Journal of Computational Mathematics, Vol.13, No.4, 1995, 291–305.

A MULTI-GRID ALGORITHM FOR STOKES PROBLEM[*1)]

Z. Huang
(*Department of Applied Mathematics, Tongji University, Shanghai, China*)

Abstract

In this paper we describe a multi-grid algorithm for the penalty procedure of Stokes problem. It is proved that the convergence rate of the algorithm is bounded away from 1 independently of the meshsize. For convenience, we only discuss Jacobi relaxation as smoothing operator in detail.

1. Introduction

Consider the Stokes problem

$$\begin{cases} -\mu \triangle \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^d, $d = 2$ or 3. Since, within a code for the numerical solution of the Navier-Stokes equations, one needs an efficient Stokes-solver, the multigrid method is very attractive for the solution of the discrete analogue of (1.1).

Brezzi and Douglas[6] have applied a penalty procedure for (1.1) with the C^0-piecewise linear element of velocity and pressure and achieved an optimal convergence rate. In this paper we establish a multi-grid algorithm for the penalty procedure of Stokes problem and show that the convergence rate of the algorithm is bounded away from 1 independently of the meshsize.

The general structure of our convergence analysis for the multi-grid algorithm is similar to that of Bank and Dupont[2,3] and Hackbusch[8]. The smoothing properties are given in terms of a mesh-dependent norm. The approximation properties are obtained from error estimates in terms of Sobolev spaces. The connection between the associated scales of Sobolev spaces, however, requires some special considerations. It is performed via the duality technique of Aubin-Nitsche. To simplify the analysis we only consider Jacobi relaxation as smoothing procedure in detail.

* Received November 24, 1992.
1) This Work was Supported in Part by China State Major Key Project for Basic Researches and in Part by Shanghai Natural Science Foundation.

2. A Multi-Grid Algorithm

A mixed formulation of (1.1) is given by the finding of $[\mathbf{u}, p] \in \mathbf{H}_0^1(\Omega) \times \hat{L}^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{u},q) \qquad\qquad = 0 & \forall q \in \hat{L}^2(\Omega) \end{cases} \tag{2.1}$$

with the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \mu \sum_{i=1}^{k} (\bigtriangledown\, u_i, \bigtriangledown\, v_i) = \mu \sum_{i,j=1}^{k} (\frac{\partial\, u_i}{\partial\, x_j}, \frac{\partial\, v_i}{\partial\, x_j}),$$
$$b(\mathbf{v},q) = -(\mathrm{div}\ \mathbf{v}, q)$$

on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$, $\mathbf{H}_0^1(\Omega) \times \hat{L}^1(\Omega)$. Here, $(\cdot,\cdot)$ is the inner product in L^2. Moreover, $H^k(\Omega)$, $k \in N$, and $L^2(\Omega) = H^0(\Omega)$ are the usual Sobolev and Lebesgue spaces equipped with the norms[1]

$$\|u\|_k = \{\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^2\ dx\}^{\frac{1}{2}}.$$

Furthermore, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^2$. We use a circumflex " $\hat{.}$ " above a function space to denote the subspace of the elements with mean value zero.

Let T_0 be a partition of Ω into d-simplices and h_0 be the longest side of the simplices of T_0. We suppose that the simplices of T_0 satisfy the usual regularity assumptions for finite elements[7] and that

$$\begin{array}{lll} h_K & \le & c_0 \rho_K, \qquad\qquad \forall K \in T_0, \\ h_K & := & \mathrm{diam}(K), \\ \rho_K & := & \sup\,\{\mathrm{diam}(B) \mid B \text{ is a ball contained in } K\}, \end{array} \tag{2.2}$$

where c_0 is not large. The partitions T_k, $1 \le k \le R$, are defined by dividing each $K \in T_{k-1}$ into 2^d d-simplices by joining the midpoints of the sides (cf. Fig. 1). Then $h_k = 2^{-k} h_0$, and the partitions T_k satisfy the regularity assumption (2.2) with the same constant c_0.

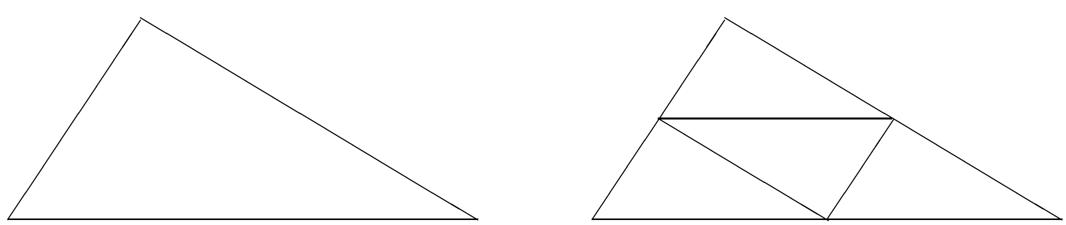

Fig. 1. Subdivision of triangles in the construction of T_k from T_{k-1}

Let S_k be the spaces of continuous, piecewise linear finite elements corresponding to T_k. Following Brezzi and Douglas[6] we define the finite element spaces:

$$\begin{aligned}\mathbf{X}_k &:= \mathbf{S}_k \cap \mathbf{H}_0^1(\Omega),\\ M_k &:= S_k \cap \hat{H}^1(\Omega),\\ D_k &:= \mathbf{X}_k \times M_k.\end{aligned}$$

Brezzi and Douglas[6] considered the modification of (2.1) given by the finding of $[\mathbf{u}^k, p^k] \in D$ such that

$$L_k([\mathbf{u}^k, p^k]; [\mathbf{v}, q]) = (\mathbf{f}, \mathbf{v}), \qquad \forall [\mathbf{v}, q] \in D, \tag{2.3}$$

where the space $D = \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ and the bilinear form

$$L_k([\mathbf{u}^k, p^k]; [\mathbf{v}, q]) := a(\mathbf{u}^k, \mathbf{v}) + b(\mathbf{v}, p^k) + b(\mathbf{u}^k, q) - h_k^2(\bigtriangledown p^k, \bigtriangledown q).$$

The problem (2.3) is a penalized version of the Stokes problem. Let $[\mathbf{u}_k, p_k] \in D_k$ be the solution of the equations:

$$L_k([\mathbf{u}_k, p_k]; [\mathbf{v}, q]) = (\mathbf{f}, \mathbf{v}), \qquad \forall [\mathbf{v}, q] \in D_k. \tag{2.4}$$

Following Brezzi and Douglas[6] we have the L_2-estimate

$$\|\mathbf{u} - \mathbf{u}_k\|_0 + h_k\|p - p_k\|_0 \le c h_k^2 \|\mathbf{f}\|_0,$$

where $[\mathbf{u}, p]$ and $[\mathbf{u}_k, p_k]$ are the solutions of problem (2.1) and (2.4), respectively.

