Computational Mathematics, Vol.13, No.4, 1995, 306–314.

HERMITE-TYPE METHOD FOR VOLTERRA INTEGRAL EQUATION WITH CERTAIN WEAKLY SINGULAR KERNEL*1)

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Abstract

We discuss the Hermite-type collocation method for the solution of Volterra integral equation with weakly singular kernel. The constructed approximation is a cubic spline in the continuity class C^1. We prove that this method is convergent with order of four.

1. Introduction

This paper considers the numerical solution of the second-kind Volterra integral equation

$$y(t) + (Ky)(t) = g(t), \tag{1.1}$$

where $y(t)$ is the unknown solution, $g(t)$ is a given function and K is the integral operator for some given kernel function K,

$$(Ky)(t) = \int_0^t K(\frac{t}{s})y(s)\frac{1}{s}ds. \tag{1.2}$$

Such equations arise from certain diffusion problems. Because K is not compact, so the standard stability proofs for numerical methods do not fit.

Many people have worked on Hermite-type collocation methods for second-kind Volterra integral equations with smooth kernels[3,4,5,6], but very few deal with weakly singular kernels. Papatheodorou & Jesanis (1980) considered Volterra integro-differential equations with weakly singular kernels. Diogo, Mckee & Tang (1991) investigated a Hermite-type collocation method for (1.1) with a singular kernel of the form $K(\sigma) = \frac{1}{\sqrt{\pi}\sqrt{\ell_n\sigma}\sigma^\mu}$, $\mu > 1$. They also considered two low-order product integration methods for the solution of (1.1) with a singular kernel of the form $K(\sigma) = \frac{1}{\sqrt{\pi}\sqrt{\ell_n\sigma}\sigma^\mu}$[10]. For general kernel $K(\sigma)$, no papers have appeared to discuss it.

* Received March 26, 1993.
1) The Project was Supported by National Natural Science Foundation of China.

In this paper, first we would to show that a unique smooth solution exists when $\alpha = \int_1^\infty \frac{|K(\sigma)|}{\sigma} d\sigma < 1$. The basic idea is to derive two (linear) Volterra equations for $y(t)$ and $y'(t)$ by transforming the original integral equation. Having the coupled equations for both $y(t)$ and $y'(t)$, we can then employ piecewise cubic Hermite polynomials to obtain numerical solution of (1.1). Finally, the convergence analysis is given.

2. Preliminaries

Let $C^m[0.T]$ denote the Banach space of mth order derivative continuous real-valued functions with the uniform norm

$$|| \, u \, ||_{m,\infty} = \max_{0\leq j\leq m} \max_{0\leq t\leq T} |u^{(j)}(t)|.$$

Our assumption on K is

$$\alpha = \int_1^\infty \frac{|\, K(\sigma) \,|}{\sigma} d\sigma < 1. \tag{2.1}$$

Lemma 1. *If $g \in C^m[0,T]$ and (2.1) is satisfied, then (1.1) possesses a unique solution $y \in C^m[0,T]$.*

Proof: Choosing an arbitrary function $v(t) \in C^m[0,T]$, and defining $u = S(v)$ such that

$$u(t) + \int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds = g(t), \quad t \in [0,T] \tag{2.2}$$

where $S(v) = -\int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds + g(t)$.
Setting $s = \lambda t$ we have

$$\int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds = \int_0^1 K(\frac{1}{\lambda}) v(\lambda t) \frac{1}{\lambda} d\lambda \; . \tag{2.3}$$

Since $v \in C^m[0,T]$ and $g \in C^m[0,T]$, we obtain from (2.2) and (2.3) that

$$u^{(j)}(t) = -\int_0^1 K(\frac{1}{\lambda}) v^{(j)}(\lambda t) \lambda^{j-1} d\lambda + g^{(j)}(t), \tag{2.4}$$

where $0 \leq j \leq m$. If $u_1 = S(v_1)$ and $u_2 = S(v_2)$, we have

$$\begin{aligned} |u_1^{(j)} - u_2^{(j)}| \quad &\leq \textstyle\int_0^1 |K(\frac{1}{\lambda})| \lambda^{j-1} |v_1^{(j)}(\lambda t) - v_2^{(j)}(\lambda t)| d\lambda \\ &\leq \textstyle\int_0^1 |K(\frac{1}{\lambda})| \lambda^{-1} d\lambda \cdot \, || \, v_1 - v_2 \, ||_{m,\infty} \; . \end{aligned} \tag{2.5}$$

