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HIGH-ACCURACY P-STABLE METHODS WITH MINIMAL PHASE-LAG FOR $y'' = f(t,y)$ *

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Abstract

In this paper, we develop a one-parameter family of P-stable sixth-order and eighth-order two-step methods with minimal phase-lag errors for numerical integration of second order periodic initial value problems:

$$y'' = f(t,y), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

We determine the parameters so that the phase-lag (frequency distortion) of these methods are minimal. The resulting methods are P-stable methods with minimal phase-lag errors. The superiority of our present P-stable methods over the P-stable methods in [1–4] is given by comparative studying of the phase-lag errors and illustrated with numerical examples.

1. Introduction

The development of numerical integration formulae for the direct integration of the periodic initial-value problem

$$y'' = f(t,y), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' , \tag{1.1}$$

which arises in the theory of orbital mechanics and in the study of wave equations, has created considerable interest in the recent years.

Usually, the Numerov's method

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{12}(f_{n+1} + 10f_n + f_{n-1}) \tag{1.2}$$

is the most popular method. Although, Numerov's method has phase-lag of order four and possess only a finite interval of periodicity $(0, 2.449^2)$. Recently Chawla and Rao[2,3] developed fourth-order and sixth-order P-stable methods with phase-lag of order six.

Ananthakrishnaiah[4] obtained a two-parameter family of second order P-stable methods $M_2(\alpha,\beta)$ with phase-lag of order six. It is therefore natural to ask whether we can obtain P-stable methods with phase-lag order and accuracy order higher than the

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methods in [1–4]. The purpose of this paper is by modificating the methods in [1–4] and selecting parameters suitably, to obtain a new family of methods with sixth-order and eighth-order. Comparing with the methods in [1–4], our methods are more useful when a large step-size is used, that is , when a modest accuracy is sufficient or the solution which consists of a slowly varying oscillation with a high-frequency oscillation superimposed, has a small amplitude. At the end of this paper we give two examples to demonstrate that our methods are better than the methods in [1–4].

2. Basic Theory

When we apply an symmetry implicit two-step method to the test equation

$$y'' = -\lambda^2 y, \quad \lambda > 0 , \tag{2.1}$$

we obtain the polynomial

$$\Omega(\xi, H^2) = A(H)\xi^2 - 2B(H)\xi + A(H), \quad H = \lambda h . \tag{2.2}$$

It is stability and $\Omega(\xi, H^2) = 0$ is characteristic equation, $A(H)$ and $B(H)$ are polynomials of $H = \lambda h$.

Definition 1. (Lambert and Watson[5]) *The method with stability polynomial* (2.2) *is said to have interval of periodicity* $(0, H_p^2)$ *if for all* $H^2 \in (0, H_p^2)$*, the roots* $\xi_{1,2}$ *of* $\Omega(\xi, H^2)$ *satisfy*

$$\xi_{1,2} = e^{\pm i\theta(H)} \tag{2.3}$$

for some real valued function $\theta(H)$.

Definition 2. *The method with stability polynomial* (2.2) *is said to be P-stable if its interval of periodicity is* $(0, \infty)$.

It is easy to see that the roots of (2.2) are complex and of module one if

$$\Big|\frac{B(H)}{A(H)}\Big| < 1 . \tag{2.4}$$

Thus, the P-stability condition is satisfied if

$$A(H) + B(H) > 0 \text{ and } A(H) - B(H) > 0, \text{ for all } H^2 \in (0, \infty) . \tag{2.5}$$

The exact solution of the test equation (2.1) with the initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$ is given by

$$y(t) = y_0 \cos \lambda t + \frac{y'_0}{\lambda} \sin \lambda t . \tag{2.6}$$

Evaluating (2.6) at t_{n+1}, t_n and t_{n-1} and eliminating y_0 and y'_0, we obtain

$$y(t_{n+1}) - 2\cos \lambda h y(t_n) + y(t_{n-1}) = 0 , \tag{2.7}$$

its characteristic equation is

$$\xi^2 - 2\cos\theta(H)\xi + 1 = 0, \quad H = \lambda h \ . \tag{2.8}$$

Now the characteristic equation of (2.2) is written as

$$\xi^2 - 2\cos\theta(H)\xi + 1 = 0 \ , \tag{2.9}$$

where $\cos\theta(H) = B(H)/A(H)$.