In order to apply a multigrid procedure to the problem (2.4), we have to consider the slightly more general problem :

Problem ($\mathbf{A}_k$). Find $[\mathbf{u}_k, p_k] \in D_k$ such that

$$L_k([\mathbf{u}_k, p_k]; [\mathbf{v}, q]) = G_k([\mathbf{v}, q]), \qquad \forall [\mathbf{v}, q] \in D_k.$$

Here, the linear forms $G_k : D_k \longrightarrow \mathrm{R}$ are defined recursively with

$$G_R([\mathbf{v}, q]) := (\mathbf{f}, \mathbf{v}) \qquad \forall [\mathbf{v}, q] \in D_R.$$

Note that actually we want to solve Problem (A_k) at level $k = R$ and that the other levels are only auxiliary ones.

Let $\{\gamma_k^j\}_{j=1}^{N_k}$ and $\{\sigma_k^j\}_{j=1}^{N'_k}$ be the orthogonal basis of $\mathbf{X}_k$ and M_k with respect to the $(\cdot,\cdot)_{l_2}$-product. Recalling $D_k = \mathbf{X}_k \times M_k$ we know that

$$\{\gamma_k^j, 0\}_{j=1}^{N_k} \cup \{0, \sigma_k^j\}_{j=1}^{N'_k}$$

is a basis in D_k. Corresponding to this basis, Problem (A_k) may written in matrix-vector notation as $U_k z_k = d_k$ with the indefinite matrix

$$U_k = \begin{bmatrix} A_k & B'_k \\ B_k & -h_k^2 C_k \end{bmatrix}.$$

We will perform the multi-grid algorithm under the $(\cdot,\cdot)_{l_2}$-product. In smoothing step, m steps of a Jacobi relaxation will be applied to the squared system after a renormalization. The relaxation factor ω_k in the smoothing steps below has to be greater than or equal to the spectral radius of U_k, which is proportional to h_k^{-2}.

Algorithm 2.1. (one iteration at level k, $1 \le k \le R$, with m smoothing steps)

1. Smoothing : *Let* $[\mathbf{u}_k^0, p_k^0] \in D_k$ *be a given approximation of the solution of Problem* (A_k). *For* $l = 1, 2, \cdots, m$, *compute first* $[\xi_k^l, \eta_k^l]$ *and then* $[\mathbf{u}_k^l, p_k^l]$ *from* $[\mathbf{u}_k^{l-1}, p_k^{l-1}]$ *by solving*

$$(\xi_k^l, \mathbf{v}) + h_k^2(\eta_k^l, p) = \omega_k^{-2}\{G_k([\mathbf{v}, q]) - L_k([\mathbf{u}_k^{l-1}, p_k^{l-1}]; [\mathbf{v}, q]), \quad \forall [\mathbf{v}, q] \in D_k, \tag{2.5}$$

$$(\mathbf{u}_k^l - \mathbf{u}_k^{l-1}, \mathbf{v}) + h_k^2(p_k^l - p_k^{l-1}, q) = L_k([\xi_k^l, \eta_k^l]; [\mathbf{v}, q]), \qquad \forall [\mathbf{v}, q] \in D_k. \tag{2.6}$$

2. Coarse-Grid-Correction : *Denote by* $[\mathbf{u}_{k-1}^*, p_{k-1}^*]$ *the solution of Problem* (A_{k-1}) *with the functional :*

$$G_{k-1}([\mathbf{v}, q]) := G_k([\mathbf{v}, q]) - L_k([\mathbf{u}_k^m, p_k^m]; [\mathbf{v}, q]) \qquad \forall [\mathbf{v}, q] \in D_{k-1}.$$

If $k = 1$, *determine* $[\mathbf{u}'_{k-1}, p'_{k-1}] = [\mathbf{u}_{k-1}^*, p_{k-1}^*]$. *If* $k > 1$, *compute an approximation* $[\mathbf{u}'_{k-1}, p'_{k-1}]$ *to* $[\mathbf{u}_{k-1}^*, p_{k-1}^*]$ *by applying* $\mu = 2$ *iterations of the algorithm at level* $k-1$ *to Problem* (A_{k-1}) *with starting value zero. Set*

$$\mathbf{u}_k^{m+1} := \mathbf{u}_k^m + \mathbf{u}'_{k-1}, \qquad p_k^{m+1} := p_k^m + p'_{k-1} \ .$$

For actual computations in the smoothing steps we replace the L_2-product on $\mathbf{X}_k$ and M_k, respectively, by the $(\cdot,\cdot)_{l_2}$-product. Since the norm $\|\cdot\|_{l_2}$ is equivalent to $\|\cdot\|_0$ on $\mathbf{X}_k$ and M_k, our analysis in Section 3 will also hold if the L_2-product are replaced by the product above.

3. Convergence Analysis

In this section we prove the convergence of Algorithm 2.1. In subsection 3.1 we give some properties of the mapping defined in problem (2.3) and of the finite element spaces, which are the bases of our convergence proof. The convergence rate of the algorithm is measured in a mesh-dependent norm. Then we state the convergence result in Theorem 3.1. It is pointed out that we only need to show a smoothing property and an approximation property corresponding to the two-grid procedure of Algorithm 2.1 for the proof of Theorem 3.1. Subsection 3.2 is devoted to the smoothing property. In subsection 3.3 we give some lemmas, which will be used in the proof of the approximation property in subsection 3.4.