Noting that the coefficient of the last term of (2.5) equals α, it follows that

$$|| \, u_1 - u_2 \, ||_{m,\infty} \leq \alpha \, || \, v_1 - v_2 \, ||_{m,\infty} \; . \tag{2.6}$$

The inequality (2.6) implies that the operator S is a contraction mapping. Since C^m is a complete normed space, S has a unique fixed point $y(t) \in C^m[0,T]$ such that $y = S(y)$. This completes the proof.

Lemma 2. *Equation* (1.1) *can be transformed into the equivalent equation*

$$y(t) - \int_0^t \rho(\frac{s}{t}) y(s) \frac{1}{s} ds = g_1(t), \tag{2.7a}$$

where

$$\rho(\sigma) = \int_\sigma^1 K(\frac{1}{\eta}) K(\frac{\eta}{\sigma}) \frac{1}{\eta} d\eta, \tag{2.7b}$$

$$g_1(t) = -\int_0^t K(\frac{t}{s}) g(s) \frac{1}{s} ds + g(t) = g(t) - K(g)(t). \tag{2.7c}$$

Proof: Consider

$$y(s) + \int_0^s K(\frac{s}{\lambda}) y(\lambda) \frac{1}{\lambda} d\lambda = g(s). \tag{2.8}$$

Multiplying both sides of (2.8) by $K(t/s)1/s$ and integrating the resulting equation from 0 to t, we obtain

$$\int_0^t K(\frac{t}{s}) y(s) \frac{1}{s} ds + \int_0^t \frac{1}{s} K(\frac{t}{s}) \int_0^s K(\frac{s}{\lambda}) y(\lambda) \frac{1}{\lambda} d\lambda ds = \int_0^t K(\frac{t}{s}) g(s) \frac{1}{s} ds. \tag{2.9}$$

Using Dirichlet's formula we have

$$\int_0^t K(\frac{t}{s}) y(s) \frac{ds}{s} + \int_0^t (\int_\lambda^t K(\frac{t}{s}) K(\frac{s}{\lambda}) \frac{ds}{s}) y(\lambda) \frac{d\lambda}{\lambda} = \int_0^t K(\frac{t}{s}) g(s) \frac{1}{s} ds\ . \tag{2.10}$$

From (1.1) we obtain

$$y(t) - \int_0^t \int_s^t K(\frac{t}{\lambda}) K(\frac{\lambda}{s}) \frac{d\lambda}{\lambda} y(s) \frac{ds}{s} = g(t) - \int_0^t K(\frac{t}{s}) g(s) \frac{ds}{s}\ . \tag{2.11}$$

Setting

$$p_1(t,s) = \int_s^t K(\frac{t}{\lambda}) K(\frac{\lambda}{s}) \frac{1}{\lambda} d\lambda, \tag{2.12}$$

the equation (2.11) becomes

$$y(t) - \int_0^t p_1(t,s) y(s) \frac{1}{s} ds = g(t) - \int_0^t K(\frac{t}{s}) g(s) \frac{1}{s} ds. \tag{2.13}$$

We now calculate $p_1(t,s)$. Setting $s = \sigma t$, we have

$$p_1(t,s) = \int_{\sigma t}^t K(\frac{t}{\lambda}) K(\frac{\lambda}{\sigma t}) \frac{1}{\lambda} d\lambda,$$

setting $\lambda = t\eta$ we obtain

$$\begin{aligned} p_1(t,s) &= \int_\sigma^1 \frac{1}{t\eta} K(\frac{1}{\eta}) K(\frac{\eta}{\sigma}) t d\eta \\ &= \int_\sigma^1 K(\frac{1}{\eta}) K(\frac{\eta}{\sigma}) \frac{1}{\eta} d\eta = \rho(\sigma) = \rho(\frac{s}{t}), \end{aligned} \tag{2.14}$$

so we have

$$y(t)-\int_0^t \rho(\frac{s}{t})y(s)\frac{1}{s}ds=g_1(t).$$

This completes the proof .