Definition 3. (Ananthakrishnaiah[6]) *We define the phase-lag error of the method with stability polynomial* (2.2) *as the leading coefficient in the expansion*

$$P(H) = \Big|\frac{A(H)\cos(H) - B(H)}{H^2}\Big| \ . \tag{2.10}$$

The motivation of Definition 3 can be easily derived from the difference in the frequency distortion of the characteristic equation (2.8) and (2.9).

3. A Family of Sixth-Order P-Stable Methods with Minimal Phase-Lag

For the numerical integration of the second-order periodic initial-value problem (1.1), we consider a family of implicit two-step sixth-order methods

$$y_n^{[i]} = y_n - \alpha_i h^2(f_{n+1} - 2f_n^{[i-1]} + f_{n-1}) \ , \quad i = 1, 2, ..., m, \quad f_n^{[0]} = f_n \ , \tag{3.1}$$

$$\bar{y}_{n+\frac{1}{2}} = \frac{3}{8}y_{n+1} + \frac{3}{4}y_n - \frac{1}{8}y_{n-1} - \frac{h^2}{128}(5f_{n-1} - 2f_n^{[m]} - 3f_{n-1}) \ , \tag{3.2}$$

$$\bar{y}_{n-\frac{1}{2}} = -\frac{1}{8}y_{n+1} + \frac{3}{4}y_n + \frac{3}{8}y_{n-1} - \frac{h^2}{128}(-3f_{n+1} - 2f_n^{[m]} + 5f_{n-1}) \ . \tag{3.3}$$

Then, for $n \geq 1$, the m-parameter family of sixth-order discretization for $y'' = f(t, y)$ is given by

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{60}[f_{n+1} + 26f_n + f_{n-1} + 16(\bar{f}_{n+\frac{1}{2}} + \bar{f}_{n-\frac{1}{2}})] \ , \tag{3.4}$$

where

$$f_n^{[i]} = f(t_n, y_n^{[i]}), \quad i = 1, 2, ..., m \ ,$$
$$\bar{f}_{n\pm\frac{1}{2}} = f(t_{n\pm\frac{1}{2}}, \bar{y}_{n\pm\frac{1}{2}}), \quad f_n = f(t_n, y_n) \ ,$$
$$t_{n\pm\frac{1}{2}} = t_n + \frac{h}{2}, \quad h > 0$$

and $\alpha_i, \quad i = 1, 2, ..., m$ are free parameters.

The local truncation errors of (3.1) – (3.4) are given by

$$\alpha_i h^4 y^{(4)}(x_n) + o(h^6), \quad i = 1, 2, ..., m \ ,$$
$$\frac{5}{768} h^5 y^{(5)}(x_n) + \frac{h^6}{3072} y^{(6)}(x_n) + o(h^7) \ ,$$
$$-\frac{5}{768} h^5 y^{(5)}(x_n) + \frac{h^6}{3072} y^{(6)}(x_n) + o(h^7) \ ,$$
$$-\frac{h^8}{120960} y^{(8)}(x_n) + o(h^{10}) \ .$$

Note that our discretizations given by (3.4) are based on $m+2$ evaluations of f per step. Note also that there is no need to compute $\bar{f}_{n-\frac{1}{2}}$ because it is available from the previous calculation that determined y_n.

If we apply the method (3.4) to the test equation (2.1), we have the stability polynomial

$$\Omega(\xi, H^2) = A(H)\xi^2 - 2B(H)\xi + A(H) \ , \tag{3.5}$$

where

$$A(H) = 1 + \frac{H^2}{12} + \frac{H^4}{240} - \frac{1}{120}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\alpha_{m-k+1}\cdots\alpha_m H^{2k+4} \ , \tag{3.6}$$

$$B(H) = 1 - \frac{5}{12}H^2 + \frac{H^4}{240} - \frac{1}{120}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\alpha_{m-k+1}\cdots\alpha_m H^{2k+4} \ . \tag{3.7}$$

Thus

$$A(H) - B(H) = \frac{H^2}{2} > 0 \ , \qquad \text{for all } H^2 \in (0, \infty) \ , \tag{3.8}$$

$$A(H) + B(H) = 2 - \frac{H^2}{3} + \frac{H^4}{120} - \frac{1}{60}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\alpha_{m-k+1}\cdots\alpha_m H^{2k+4} \ . \tag{3.9}$$