3.1. The Convergence of Algorithm 2.1

Our convergence analysis will be based on the following properties. The properties refer to the bilinear forms a, b defined in section 2. The two bilinear forms a, b satisfy the following continuity, coercively and inf-sup condition[8]

$(P1)$ $\quad a(\mathbf{w},\mathbf{v}) \le c_1\|\mathbf{w}\|_1\|\mathbf{v}\|_1, \qquad \forall \mathbf{w},\ \mathbf{v} \in \mathbf{H}_0^1(\Omega);$

$(P2)$ $\quad a(\mathbf{v},\mathbf{v}) \ge \alpha\|\mathbf{v}\|_1^2, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$ with $\alpha \ge 0;$

$(P3)$ $\quad b(\mathbf{v},q) \le c_2\|\mathbf{v}\|_1\|q\|_0, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),\ q \in \hat{H}^1(\Omega);$

$(P4)$ $\quad \inf\limits_{q\in\hat{H}^1(\Omega)} \sup\limits_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \dfrac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|_0} \ge \beta_1,$ with a constant $\beta_1 > 0$.

The finite element spaces $\mathbf{X_k}$ and M_k satisfy the usual approximation properties and inverse estimates[7]:

$$(P5)\quad \left\{\begin{array}{ll} \inf_{\mathbf{v}_k\in\mathbf{X}_k} \|\mathbf{v}-\mathbf{v}_k\|_\alpha \le ch_k^{\beta-\alpha}\|\mathbf{v}\|_\beta, & \forall \mathbf{v}\in\mathbf{H}^\beta(\Omega) \\ \inf_{q_k\in M_k} \|q-q_k\|_\alpha \le ch_k^{\beta-\alpha}\|q\|_\beta & \forall q\in\hat{H}^\beta(\Omega) \end{array}\right., \quad 0\le\alpha\le 1\le\beta\le 2,$$

$$(P6)\qquad \left\{\begin{array}{ll} \|\mathbf{v}_k\|_1 \le ch_k^{-1}\|\mathbf{v}_k\|_0, & \forall \mathbf{v}\in\mathbf{X}_k; \\ \|q_k\|_1 \le ch_k^{-1}\|q_k\|_0, & \forall q\in M_k. \end{array}\right.$$

The properties $(P5)$ will be used in the application of the duality technique and the properties $(P6)$ in the estimate of the largest eigenvalue of the operator L_k on D_k.

Now we define mesh-dependent norms on D_k. Put $\Delta_k := \dim D_k$. Since $L_k([\mathbf{u},p];[\mathbf{v},q])$ is a symmetric bilinear form on $D_k \times D_k$, there is a complete set of eigenfunctions $[\phi_k^j,\psi_k^j] \in D_k$, $1 \le j \le \Delta_k$, such that

$$L_k([\phi_k^j,\psi_k^j];[\mathbf{v},q]) = \lambda_j\{(\phi_k^j,\mathbf{v}) + h_k^2(\psi_k^j,q)\}, \qquad \forall [\mathbf{v},q]\in D_k. \tag{3.1}$$

They are assumed to be normalized by

$$(\phi_k^i,\phi_k^j) + h_k^2(\psi_k^i,\psi_k^j) = \delta_{ij}, \qquad \forall 1\le i,j\le\Delta_k. \tag{3.2}$$

From Brezzi and Douglas [6] the eigenvalues are nonzero, and we can arrange them such that

$$0 < |\lambda_1| \le \cdots \le |\lambda_{\Delta_k}| =: \Lambda_k.$$

Given

$$[\mathbf{v},q] = \sum_{j=1}^{\Delta_k} w_j[\phi_k^j,\psi_k^j] \in D_k,$$

we define the $|||\cdot|||_{s,k}$-norm, $s\in N$, by

$$|||[\mathbf{v},q]|||_{s,k} := \{\sum_{j=1}^{\Delta_k} |\lambda_j|^s w_j^2\}^{\frac{1}{2}}.$$

Note that for $s=0$ we have

$$|||[\mathbf{v},q]|||_{0,k} = \{\|\mathbf{v}\|_0^2 + h_k^2\|q\|_0^2\}^{\frac{1}{2}}. \tag{3.3}$$

Now we state the main theorem of this paper:

Theorem 3.1. *Let Ω be a convex polyhedron domain and T_k be a regular partition of Ω. Suppose that $\Lambda_k \le \omega_k \le ch_k^{-2}$ in Algorithm 2.1. Let $\delta_{m,k}$ be the convergence rate measured in the $|||\cdot|||_{0,k}$-norm of one iteration of Algorithm 2.1 at level k with m smoothing steps. Then, for every $\kappa \in (0, 4^{-\frac{1}{\mu-1}})$, there is a number m_k which only depends on κ such that*

$$\delta_{k,m} \le \kappa \qquad \forall k \in N, \quad m \ge m_k. \tag{3.4}$$

The proof of Theorem 3.1 will be arranged as follows. Let $[\mathbf{u}_k^*, p_k^*] \in D_k$ denote the exact solution of Problem (A_k) and

$$\mathbf{e}_k^l := \mathbf{u}_k^* - \mathbf{u}_k^l, \qquad \epsilon_k^l := p_k^* - p_k^l \qquad \text{for} \quad l = 0, 1, ..., m$$

be the error of the l-th iterate of Algorithm 2.1 with m smoothing steps. Using the mesh-dependent norm $|||\cdot|||_{2,k}$ on D_k, we will establish a smoothing property

$$|||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \le \frac{c}{\sqrt{2m+1}} h_k^{-2} |||[\mathbf{e}_k^0, \epsilon_k^0]|||_{0,k}, \tag{3.5}$$

and an approximation property

$$|||[\mathbf{e}_k^m - \mathbf{u}_{k-1}^*, \epsilon_k^m - p_{k-1}^*]|||_{0,k} \le ch_k^2 |||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \ . \tag{3.6}$$

They immediately yield the upper bound of the two-grid convergence rate

$$\delta_{1,m} \le \frac{c}{\sqrt{2m+1}} \ . \tag{3.7}$$

Then, referring to Hackbusch [8] and the idea of the proof of Theorem 4.1 in Verfürth [10], by the smoothing property (3.5) and the approximation property (3.6), we obtain the two-grid convergence rate (3.7) and the multi-grid convergence rate (3.4).

Therefore, we will focus on the proof of the smoothing property and the approximation property, respectively.