Lemma 3. *The derivative of y, the solution of* (1.1)*, satisfies*

$$y'(t)-\int_0^t \rho(\frac{s}{t})\frac{s}{t}y'(s)\frac{1}{s}ds=g_2(t), \tag{2.15a}$$

where

$$g_2(t)=-\int_0^t K(\frac{t}{s})\frac{s}{t}g'(s)\frac{1}{s}ds+g'(t), \tag{2.15b}$$

$\rho(\sigma)$ is given in Lemma 2.

Proof: By a change of variables, (2.7a) can be written in the form

$$y(t)-\int_0^1 \rho(\lambda)y(\lambda t)\frac{d\lambda}{\lambda}=-\int_0^1 K(\frac{1}{\lambda})g(\lambda t)\frac{d\lambda}{\lambda}+g(t). \tag{2.16}$$

Differentiating both sides of (2.16) with respect to t yields (2.15).

3. Hermite-Type Collocation Method

Let Δ_N denote an equidistant partition of [0,T]

$$\Delta_N: \quad 0=t_0<t_1<\ldots<t_N=T,$$

and let $h=t_{i+1}-t_i=T/N$. The subintervals generated by this partition of Δ_N are denoted by I_n; i.e., $I_0=[t_0,t_1], I_n=(t_n,t_{n+1}](n=1,2,\ldots,N-1)$. In the following we shall be concerned with the approximating spaces

$$S_3^1(\Delta_N)=\{u: u\in C^1[0,T], u\mid_{I_n}\in\pi_3, n=0,1,\ldots,N-1\}.$$

For $n=0,1,\ldots,N-1$, let

$$\phi_{1n}(t)=(t-t_{n+1})^2[h+2(t-t_n)]/h^3, \tag{3.1a}$$

$$\phi_{2n}(t)=(t-t_n)^2[h+2(t_{n+1}-t)]/h^3, \tag{3.1b}$$

$$\psi_{1n}(t)=(t-t_n)(t-t_{n+1})^2/h^2, \tag{3.1c}$$

$$\psi_{2n}(t)=(t-t_n)^2(t-t_{n+1})/h^2, \tag{3.1d}$$

then for any $u(t)\in S_3^1(\Delta_N)$, $u(t)$ can be expressed as

$$u(t)=u_i\phi_{1i}(t)+u_{i+1}\phi_{2i}(t)+u_i'\psi_{1i}(t)+u_{i+1}'\psi_{2i}(t), \tag{3.2}$$

$$t\in[t_i,t_{i+1}],\ i=0,1,,\ldots,N-1.$$

Definition. *Let $f \in C^m[0,T]$, $m \geq 1$, be a given function. Then $H_f(t) \in S_3^1(\Delta_N)$ is the Hermite cubic interpolant of f if*

$$H_f(t_n) = f(t_n), \quad H_f'(t_n) = f'(t_n), \quad 0 \leq n \leq N.$$

Lemma 4. (Schultz, 1973) *Assume $f \in C^4[0,T]$. Then*

$$\| f - H_f \|_\infty \leq \frac{1}{384} h^4 \| f^{(4)} \|_\infty, \tag{3.3a}$$

$$\| f' - H_f' \|_\infty \leq \frac{\sqrt{3}}{216} h^3 \| f^{(4)} \|_\infty \quad . \tag{3.3b}$$