From (3.6), (3.7) and Definition 3, we have

$$\frac{A(H)\cos(H) - B(H)}{H^2} = \sum_{k=2}^{m+1}(-1)^k\Big[\frac{1}{(2k+4)!} - \frac{1}{12(2k+2)!} + \frac{1}{(2k)!}\cdot\frac{1}{240} + \frac{1}{120}\sum_{l=1}^{k-1}\frac{2^{l-1}}{[2(k-l)]!}\alpha_{m-l+1}\cdots\alpha_m\Big]H^{2k+2} \ . \tag{3.10}$$

From (3.10) and Definition 3, we have

Theorem 1. *If the parameters $\alpha_2, \alpha_3, \cdots, \alpha_m$ of (3.1) are given by*

$$\alpha_m = -\frac{5}{252} \ , \qquad m \geq 2 \ ,$$
$$\alpha_{m-k+2} = -\frac{120}{2^{k-3}\alpha_{m-k+3}\cdots\alpha_m}\Big[\frac{1}{(2k+4)!} - \frac{1}{12(2k+2)!} + \frac{1}{240}\cdot\frac{1}{(2k)!} + \frac{1}{120}\sum_{l=1}^{k-2}\frac{2^{l-1}}{[2(k-l)]!}\alpha_{m-l+1}\cdot\alpha_m\Big] \ , \qquad k = 3, \cdots, m \ , \tag{3.11}$$

then, the sixth-order implicit methods (3.4) *possess minimal phase-lag errors*

$$P(H) = \Big| \frac{1}{2(2m+6)!} - \frac{1}{12(2m+4)!} + \frac{1}{240} \cdot \frac{1}{(2m+2)!} + \frac{1}{120} \sum_{l=1}^{m} \frac{2^{l-1}}{[2(m-l+1)]!} \alpha_{m-l+1} \cdots \alpha_m \Big| H^{2m+4} \ . \tag{3.12}$$

Thus , if $\alpha_2, \alpha_3, \cdots, \alpha_m$ are given by (3.11), we obtained one -parameter α_1 family of sixth-order implicit methods (3.4) with minimal phase-lag error order $2m+4$, which is denoted as $M_6(\alpha_1)$.

Selecting parameter α_1 of methods $M_6(\alpha_1)$ suitably, we have

$$A(H) + B(H) = 2 - \frac{H^2}{3} + \frac{H^4}{120} - \frac{1}{60} \sum_{k=1}^{m} (-1)^{k+1} 2^{k-1} \alpha_{m-k+1} \cdots \alpha_m H^{2k+4} > 0 \ , \quad \text{for all } H^2 \in (0, \infty) \ .$$

From (3.8), (3.9) and (2.5) , a family of sixth-order P-stable methods $M_6(\alpha_1)$ with minimal phase-lag error order $2m+4$ can be obtained by selecting parameter α_1 suitably. From (3.8), (3.9), (3.11) and (2.5), we obtain Table 1 as following:

Table 1

P-stable sixth-order method $M_6(\alpha_1)$

m	$M_6(\alpha_1)$	α_1	α_2	α_3	α_4	Phase-lag error P(H)
1	Method 1	$\frac{-1}{972}(13 + 2\sqrt{55})$				$\frac{2}{3\times 8!}\lvert 5 + 252\alpha_1\rvert H^6$
2	Method 2	$< -2.560009E - 02$	$-\frac{5}{252}$			$\frac{3}{2\times 10!}\lvert 7 + 400\alpha_1\rvert H^8$
3	Method 3	< 0	$-\frac{7}{400}$	$-\frac{5}{252}$		$\frac{9}{12!}\lvert 5 + 308\alpha_1\rvert H^{10}$
4	Method 4	$< -2.187734E - 02$	$-\frac{5}{308}$	$-\frac{7}{400}$	$-\frac{5}{252}$	$\frac{1}{30\times 14!}\lvert 7061 + 491400\alpha_1\rvert H^{12}$

From Table 1 we conclude that new sixth-order P-stable method $M_6(\alpha_1)$ have smaller phase-lag error than the same order implicit methods in [1–4].