3.2. Smoothing Property

Proof of the smoothing property (3.5). To simplify the notation, we assume that $\omega_k = \Lambda_k$ in Algorithm 2.1, although the analysis still holds under the weaker assumption $\Lambda_k \le \omega_k \le c\Lambda_k$. Using the eigenfunctions and the eigenvalues of (3.1) and (3.2), we establish an upper bound of the eigenvalues

$$\begin{aligned} |\lambda_j| &= |L_k([\phi_k^j, \psi_k^j]; [\phi_k^j, \psi_k^j])| \\ &\le c_1 \|\phi_k^j\|_1^2 + 2c_2 \|\phi_k^j\|_1 \|\psi_k^j\|_0 + c_3 h_k^2 \|\psi_k^j\|_1^2 \ . \end{aligned}$$

From the inverse inequality (P6), Young′s inequality and the normalization (3.2), we obtain

$$\begin{aligned} |\lambda_j| &\le c' \{h_k^{-2} \|\phi_k^j\|_0^2 + (h_k^{-1} \|\phi_k^j\|_0) \|\psi_k^j\|_0 + \|\psi_k^j\|_0^2\} \\ &\le ch_k^{-2} (\|\phi_k^j\|_0^2 + h_k^2 \|\psi_k^j\|_0^2) \\ &= ch_k^{-2} \ . \end{aligned}$$

Hence, $\Lambda_k := \max_j\{|\lambda_j|\} \le ch_k^{-2}$. A standard argument[8] then yields

$$\begin{aligned} |||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} &\le \Lambda_k \max_{-1\le x\le 1} |x(1-x^2)^m| \cdot |||[\mathbf{e}_k^0, \epsilon_k^0]|||_{0,k} \\ &\le ch_k^{-2} \frac{1}{\sqrt{2m+1}} |||[\mathbf{e}_k^0, \epsilon_k^0]|||_{0,k} \ . \end{aligned}$$

This completes the proof of the smoothing property.

3.3. Some Lemmas for the Proof of the Approximation Property

In this subsection we give some lemmas for the proof of the approximation property. Lemma 3.1 shows that the operator $L_k : D \longrightarrow D'$ is an isomorphism and satisfies the Babuška condition. The proof of the approximation property will heavily depend on the L_2-error estimate for the finite element approximation. In Lemma 3.3 we shall obtain this estimate by the duality technique. To this end we have to acquire the corresponding regularity estimate for the more general problem by finding of $[\mathbf{u}^k, p^k] \in D$ such that

$$L_k([\mathbf{u}^k, p^k]; [\mathbf{v}, q]) =< \mathbf{l}, \mathbf{v} > + < g, q >, \tag{3.8}$$

with $\mathbf{l} \in (\mathbf{H}_0^1(\Omega))'$ and $g \in (\hat{H}^1(\Omega))'$. Lemma 3.2 gives this result. Since a penalty procedure is used in Algorithm 2.1, we have to need an L_2-estimate between mixed problem with and without penalty term. This is obtained in Lemma 3.4.

The proofs of Lemma 3.1 and 3.2 can be found in Huang [9].

Lemma 3.1. *Let Ω be a convex polyhedron. Then the operator*

$$L_k : \ [\xi, \eta] \in D \longrightarrow [\mathbf{l}, g] \in D'$$

defined by

$$L_k([\xi, \eta], [\mathbf{v}, q]) =< \mathbf{l}, \mathbf{v} > + < g, q > \ , \qquad \forall [\mathbf{v}, q] \in D$$

is an isomorphism, if D is equipped with the norm

$$\|[\mathbf{v}, q]\|_D := \|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1 \ .$$

Moreover, the Babuška condition

$$\|\xi\|_1 + \|\eta\|_0 + h_k\|\eta\|_1 \le c \sup_{[\mathbf{v},q]\in D} \frac{L_k([\xi, \eta]; [\mathbf{v}, q])}{\|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1} \ .$$

holds.

Lemma 3.2. *(Regularity) Suppose that Ω is a convex polyhedron and that $\mathbf{l} \in \mathbf{L}^2(\Omega)$, $g \in \hat{L}^2(\Omega)$. Then there is a unique pair $[\mathbf{u}^k, p^k] \in \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ solving (3.8). Moreover, $\mathbf{u}^k \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, $p^k \in \hat{H}_0^1(\Omega) \cap H^2(\Omega)$ and*

$$\|\mathbf{u}^k\|_2 + \|p^k\|_1 + h_k\|p^k\|_2 \le c(\|\mathbf{l}\|_0 + h_k^{-1}\|g\|_0) \ .$$

Lemma 3.3. *(L_2-error estimate) Assume that Ω is a convex polyhedron. Given $[\chi,\nu] \in D$, let $[\chi_k,\nu_k] = R_k[\chi,\nu] \in D_k$ be the finite element approximation of problem (3.8) on the space D_k, i.e.,*

$$L_k([\chi-\chi_k,\nu-\nu_k];[\mathbf{v},q]) = 0, \qquad \forall [\mathbf{v},q] \in D_k. \tag{3.9}$$

Then we have the error estimate

$$|||[\chi-\chi_k,\nu-\nu_k]|||_{0,k} \le ch_k\{\|\chi-\chi_k\|_1 + h_k\|\nu-\nu_k\|_1\}, \tag{3.10}$$

where c is a constant independent of h_k.

Proof. First we rewrite (3.9) as

$$\begin{cases} a(\chi-\chi_k,\mathbf{v}) + b(\mathbf{v},\nu-\nu_k) & = 0\ , \\ b(\chi-\chi_k,q) - h_k^2(\nabla(\nu-\nu_k),\nabla q) & = 0\ , \end{cases} \qquad \forall [\mathbf{v},q] \in D_k. \tag{3.11}$$

Now we use the duality technique to prove (3.10). Let $[\xi,\eta] \in D$ be the solution of

$$\begin{cases} a(\xi,\mathbf{v}) + b(\mathbf{v},\eta) & = (\chi-\chi_k,\mathbf{v}) \\ b(\xi,q) - h_k^2(\nabla\eta,\nabla q) & = h_k^2(\nu-\nu_k,q) \end{cases}, \qquad \forall [\mathbf{v},q] \in D_k. \tag{3.12}$$