The Hermite-type collocation method of (1.1) is looking for $u \in S_3^1(\Delta_N)$ satisfying

$$u(t_n) - \int_0^{t_n} \rho(\frac{s}{t_n}) u(s) \frac{1}{s} ds = \tilde{g}_1(t_n), \tag{3.4a}$$

$$u'(t_n) - \int_0^{t_n} \rho(\frac{s}{t_n}) \frac{s}{t_n} u(s) \frac{1}{s} ds = \tilde{g}_2(t_n), \quad 1 \leq n \leq N, \tag{3.4b}$$

with

$$u(0) = y(0) = g(0)/(1+\tilde{\alpha}), \tag{3.4c}$$

$$u'(0) = y'(0) = g'(0)/(1+\tilde{\beta}), \tag{3.4d}$$

where

$$\tilde{\alpha} = \int_1^\infty \frac{K(\sigma)}{\sigma} d\sigma, \tag{3.4e}$$

$$\tilde{\beta} = \int_1^\infty \frac{K(\sigma)}{\sigma^2} d\sigma, \tag{3.4f}$$

$$\tilde{g}_1(t_n) = -\int_0^{t_n} K(\frac{t_n}{s}) \tilde{g}(s) \frac{1}{s} ds + g(t_n), \tag{3.4g}$$

$$\tilde{g}_2(t_n) = -\int_0^{t_n} K(\frac{t_n}{s}) \frac{s}{t_n} \tilde{g}'(s) \frac{1}{s} ds + g'(t_n), \tag{3.4h}$$

Here the function $\tilde{g} \in S_s^1(\Delta_N)$ is the Hermite cubic interpolant to g. The approximations $\tilde{g}_1(t_n)$ and $\tilde{g}_2(t_n)$ will be replaced by the exact values $g_1(t_n)$ and $g_2(t_n)$ if the integrals involved can be calculated analytically.

4. Convergence

Lemma 5. *For $t \in [t_n, t_{n+1}]$, $0 \leq n \leq N-1$, we have*

$$| \phi_{1n}(t) | + | \phi_{2n}(t) | = 1, \quad | \psi_{1n}(t) | + | \psi_{2n}(t) | \leq \frac{h}{4}, \tag{4.1a}$$

$$| \phi_{1n}'(t) | + | \phi_{2n}'(t) | \leq \frac{3}{h}, \quad | \psi_{1n}'(t) | + | \psi_{2n}'(t) | \leq 1. \tag{4.1b}$$

Define the error function

$$e(t) = y(t) - u(t), \quad t \in [0, T].$$

Set $t = t_n$ in (2.7a) and subtract (3.4a) from the resulting equation . It follows that $e(t)$ satisfies the error equations

$$e(t_n) - \int_0^{t_n} \rho(\frac{s}{t_n}) \cdot e(s)\frac{1}{s}ds = g_1(t_n) - \tilde{g}_1(t_n). \tag{4.2a}$$

Similarly, we can obtain

$$e'(t_n) - \int_0^{t_n} \rho(\frac{s}{t_n})\frac{s}{t_n}e'(s)\frac{1}{s}ds = g_2(t_n) - \tilde{g}_2(t_n), \tag{4.2b}$$

where $n = 0, 1, \ldots, N$.

Lemma 6. *If $g \in C^4[0, T]$, then*

(i) $$| g_1(t_n) - \tilde{g}_1(t_n) |\leq \frac{\alpha}{384}h^4 \| g^{(4)} \|_\infty, \tag{4.3a}$$

(ii) $$| g_2(t_n) - \tilde{g}_2(t_n) |\leq \frac{\sqrt{3}\beta}{216}h^3 \| g^{(4)} \|_\infty, \tag{4.3b}$$

where

$$\alpha = \int_1^\infty \frac{| K(\sigma) |}{\sigma}d\sigma \quad , \quad \beta = \int_1^\infty \frac{| K(\sigma) |}{\sigma^2}d\sigma \ .$$

Proof: (i)

$$\begin{aligned} | g_1(t_n) - \tilde{g}_1(t_n) | &= | \int_0^{t_n} K(\frac{t_n}{s})(\tilde{g}(s) - g(s))\frac{1}{s}ds | \\ &\leq \int_0^{t_n} | K(\frac{t_n}{s})\frac{1}{s}ds \cdot \max_{0\leq s\leq T} | g(s) - \tilde{g}(s) | \\ &\leq \alpha \max_{0\leq s\leq T} | g(s) - \tilde{g}(s) | \ . \end{aligned}$$

Applying (3.3a), we obtain (4.3a).