4. A Family of Eighth-order P-stable Method $M_8(\beta_1)$

For the numerical integration of the second order periodic initial problem (1.1) we consider the m-parameter family of implicit two-step eighth-order methods:

$$y_n^{[i]} = y_n - \beta_i h^2(f_{n+1} - 2f_n^{[i-1]} + f_{n-1}) \ , \quad i = 1, 2, \cdots, m, \quad f_n^{[0]} = f_n \ , \tag{4.1}$$

$$\hat{y}_{n+\alpha} + a_1 y_{n+1} + a_0 y_n + a_{-1} y_{n-1} = h^2(b_1 f_{n+1} + b_0 f_n^{[m]} + b_{-1} f_{n-1}) \ , \tag{4.2}$$

$$\hat{y}_{n-\alpha} + a_{-1} y_{n+1} + a_0 y_n + a_1 y_{n-1} = h^2(b_1 f_{n+1} + b_0 f_n^{[m]} + b_1 f_{n-1}) \ , \tag{4.3}$$

$$\begin{aligned}\tilde{y}_{n+\alpha} + c_1 y_{n+1} + c_0 y_n + c_{-1} y_{n-1} = h^2(&d_1 f_{n+1} + d_0 f_n + d_{-1} f_{n-1} \\ &+ e_1 \hat{f}_{n+\alpha} + e_{-1} \hat{f}_{n-\alpha}) \ ,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\tilde{y}_{n-\alpha} + c_{-1} y_{n+1} + c_0 y_n + c_1 y_{n-1} = h^2(&d_{-1} f_{n+1} + d_0 f_n + d_1 f_{n-1} \\ &+ e_{-1} \hat{f}_{n+\alpha} + e_1 \hat{f}_{n-\alpha}) \ .\end{aligned} \tag{4.5}$$

Then for $n \geq 1$, our m-parameter family of eighth-order discretization for $y'' = f(t, y)$ is given by

$$\begin{aligned}y_{n+1} =& 2y_n - y_{n-1} + \frac{19}{1740} h^2(f_{n+1} + f_{n-1}) + \frac{199}{390} h^2 f_n \\ &+ \frac{441}{1885} h^2(\tilde{f}_{n+\alpha} + \tilde{f}_{n-\alpha}) \ ,\end{aligned} \tag{4.6}$$

where

$$\alpha = \sqrt{\frac{13}{42}} \ , \quad f_n^{[i]} = f(t_n, y_n^{[i]} \ , \quad i = 1, 2, \cdots, m \ ,$$
$$\hat{f}_{n\pm\alpha} = f(t_{n\pm\alpha}, \hat{y}_{n\pm\alpha}), \quad \tilde{f}_{n\pm\alpha} = f(t_{n\pm\alpha}, \tilde{y}_{n\pm\alpha}) \ ,$$

and β_1, β_2, $\cdots$, β_m are free parameters and the local truncation error of (4.6) is given by

$$\frac{h^{10}}{8! \times 2216760} y^{(10)}(x_n) + o(h^{12}) \ . \tag{4.7}$$

In order to use these approximation with (4.6), we need $o(h^8)$ - approximation for $\hat{y}_{n\pm\alpha}$ and $o(h^6)$ - approximation for $\hat{y}_{n\pm\alpha}$. We note that because of symmetry it is sufficient to have $o(h^7)$ - approximations for $\tilde{y}_{n\pm\alpha}$. The following order conditions are satisfied:

$$\begin{cases} 1 + a_1 + a_0 + a_{-1} = 0 \\ \alpha + a_1 - a_{-1} = 0 \\ \frac{1}{2}\alpha^2 + \frac{1}{2}(a_1 + a_{-1}) = b_1 + b_0 + b_{-1} \\ \frac{1}{3!}\alpha^3 + \frac{1}{3!}(a_1 - a_{-1}) = b_1 - b_{-1} \\ \frac{1}{4!}\alpha^4 + \frac{1}{4!}(a_1 + a_{-1}) = \frac{1}{2}(b_1 + b_{-1}) \end{cases} \tag{4.8}$$

where $\alpha = \sqrt{\frac{13}{42}}$.