Choosing $\mathbf{v} = \chi-\chi_k$, $q = \nu-\nu_k$ in (3.12) and subtracting (3.11) from (3.12) we have for all $[\mathbf{v},q] \in D_k$,

$$\begin{aligned} \|\chi-\chi_k\|_0^2 + h_k^2\|\nu-\nu_k\|_0^2 &= a(\xi-\mathbf{v},\chi-\chi_k) + b(\chi-\chi_k,\eta-q) \\ &\quad + b(\xi-\mathbf{v},\nu-\nu_k) - h_k^2(\nabla(\eta-q),\nabla(\nu-\nu_k)) \\ &\le c(\|\xi-\mathbf{v}\|_1 + \|\eta-q\|_0 + h_k\|\eta-q\|_1) \\ &\quad \cdot(\|\chi-\chi_k\|_1 + \|\nu-\nu_k\|_0 + h_k\|\nu-\nu_k\|_1)\ . \end{aligned}$$

Applying the approximation property (P5) and Lemma 3.2 we have

$$\begin{aligned} &\|\chi-\chi_k\|_0^2 + h_k^2\|\nu-\nu_k\|_0^2 \\ \le\ & ch_k(\|\xi\|_2 + \|\eta\|_1 + h_k\|\eta\|_2)(\|\chi-\chi_k\|_1 + \|\nu-\nu_k\|_0 + h_k\|\nu-\nu_k\|_1) \\ \le\ & ch_k(\|\chi-\chi_k\|_0 + h_k\|\nu-\nu_k\|_0)(\|\chi-\chi_k\|_1 + \|\nu-\nu_k\|_0 + h_k\|\nu-\nu_k\|_1)\ . \end{aligned}$$

From this it follows that

$$\|\chi-\chi_k\|_0 + h_k\|\nu-\nu_k\|_0 \le ch_k(\|\chi-\chi_k\|_1 + \|\nu-\nu_k\|_0 + h_k\|\nu-\nu_k\|_1)\ . \tag{3.13}$$

Next, we use again the duality technique to estimate $\|\nu-\nu_k\|_0$. Let $[\psi,\theta]$ be the solution of

$$\begin{cases} a(\psi,\mathbf{v}) + b(\mathbf{v},\theta) & = 0 \\ b(\psi,q) & = (\nu-\nu_k,q) \end{cases}, \qquad \forall [\mathbf{v},q] \in D\ . \tag{3.14}$$

Then

$$\|\psi\|_1 + \|\theta\|_0 \le c\|\nu - \nu_k\|_0 \ . \tag{3.15}$$

From (3.14) and (3.11) we have for any $\mathbf{v} \in \mathbf{X}_k \subset \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned} \|\nu - \nu_k\|_0^2 &= b(\psi, \nu - \nu_k) \\ &= b(\psi - \mathbf{v}, \nu - \nu_k) - a(\chi - \chi_k, \mathbf{v}) \\ &\le c\|\psi - \mathbf{v}\|_0 \|\nu - \nu_k\|_1 + \|\chi - \chi_k\|_1 \|\mathbf{v}\|_1 \ . \end{aligned}$$

Referring (P5) and (3.15) we obtain

$$\begin{aligned} \|\nu - \nu_k\|_0^2 &\le ch_k\|\psi\|_1 \|\nu - \nu_k\|_1 + \|\chi - \chi_k\|_1 \|\psi\|_1 \\ &\le c\|\nu - \nu_k\|_0 (h_k\|\nu - \nu_k\|_1 + \|\chi - \chi_k\|_1) \ . \end{aligned}$$

From this it follows that

$$\|\nu - \nu_k\|_0 \le c(h_k\|\nu - \nu_k\|_1 + \|\chi - \chi_k\|_1). \tag{3.16}$$

Now the L_2-error estimate (3.10) follows immediately from (3.13) and (3.16).

Lemma 3.4. *Suppose that Ω is a convex polyhedron and that $\mathbf{l} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$. Let $[\chi, \nu] \in \mathbf{H}_0^1(\Omega) \times \hat{L}^2(\Omega)$ and $[\chi^k, \nu^k] \in \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ be the solutions of*

$$\begin{cases} a(\chi, \mathbf{v}) + b(\mathbf{v}, \nu) &= (\mathbf{r}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\chi, q) &= h_k^2(s, q), \qquad \forall q \in \hat{L}^2(\Omega), \end{cases} \tag{3.17}$$

and

$$\begin{cases} a(\chi^k, \mathbf{v}) + b(\mathbf{v}, \nu^k) &= (\mathbf{r}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\chi^k, q) - h_k^2(\nabla \nu^k, \nabla q) &= h_k^2(s, q), \qquad \forall q \in \hat{H}^1(\Omega), \end{cases} \tag{3.18}$$

respectively. Then we have the estimate

$$\|\chi^k - \chi\|_0 + h_k\|\nu^k - \nu\|_0 \le ch_k^2(\|\mathbf{r}\|_0 + h_k\|s\|_0) \ . \tag{3.19}$$

Proof. We use the duality technique to estimate $\|\chi^k - \chi\|_0$. Let $[\psi, \rho] \in \mathbf{H}_0^1(\Omega) \times \hat{L}^2(\Omega)$ be the solution of

$$\begin{cases} a(\psi, \mathbf{v}) + b(\mathbf{v}, \rho) &= (\chi^k - \chi, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\psi, q) &= 0, \qquad \forall q \in \hat{L}^2(\Omega) \ . \end{cases} \tag{3.20}$$

Choosing $\mathbf{v} = \chi^k - \chi$, $q = \nu^k - \nu$ in (3.20) and subtracting (3.17) from (3.18) we conclude that

$$\|\chi^k - \chi\|_0^2 = a(\psi, \chi^k - \chi) + b(\chi^k - \chi, \rho) + b(\psi, \nu^k - \nu) = h_k^2(\nabla \nu^k, \nabla \rho).$$

Applying the regularity estimate of Stokes problem

$$\|\psi\|_2 + \|\rho\|_1 \le c\|\chi^k - \chi\|_0$$

and Lemma 3.2 we have

$$\|\chi^k - \chi\|_0^2 \le h_k^2 \|\nu^k\|_1 \|\rho\|_1 \le c h_k^2 \|\chi^k - \chi\|_0 (\|\mathbf{r}\|_0 + h_k \|s\|_0) \ .$$