(ii) Similarly, we have

$$| g_2(t_n) - \tilde{g}_2(t_n) |\leq \beta \max_{0\leq s\leq T} | g'(s) - \tilde{g}'(s) | \ .$$

Applying (3.3b), we can obtain (4.3b).

Theorem 1. *If $g \in C^4[0, T]$, $\alpha < \sqrt{3} - 1$, then the error function $e(t) = y(t) - u(t)$ satisfies*

$$\| e^{(i)} \|_\infty = \max_{0\leq t\leq T} | e^{(i)}(t) | = 0(h^{4-i}), \quad i = 0, 1.$$

Proof: Let $E_1 = \max_{1\leq n\leq N} \mid e(t_n) \mid$, and $E_2 = \max_{1\leq n\leq N} \mid he'(t_n) \mid$.
For $t \in [t_n, t_{n+1}]$, we have

$$u(t) = \phi_{1n}(t)u_n + \phi_{2n}(t)u_{n+1} + \psi_{1n}(t)u'_n + \psi_{2n}(t)u'_{n+1}. \tag{4.4}$$

As $g \in C^4[0,T]$, by Lemma 1, $y(t) \in C^4[0,T]$. It follows from Lemma 4 that

$$y(t) = \phi_{1n}(t)y(t_n) + \phi_{2n}(t)y(t_{n+1}) + \psi_{1n}(t)y'(t_n) + \psi_{2n}(t)y'(t_{n+1}) + 0(h^4). \tag{4.5}$$

Subtracting (4.4) from (4.5) and using (4.1a), we obtain

$$\mid e(t) \mid\leq E_1 + \frac{1}{4}E_2 + 0(h^4). \tag{4.6}$$

From equation (4.2a) we have

$$\begin{aligned} \mid e(t_n) \mid &\leq \int_0^{t_n} \mid \rho(\frac{s}{t_n}) \mid \cdot \mid e(s) \mid \frac{1}{s}ds + 0(h^4) \\ &\leq (E_1 + \frac{1}{4}E_2 + 0(h^4)) \int_0^{t_n} \mid \rho(\frac{s}{t_n}) \mid \frac{1}{s}ds + 0(h^4). \end{aligned} \tag{4.7}$$

Noting that

$$\begin{aligned} \int_0^{t_n} \mid \rho(\frac{s}{t_n}) \mid \frac{1}{s}ds &= \int_0^1 \mid \rho(z) \mid \frac{1}{z}dz = \int_0^1 \frac{1}{z} \mid \int_z^1 K(\frac{1}{\eta})K(\frac{\eta}{z})\frac{1}{\eta}d\eta \mid dz \\ &= \int_0^1 \frac{1}{z} \int_z^1 \mid K(\frac{1}{\eta})K(\frac{\eta}{z})\frac{1}{\eta} \mid d\eta dz \\ &= \int_0^1 \frac{1}{\eta} \mid K(\frac{1}{\eta}) \mid \int_0^\eta \frac{1}{z} \mid K(\frac{\eta}{z}) \mid dz d\eta \\ &= (\int_0^1 \frac{1}{\eta} \mid K(\frac{1}{\eta}) \mid d\eta)^2 = \alpha^2. \end{aligned}$$

We have

$$E_1 \leq \alpha^2(E_1 + \frac{1}{4}E_2) + 0(h^4). \tag{4.8}$$

Thus

$$E_1 \leq \frac{\alpha^2}{4(1-\alpha^2)}E_2 + 0(h^4). \tag{4.9}$$

On the other hand

$$\mid e'(t) \mid\leq (\mid \phi'_{1n}(t) \mid + \mid \phi'_{2n}(t) \mid)E_1 + \frac{1}{h}(\mid \psi'_{1n}(t) \mid + \mid {\psi'}_{2n}(t) \mid)E_2 + 0(h^3). \tag{4.10}$$