Taking $a_1 + a_{-1} = -\frac{13}{42}$, we obtain the solution of (4.8) :

$$\begin{cases} a_0 = -\frac{29}{42}, \quad b_0 = \frac{377}{21168} \\ a_1 = -\frac{1}{2}\alpha(1+\alpha), \quad a_{-1} = \frac{1}{2}\alpha(1-\alpha) \\ b_1 = -\frac{29}{504}\alpha(1+\frac{1}{2}\alpha), \quad b_{-1} = \frac{29}{504}\alpha(1-\frac{1}{2}\alpha) \\ \alpha = \sqrt{\frac{13}{42}} \, . \end{cases} \tag{4.9}$$

The local truncation error for sum formula of (4.2) and (4.3) is given by

$$\frac{1885}{222264} h^6 y^{(6)}(x_n) + o(h^8) \, .$$

In order to get $o(h^7)$ - approximations for $\tilde{y}_{n\pm\alpha}$, based on the order conditions, we have

$$\begin{cases} 1 + c_1 + c_0 + c_{-1} = 0 \\ \alpha + c_1 - c_{-1} = 0\frac{1}{2}\alpha^2 + \frac{1}{2}(c_1 + c_{-1}) = d_1 + d_0 + d_{-1} + e_1 + e_{-1} \\ \frac{1}{3!}\alpha^3 + \frac{1}{3!}(c_1 - c_{-1}) = d_1 - d_{-1} + \alpha(e_1 - e_{-1}) \\ \frac{1}{4!}\alpha^4 + \frac{1}{4!}(c_1 + c_{-1}) = \frac{1}{2}(d_1 + d_{-1}) + \frac{\alpha^2}{2}(e_1 + e_{-1}) \\ \frac{1}{5!}\alpha^5 + \frac{1}{5!}(c_1 - c_{-1}) = \frac{1}{3!}(d_1 - d_{-1}) + \frac{\alpha^3}{3!}(e_1 - e_{-1}) \\ \frac{1}{6!}\alpha^6 + \frac{1}{6!}(c_1 + c_{-1}) = \frac{1}{4!}(d_1 + d_{-1}) + \frac{1}{4!}\alpha^4(e_1 + e_{-1}) \\ \alpha = \sqrt{\frac{13}{42}} \, . \end{cases} \tag{4.10}$$

Let $c_1 + c_{-1} = -\frac{13}{42}$, we obtain the solution of (4.10) as following

$$\begin{cases} c_0 = -\frac{29}{42}, \quad c_1 = -\frac{1}{2}\alpha(1+\alpha), \quad c_{-1} = \frac{1}{2}\alpha(1-\alpha) \\ d_0 = \frac{2233}{49392}, \quad d_1 = -\frac{1}{144}\alpha(1+\frac{9}{7}\alpha), \quad d_{-1} = \frac{1}{144}\alpha(1-\frac{9}{7}\alpha) \\ \alpha = \sqrt{\frac{13}{42}} \, , \end{cases} \tag{4.11}$$

The local truncation errors for sum formula of (4.4) and (4.5) are given by

$$\frac{700843}{8! \times 467544} h^8 y^{(8)}(x_n) + o(h^{10}) \, .$$

We apply methods (4.6) to the test (2.2),

$$A(H) = 1 + \frac{H^2}{12} + \frac{H^4}{240} + \frac{H^6}{6048} + \frac{1}{3024}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\beta_{m-k+1}\cdots\beta_m H^{2k+6} \ ,$$

$$B(H) = 1 - \frac{5}{12}H^2 + \frac{H^4}{240} + \frac{H^6}{6048} + \frac{1}{3024}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\beta_{m-k+1}\cdots\beta_m H^{2k+6} \ . \quad (4.12)$$

Thus, we have

$$A(H) - B(H) = \frac{H^2}{2} > 0, \quad \text{for all } H^2 \in (0,\infty) \ , \quad (4.13)$$

$$A(H) + B(H) = 2 - \frac{H^2}{3} + \frac{H^4}{120} + \frac{H^6}{3024} + \frac{1}{1512}\sum_{k=1}^{m}(-1)^{k+1}2^{k-1}\beta_{m-k+1}\cdots\beta_m H^{2k+6} \quad (4.14)$$

$$\frac{A(H)\cos(H) - B(H)}{H^2} = \sum_{k=2}^{m+1}(-1)^{k+1}\Big[\frac{1}{(2k+6)!} - \frac{1}{12(2k+4)!} + \frac{1}{240(2k+2)!} - \frac{1}{6048(2k)!} - \frac{1}{3024}\sum_{l=1}^{k-1}\frac{2^{l-1}}{[2(k-l)]!}\beta_{m-l+1}\cdots\beta_m\Big]H^{2k+4} \ . \quad (4.15)$$