From this it follows that

$$\|\chi^k - \chi\|_0 \le c h_k^2 (\|\mathbf{r}\|_0 + h_k \|s\|_0) \ . \tag{3.21}$$

Now we estimate $h_k\|\nu - \nu_k\|_0$ in (3.19). Subtracting (3.17) from (3.18) we have

$$\begin{array}{rcll} a(\chi^k - \chi, \mathbf{v}) + b(\mathbf{v}, \nu^k - \nu) & = & 0, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\chi^k - \chi, q) - h_k^2 (\nabla \nu^k, \nabla q) & = & 0, & \forall q \in \hat{H}^1(\Omega). \end{array}$$

From this and inf-sup condition (P4) it follows that

$$\|\nu^k - \nu\|_0 \le \beta \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v}, \nu^k - \nu)}{\|\mathbf{v}\|_1} \le \beta \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{-a(\chi^k - \chi, \mathbf{v})}{\|\mathbf{v}\|_1} \le c \|\chi^k - \chi\|_1 \tag{3.22}$$

and

$$\|\chi^k - \chi\|_1 \le \beta \sup_{q \in \hat{H}^1(\Omega)} \frac{b(\chi^k - \chi, q)}{\|q\|_0} \le \beta \sup_{q \in \hat{H}^1(\Omega)} \frac{h_k^2 (\nabla \nu^k, \nabla q)}{\|q\|_0} \ . \tag{3.23}$$

Since (3.18) is the variational problem of

$$\left\{ \begin{array}{rl} -\Delta \chi^k + \bigtriangledown \nu^k = \mathbf{r}, & \text{in } \Omega, \\ \operatorname{div} \chi^k - h_k^2 \Delta \nu^k = h_k^2 s, & \text{in } \Omega, \\ \chi^k = 0, & \text{on } \partial\Omega, \\ \frac{\partial \nu^k}{\partial n} = 0, & \text{on } \partial\Omega, \end{array} \right.$$

by first Green's formula we obtain

$$\int_\Omega \Delta \nu^k \cdot \Delta q = -\int_\Omega (\Delta \nu^k) \cdot q + \int_\Gamma \frac{\partial \nu^k}{\partial n} \cdot q \, d\gamma \le \|\nu^k\|_2 \cdot \|q\|_0 \ .$$

From this and (3.22), (3.23) we have

$$\|\nu^k - \nu\|_0 \le c h_k^2 \|\nu^k\|_2 \ .$$

Recalling (3.18) and applying Lemma 3.2 it follows that

$$h_k \|\nu^k - \nu\|_0 \le c h_k^3 \|\nu^k\|_2 \le c h_k^2 (\|\mathbf{r}\|_0 + h_k \|s\|_0) \ .$$

Combining this and (3.21) we obtain the estimate (3.19).

3.4. Approximation Property

Proof of the approximation property (3.6)*:* The proof of the approximation property is as follows. $[\mathbf{u}_{k-1}^*, p_{k-1}^*]$ and $[\mathbf{e}_k^m, \epsilon_k^m]$ will be considered as the finite element solutions of an auxiliary problem in D_{k-1} and D_k, respectively. We will estimate the auxiliary problem between the finite element solutions and the solution of the auxiliary problem by a duality argument. This technique can be seen in Hackbusch [8], Braess and Verfürth [4] and Huang [9].

Following the technique in Huang [9], consider the auxiliary function $[\mathbf{r}_k, s_k] \in D_k$ defined by

$$(\mathbf{r}_k, \mathbf{v}) + h_k^2(s_k, q) = L_k([\mathbf{e}_k^m, \epsilon_k^m]; [\mathbf{v}, q]), \qquad \forall [\mathbf{v}, q] \in D_k \ . \tag{3.24}$$

From the definition of the norms $|||\cdot|||_{s,k}$ it follows that

$$\|\mathbf{r}_k\|_0^2 + h_k^2\|s_k\|_0^2 = L_k([\mathbf{e}_k^m, \epsilon_k^m]; [\mathbf{r}_k, s_k]) \leq |||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \cdot |||[\mathbf{r}_k, s_k]|||_{0,k} \ .$$

Then, we obtain the estimate for the residual $[\mathbf{r}_k, s_k]$

$$\|\mathbf{r}_k\|_0 + h_k\|s_k\|_0 \leq c|||[e_k^m, \epsilon_k^m]|||_{2,k} \ . \tag{3.25}$$

Next, we use the estimate of the residual to make an L_2-estimate for the dual problem.

By the Hahn-Banach extension theorem, there exists two linear continuous functional $\mathbf{r}'$ on $(\mathbf{L}^2(\Omega), \|\cdot\|_0)$ and s' on $(L^2(\Omega), \|\cdot\|_0)$ satisfying

$$< \mathbf{r}', \mathbf{v} >= (\mathbf{r}_k, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}_k, \qquad \|\mathbf{r}'\| = \sup_{\mathbf{v}\in\mathbf{X}_k} \frac{(\mathbf{r}_k, \mathbf{v})}{\|\mathbf{v}\|_0} \leq \|\mathbf{r}_k\|_0 \tag{3.26}$$

and

$$< s', q >= (s_k, q) \quad \forall q \in M_k, \qquad \|s'\| = \sup_{q\in M_k} \frac{(s_k, q)}{\|q\|_0} \leq \|s_k\|_0 \ . \tag{3.27}$$

Let $[\chi^k, \nu^k]$ and $[\chi^{k-1}, \nu^{k-1}]$ in D be the solutions of

$$L_k([\chi^k, \nu^k]; [\mathbf{v}, q]) =< \mathbf{r}', \mathbf{v} > +h_k^2 < s', q >, \qquad \forall [\mathbf{v}, q] \in D \tag{3.28}$$

and

$$L_{k-1}([\chi^{k-1}, \nu^{k-1}]; [\mathbf{v}, q]) =< \mathbf{r}', \mathbf{v} > +h_k^2 < s', q >, \quad \forall [\mathbf{v}, q] \in D \ . \tag{3.29}$$

From Eqs.(3.24), (3.26) and (3.27) we obtain

$$L_k([\mathbf{e}_k^m, \epsilon_k^m]; [\mathbf{v}, q]) =< \mathbf{r}', \mathbf{v} > +h_k^2 < s', q >, \qquad \forall [\mathbf{v}, q] \in D_k \ .$$