Using (4.1b) we obtain

$$h \mid e'(t) \mid\leq 3E_1 + E_2 + 0(h^4). \tag{4.11}$$

Combining this with (4.2b) we have

$$\begin{aligned} | he'(t_n) | &\le \int_0^{t_n} | \rho(\frac{s}{t_n}) | \frac{s}{t_n} | he'(s) | \frac{1}{s} ds + 0(h^4). \\ &\le (3E_1 + E_2 + 0(h^4)) \int_0^{t_n} | \rho(\frac{s}{t_n}) | \frac{1}{t_n} ds + 0(h^4). \end{aligned}$$

Since

$$\begin{aligned} \int_0^{t_n} | \rho(\frac{s}{t_n}) | \frac{1}{t_n} ds &= \int_0^1 | \rho(z) | dz = \int_0^1 | \int_z^1 K(\frac{1}{\eta}) K(\frac{\eta}{z}) \frac{1}{\eta} d\eta | dz \\ &\le \int_0^1 \int_z^1 | K(\frac{1}{\eta}) K(\frac{\eta}{z}) \frac{1}{\eta} | d\eta dz \\ &= \int_0^1 \frac{1}{\eta} | K(\frac{1}{\eta}) | \int_0^1 | K(\frac{1}{\lambda}) | \eta d\lambda d\eta \\ &= (\int_0^1 | K(\frac{1}{\lambda}) | d\lambda)^2 = (\int_1^\infty \frac{| K(\sigma) |}{\sigma^2} d\sigma)^2 = \beta^2. \end{aligned}$$

We have

$$E_2 \le \beta^2 (3E_1 + E_2) + 0(h^4). \tag{4.12}$$

This implies

$$E_2 \le \frac{3\beta^2}{1-\beta^2} E_1 + 0(h^4). \tag{4.13}$$

Substitute this inequality into (4.9) we obtain

$$E_1 \le \frac{3\alpha^2\beta^2}{4(1-\alpha^2)(1-\beta^2)} E_1 + 0(h^4). \tag{4.14}$$

Using the fact of $\beta^2 < \alpha^2$, we have

$$E_1 \le \frac{3\alpha^4}{4(1-\alpha^2)^2} E_1 + 0(h^4). \tag{4.15}$$

So

$$E_1 \le \frac{4(1-\alpha^2)^2}{\alpha^4 - 8\alpha^2 + 4} 0(h^4), \tag{4.16}$$

provided $\alpha^4 - 8\alpha^2 + 4 > 0$. Solving this inequality we obtain $\alpha < \sqrt{3} - 1$. Combining (4.16), (4.13), (4.6) and (4.11) the Theorem is thus proved.

Appendix. In the following, we show that the singular kernel of the form $K(\sigma) = \frac{1}{\sqrt{\pi}\sqrt{l_n \sigma}\sigma^\mu}$ $(\mu > 1)$, which had been considered in [2], satisfies

$$\int_1^\infty \frac{| K(\sigma) |}{\sigma} d\sigma = \frac{1}{\sqrt{\mu}} < 1.$$

From the definition of $K(\sigma)$ we have

$$\int_1^\infty \frac{|\,K(\sigma)\,|}{\sigma} d\sigma = \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{1}{\sqrt{l_n \sigma}\, \sigma^{\mu+1}} d\sigma. \qquad (*)$$

Let $\sqrt{l_n\sigma} = t$, then $\sigma = e^{t^2}$, $d\sigma = 2te^{t^2}dt$.

Substituting these expressions into (*) we have

$$\int_1^\infty \frac{|\,K(\sigma)\,|}{\sigma} d\sigma = \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{2te^{t^2}dt}{te^{t^2(\mu+1)}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\mu t^2} dt = \frac{1}{\sqrt{\mu}} < 1.$$

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