Based on (4.15) and from the definition of phase-lag error, we have following conclusion :

Theorem 2. *If the parameter $\beta_2, \beta_3, \cdots, \beta_m$ of (4.1) are given by*

$$\begin{cases} \beta_m = -\dfrac{7}{400} \\ \beta_{m-k+2} = \dfrac{3024}{2^{k-3}\beta_{m-l+1}\cdots\beta_m}\Big[\dfrac{1}{(2k+6)!} - \dfrac{1}{12(2k+4)!} + \dfrac{1}{240(2k+2)!} \\ \qquad - \dfrac{1}{6048(2k)!} - \dfrac{1}{3024}\sum_{l=1}^{k-2}\dfrac{2^{l-1}}{[2(k-l)]!}\beta_{m-l+1}\cdots\beta_m\Big] \end{cases} \quad (4.16)$$

then, we obtain a family of eighth-order implicit methods with minimal phase-lag errors

$$P(H) = \Big|\frac{1}{(2m+8)!} - \frac{1}{12(2m+6)!} + \frac{1}{240}\cdot\frac{1}{(2m+4)!} - \frac{1}{6048}\cdot\frac{1}{(2k+2)!} - \frac{1}{3024}\sum_{l=1}^{m}\frac{2^{m-1}}{[2(m-l+1)]!}\beta_{m-l+1}\cdots\beta_m\Big|H^{2m+6} \ . \quad (4.17)$$

Thus, if parameters $\beta_2, \beta_3, \cdots, \beta_m$ of (4.1) are given by (4.6), we obtain one-parameter β_1 family of eighth - order implicit methods (4.6) with minimal phase-lag error order $2m+6$, which is denoted as $M_8(\beta_1)$.

By selecting parameter β_1 of methods $M_8(\beta_1)$ suitably, so that

$$A(H) + B(H) > 0, \quad \text{for all } H^2 \in (0,\infty) \ ,$$

and considering (4.12), we obtain a family of eighth-order P-stable methods $M_8(\beta_1)$ with phase-lag error order $2m+6$.

Based on (4.13) – (4.17) and (2.5), we can obtain Table 2 as following

Table 2

P-stable eighth-order method $M_8(\beta_1)$

m	$M_8(\beta_1)$	β_1	β_2	β_3	Phase-lag error P(H)
1	Method I	$< -2.560009E-02$			$\frac{3}{2\times 10!}\lvert 7+400\beta_1\rvert H^8$
2	Method II	< 0	$-\frac{7}{400}$		$\frac{9}{12!}\lvert 5+308\beta_1\rvert H^{10}$
3	Method III	$< -2.187734E-02$	$-\frac{5}{308}$	$-\frac{7}{400}$	$\frac{1}{30\times 14!}\lvert 7601+491400\beta_1\rvert H^{12}$

From the Table 2, it is easy to see the P-stable eighth-order methods I, II, III of Table 2 possess higher accuracy and considerably smaller phase-lag error than the P-stable implicit methods of [1–4].

5. Numerical Performance

We notice that for nonlinear $f(t,y)$, all the methods we discussed above are implicit, so an iterative process for computing the solution at each step is needed. Now we apply the modified Newton's methods for this purpose.

We consider methods $M_6(\alpha_1)$, $M_8(\beta_1)$ defined in the form:

$$G(y_{n+1}) - \phi(y_{n+1}) = 0 \ . \tag{5.1}$$

Let $y_{n+1}^{(0)}$ denote an initial approximation for y_{n+1}, then the modified Newton's method for (5.1) is

$$\begin{aligned} &G(y_{n+1}^{(i)}) + G'(y_{n+1}^{(0)})\Delta y_{n+1}^{(i)} = 0 \ , \\ &y_{n+1}^{(i+1)} = y_{n+1}^{(i)} + \Delta y_{n+1}^{(i)}, \quad i = 1,2,\cdots . \end{aligned} \tag{5.2}$$

The starting value $y_{n+1}^{(0)}$ is provided by the Noumerov explicit methods

$$\begin{aligned} &\bar{y}_{n+1} = 2y_n - y_{n-1} + h^2 f_n \ , \\ &y_{n+1}^{(0)} = 2y_n - y_{n-1} + \tfrac{h^2}{12}(\bar{f}_{n+1} + 10 f_n + f_{n-1}) \ . \end{aligned} \tag{5.3}$$

The modified Newton's method converges for h sufficiently small, because $|y_{n+1}^{(0)} - y_{n+1}| = o(H^6)$, $|1 - G'(y_{n+1}^{(0)})| = o(h^4)$ and $G'(y)$ is a continuous function of y.