From this, (3.28) and the definition of R_k it follows that

$$[\mathbf{e}_k^m, \epsilon_k^m] = R_k[\chi^k, \nu^k] \ . \tag{3.30}$$

From the definitions of $[\mathbf{u}_{k-1}^*, p_{k-1}^*]$, G_{k-1}, $[\mathbf{e}_k^m, \epsilon_k^m]$, $[\mathbf{r}_k, s_k]$, and (3.26), (3.27), we conclude that for all $[\mathbf{v}, q] \in D_{k-1}$

$$\begin{aligned} L_{k-1}([\mathbf{u}_{k-1}^*, p_{k-1}^*]; [\mathbf{v}, q]) &= G_{k-1}([\mathbf{v}, q]) \\ &= L_k([\mathbf{e}_k^m, \epsilon_k^m]; [\mathbf{v}, q]) \\ &= < \mathbf{r}', \mathbf{v} > +h_k^2 < s', q > . \end{aligned}$$

From this , (3.29) and the definition of R_{k-1} it follows that

$$[\mathbf{u}^*_{k-1}, p^*_{k-1}] = R_{k-1}[\chi^{k-1}, \nu^{k-1}] \ . \tag{3.31}$$

Now we estimate $[\chi^k, \nu^k] - R_k[\chi^k, \nu^k]$ by a duality technique. Let us rename the Ritz projection $[\mathbf{e}^m_k, \epsilon^m_k] = R_k[\chi^k, \nu^k]$ by $[\chi_k, \nu_k]$. By Lemma 3.3 we have

$$|||[\chi^k, \nu^k] - R_k[\chi^k, \nu^k]|||_0 \le c h_k\{\|\chi^k - \chi_k\|_1 + h_k\|\nu^k - \nu_k\|_1\} \ . \tag{3.32}$$

On the other hand, referring to the definition of R_k and (3.28) we have for all $[\mathbf{v}, q] \in D$

$$\begin{aligned} & L_k([\chi^k, \nu^k] - R_k[\chi^k, \nu^k]; [\mathbf{v}, q]) \\ &= L_k([\chi^k, \nu^k]; [\mathbf{v}, q] - R_k[\mathbf{v}, q]) \\ &= < \mathbf{r}', \mathbf{v} - \mathbf{v}_k > + h_k^2 < s', q - q_k > \\ &\le (\|\mathbf{r}'\| + h_k\|s'\|)(\|\mathbf{v} - \mathbf{v}_k\|_0 + h_k\|q - q_k\|_0) \ . \end{aligned} \tag{3.33}$$

From Lemma 3.3 we have

$$\|\mathbf{v} - \mathbf{v}_k\|_0 + h_k\|q - q_k\|_0 \le c h_k(\|\mathbf{v} - \mathbf{v}_k\|_1 + h_k\|q - q_k\|_1) \ . \tag{3.34}$$

From (P2) and the equivalence of the norms $\|\cdot\|_1$ and $|\cdot|_1$ in $\hat{H}^1(\Omega)$, the definition R_k and (P1), (P3), Lemma 3.3, we conclude that

$$\begin{aligned} & \|\mathbf{v} - \mathbf{v}_k\|_1^2 + h_k^2\|q - q_k\|_1^2 \\ \le\ & c\{[a(\mathbf{v} - \mathbf{v}_k, \mathbf{v} - \mathbf{v}_k) + b(\mathbf{v} - \mathbf{v}_k, q - q_k)] \\ & -[b(\mathbf{v} - \mathbf{v}_k, q - q_k) - h_k^2(\nabla(q - q_k), \nabla(q - q_k))]\} \\ \le\ & c\{[a(\mathbf{v} - \mathbf{v}_k, \mathbf{v}) + b(\mathbf{v}, q - q_k)] \\ & -[b(\mathbf{v} - \mathbf{v}_k, q) - h_k^2(\nabla(q - q_k), \nabla q)]\} \\ \le\ & c(\|\mathbf{v} - \mathbf{v}_k\|_1 + \|q - q_k\|_0 + h_k\|q - q_k\|_1)(\|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1) \\ \le\ & c(\|\mathbf{v} - \mathbf{v}_k\|_1 + h_k\|q - q_k\|_1)(\|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1) \ . \end{aligned}$$

Then

$$\|\mathbf{v} - \mathbf{v}_k\|_1 + h_k\|q - q_k\|_1 \le c(\|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1) \tag{3.35}$$

holds.

Recalling (3.33), (3.34), (3.35) and Lemma 3.1 we obtain

$$\begin{aligned} & \|\chi^k - \chi_k\|_1 + \|\nu^k - \nu_k\|_0 + h_k\|\nu^k - \nu_k\|_1 \\ \le\ & c \sup_{[\mathbf{v}, q] \in D} \frac{L_k([\chi^k - \chi_k, \nu^k - \nu_k];\ [\mathbf{v}, q])}{\|\mathbf{v}\|_1 + \|q\|_0 + h_k\|q\|_1} \\ \le\ & c h_k(\|\mathbf{r}'\| + h_k\|s'\|) \ . \end{aligned}$$

Combining this and (3.32), (3.26), (3.27), (3.25), we have

$$|||[\chi^k - \chi_k, \nu^k - \nu_k]|||_{0,k} \le c h_k^2 |||[\mathbf{e}^m_k, \epsilon^m_k]|||_{2,k} \ . \tag{3.36}$$

Analogously, we rename the Ritz projection $[\mathbf{u}_{k-1}^*, p_{k-1}^*] = R_{k-1}[\chi^{k-1}, \nu^{k-1}]$ by $[\chi_{k-1}, \nu_{k-1}]$. From $h_{k-1} = 2h_k$ and (3.26), (3.27), (3.25), it follows that

$$|||[\chi^{k-1} - \chi_{k-1}, \nu^{k-1} - \nu_{k-1}]|||_{0,k} \le ch_k^2(\|\mathbf{r}'\| + h_k\|s'\|) \le ch_k^2|||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \ . \quad (3.37)$$