6. Numerical Illustration

We consider the test problem

$$y'' = -\lambda^2 y, \quad y(0) = 1, \quad y'(0) = 0, \quad \lambda = 25 \,. \tag{6.1}$$

The problem is solved by using the methods

(i) $M_4(\frac{1}{200})$ given by [1] ,

(ii) $M_6(0)$ given by [3] ,

(iii) $M_6(-\frac{5}{308})$ given by Method 3 of Table 1 ,

(iv) $M_8(-\frac{5}{308})$ given by Method II of Table 2.

Using the steplength $h = \frac{\pi}{12}$, the absolute errors in the solution $y(t)$ are tabulated for $t = 2\pi(2\pi)10\pi$, in Table 3.

Table 3

Absolute errors in $y(t)$ for the problem (6.1)

t	$M_6(-\frac{5}{308})$ of Table 1	$M_8(-\frac{5}{308})$ of Table 2	$M_4(\frac{1}{200})$ of [1]	$M_6(0)$ of [3]
π	$2.45D-07$	$1.84D-07$	$2.07D-05$	$2.21D-06$
2π	$9.22D-07$	$8.75D-07$	$9.13D-05$	$8.92D-06$
4π	$5.67D-06$	$3.09D-06$	$3.81D-04$	$5.23D-05$
6π	$9.20D-06$	$8.54D-06$	$8.70D-04$	$9.41D-05$
8π	$2.87D-05$	$2.17D-05$	$1.56D-03$	$2.8D-04$
10π	$3.42D-05$	$3.18D-05$	$2.44D-03$	$3.59D-04$

Notice that in Table 3 the absolute errors for our present sixth- order P-stable method $M_6(-\frac{5}{308})$ and eighth-order P-stable method $M_8(-\frac{5}{308})$ are much smaller than $M_4(\frac{1}{200})$ of [1] and $M_6(0)$ of [3] because our present method $M_6(-\frac{5}{308})$ and $M_8(-\frac{5}{308})$ possess minimal phase-lag error of order twelve.

We now again consider the Duffing equation forcedly by a harmonic function[7]

$$y'' + y + y^3 = F\cos(\Omega t), \quad y(0) = y_G(0), \quad y'(0) = 0 \,, \tag{6.2}$$

with $F = 0.002$, $\Omega = 1.01$, $y_G(t)$ is Galerkin's approximation of order nine to a periodic solution computed by Van Dooren:

$$y_G(t) = \sum_{i=0}^{4} a_{2i+1} \cos[(2i+1)\Omega t] \ ,$$

where $a_1 = 0.200179477536$, $a_3 = 0.246946143 \times 10^{-3}$, $a_5 = 0.304014 \times 10^{-6}$, $a_7 = 0.374 \times 10^{-9}$, $a_9 = 0.0$. We solved the problem both by our present sixth-order P-stable method $M_6(-\frac{5}{308})$ and by the present Method 3 of Table land $M_6(0)$ of [3]. We showed the absolute errors in the computation of $y(40\pi)$ for a few selections of the steplengths. From Table 4, it is easy to see that our present method $M_6(-\frac{5}{308})$ has much smaller absolute error than the method $M_6(0)$ of [3].

Table 4.

Absolute errors in the computation of $y(40\pi)$ for the problem (6.2)

h	the present $M_6(-\frac{5}{308})$ (Method 3 of Table 1)	$M_6(0)$ of [3]
$\frac{\pi}{5}$	$3.45D-05$	$1.4D-03$
$\frac{\pi}{10}$	$5.67D-07$	$2.2D-05$
$\frac{\pi}{20}$	$7.91D-09$	$3.4D-07$
$\frac{\pi}{40}$	$8.20D-11$	$5.4D-09$

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