Let $[\chi, \nu] \in \mathbf{H}_0^1(\Omega) \times \hat{H}^1(\Omega)$ be the solution of

$$\begin{cases} a(\chi, \mathbf{v}) + b(\mathbf{v}, \nu) & =< \mathbf{r}', \mathbf{v} >, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\chi, q) & = h_k^2 < s', q >, & \forall q \in \hat{L}^2(\Omega). \end{cases}$$

From Lemma 3.4 and (3.26), (3.27), (3.25) we have

$$|||[\chi^k - \chi, \nu^k - \nu]|||_{0,k} \le ch_k^2(\|\mathbf{r}'\| + h_k\|s'\|) \le ch_k^2|||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \quad (3.38)$$

and

$$|||[\chi^{k-1} - \chi, \nu^{k-1} - \nu]|||_{0,k} \le ch_k^2|||[\mathbf{e}_k^m, \epsilon_k^m]|||_{2,k} \ . \quad (3.39)$$

Finally, by (3.36), (3.37), (3.38), (3.39) we obtain the approximation property (3.6),

$$\begin{aligned} & |||[\mathbf{e}_k^m - \mathbf{u}_{k-1}^*, \epsilon_k^m - p_{k-1}^*]|||_{0,k} \\ \le\ & |||[\chi^k, \nu^k] - R_k[\chi^k, \nu^k]|||_{0,k} + |||R_k[\chi^k, \nu^k] - [\chi, \nu]|||_{0,k} \\ & + |||[\chi, \nu] - R_{k-1}[\chi^{k-1}, \nu^{k-1}]|||_{0,k} + |||R_{k-1}[\chi^{k-1}, \nu^{k-1}] - [\chi^{k-1}, \nu^{k-1}]|||_{0,k} \\ \le\ & ch_k^2|||[\mathbf{e}_k^m, \epsilon_k^m]|||_{0,k} \ . \end{aligned}$$

4. Numerical Results

We consider the Stokes problem

$$\begin{aligned} -\Delta u_1 + \frac{\partial p}{\partial x} &= -2y(y^2-1)(3x^2-1) - 3y(x^2-1)^2 \\ & \quad - \frac{2x}{(1-x^2)^2(1-y^2)} e^{-\frac{1}{(1-x^2)(1-y^2)}}, && (x,y) \in \Omega, \\ -\Delta u_2 + \frac{\partial p}{\partial y} &= 2x(x^2-1)(3y^2-1) + 3x(y^2-1)^2 \\ & \quad - \frac{2y}{(1-x^2)(1-y^2)^2} e^{-\frac{1}{(1-x^2)(1-y^2)}}, && (x,y) \in \Omega, \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0, && (x,y) \in \Omega, \\ u_1 = u_2 &= 0, && (x,y) \in \partial\Omega, \end{aligned}$$

where the domain $\Omega = (-1,1) \times (-1,1)$ and the true solution

$$\begin{aligned} u_1 &= 0.5(x^2-1)^2(y^2-1)y, \\ u_2 &= -0.5(y^2-1)^2(x^2-1)x, \\ p &= e^{-\frac{1}{(1-x^2)(1-y^2)}}. \end{aligned}$$

Let T_0 be the coarsest grid triangulation of Ω into triangles. The triangulations T_k, $1 \le k \le 3$, are defined by dividing each $K \in T_{k-1}$ into 4 triangles by joining the midpoints of the sides, as illustrated in Fig. 2.

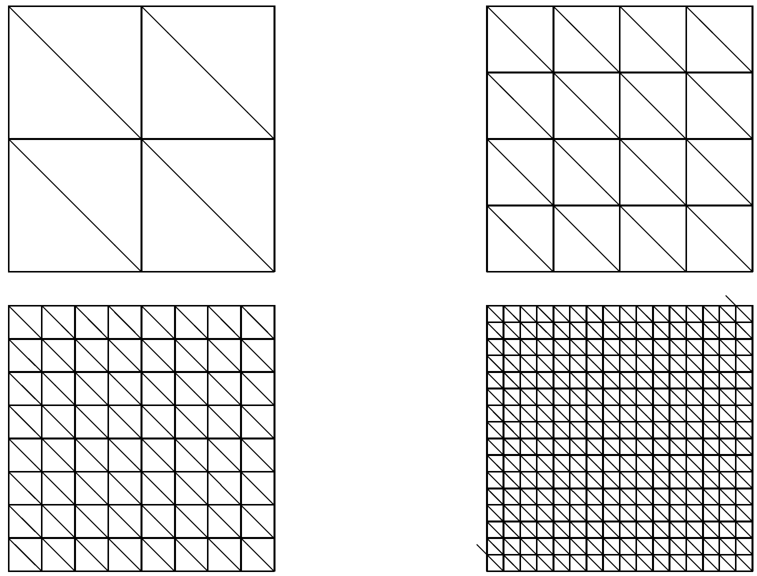

Fig. 2. The triangulations T_0, T_1, T_2 and T_3 of Ω into triangles

We use Algorithm 2.1 to compute the example. The convergence rates $\delta_{k,m}$ of two-grid method and multi-grid method are given in Table 1—3.

Table 1

The convergence rates $\delta_{k,m}$ of Algorithm 2.1
with m Jacobi relaxations in smoothing steps

$\delta_{k,m}$	$m=5$	$m=10$	$m=15$	$m=20$	$m=25$
k=1	0.90	0.88	0.85	0.82	0.80
k=2	0.94	0.93	0.91	0.88	0.87
k=3	0.98	0.95	0.95	0.92	0.90

Table 2

The convergence rates $\delta_{k,m}$ of Algorithm 2.1
with m Gauss-Seidal relaxations in smoothing steps

$\delta_{k,m}$	$m=1$	$m=2$	$m=3$	$m=5$	$m=10$
k=1	0.63	0.56	0.52	0.46	0.38
k=2	0.71	0.61	0.54	0.48	0.40
k=3	0.79	0.68	0.57	0.51	0.43

Table 3
The convergence rates $\delta_{k,m}$ of Algorithm 2.1
with m SOR relaxations in smoothing steps ($\omega = 1.133$)

$\delta_{k,m}$	$m = 1$	$m = 2$	$m = 3$	$m = 5$	$m = 10$
k=1	0.60	0.50	0.45	0.35	0.26
k=2	0.65	0.53	0.48	0.37	0.30
k=3	0.70	0.57	0.52	0.40	0.35

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