Journal of Computational Mathematics, Vol.13, No.3, 1995, 211-217.

DETERMINATION AND CORRECTION OF AN INCONSISTENT SYSTEM OF LINEAR INEQUALITIES *1)

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Abstract

In this paper the problems to determine an inconsistent system of linear inequalities and to correct its right-hand side vector are solved by using the isometric plane method for linear programming. As an example, the suitable perturbation quantity of the perturbed inequalities of ill-conditioned linear equations is determined in the numerical experiments.

1. Introduction

In mathematical models of some practical problems, such as constraint conditions of linear programming problems, the systems of linear inequalities which should be consistent may be inconsistent due to incorrect input data. It is sometimes difficult or even impossible to obtain better data. So the problems to determine and correct an inconsistent system of linear inequalities become significant.

In 1988, R.L. Mogilevskaya and P.A. Shvartsman published an algorithm to correct the right-hand side vector of inconsistent system in such a way that the new system is consistent system and is not too far from the model. The algorithm is mathematically simple and give the user the possibility to choose a correction suitable with respect to the corresponding practical problem[1].

Consider the following system of linear inequalities

$$AX > B, \tag{1.1}$$

where $A = (a_{i,j})$ is an $m \times n$ matrix $(m \geq 1, n \geq 2)$, $X = (x_1, x_2, \cdots, x_n)^T$ and $B = (b_1, \cdots, b_m)^T$ are $n-$ and $m-$ dimensional vectors respectively, $(\cdot)^T$ denotes transpose of $(\cdot)$. Note that no equality is contained within (1.1) by means of appropriate treatment.

* Received December 8, 1992.
1) This Project is Supported in part by National Natural Science Foundation of China and by Foundation of Zhongshan University Advanced Research Center.

The system (1.1) is said to be nonreducibly inconsistent if it is inconsistent but each of its proper subsystems is consistent . In [1], the main idea to correct an inconsistent system is to select nonreducibly inconsistent subsystems of the given system to perform corrections on these subsystems. For each nonreducibly inconsistent system, the set of corrections of its right- hand side vector making the system consistent may be described by a unique formula. In order to select a nonreducibly inconsistent subsystem from (1.1) the following series of LP problems must be solved

$$-\varepsilon \to \max, \quad A_kX > B_k, \quad a_{k+1}X + \varepsilon > b_{k+1} \geq 0, \tag{1.2}$$

where $A_kX > B_k$ is a consistent subsystem of (1.1) and $a_{k+1}X > b_{k+1}$ is an inequality, which does not belong to $A_kX > B_k$, but belongs to (1.1).

There is a more suitable method to determine the inconsistent system (1.1) and to correct its right-hand side vector, in authors' opinion, that is the isometric plane method for linear programming[2] which has been presented in 1988.

In the isometric plane method so-called general LP problem,

$$C^TX \to \max, \quad AX > B \tag{1.3}$$

is considered, here the constraint set is the same with (1.1) formally. The constraint set of (1.3) can form arbitrary convex polyhedron in $n-$dimensional space, specially it can be empty set that is equivalent to inconsistent of (1.1) or no solution of (1.3). In [2] a way to determine the consistence of (1.1) without considering the selection of nonreducibly inconsistent subsystems are provided. However, the isometric plane method for linear programming is also suitable to select nonreducibly inconsistent subsystems with an inconsistent system in an alternative way which will be discussed in section 2.

2. Selection of Nonreducibly Inconsistent Subsystems

In the system (1.1) an initial consistent subsystem is easy found, for example, the first inequality

$$a_1X = (a_{11}, a_{12}, \cdots, a_{1n}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} > b_1,\ n \geq 2,\ a_1 \neq 0$$

is clearly consistent and there are innumerable points, which are called interior points, to satisfy

$$a_1X > b_1.$$

Without loss of generality , assume that

$$A_kX = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} X > B_k = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}, \quad 1 \leq k < m \tag{2.1}$$

is an initial consistent subsystem of (1.1) and X^0 is an initial interior point of (2.1) , namely

$$A_k X^0 > B_k.$$

(2.1) form a convex polyhedron Ω^K in R^n. The boundary $D\Omega^K$ of Ω^K consists of all or portion of hyperplane

$$P_1 = \{X \in R^n | a_1 X = b_1,\ 1 \le i \le k\}.$$

To be different from (1.2) the LP problem

$$a_{k+1}X \to \max,\quad A_k X > B_k \tag{2.2}$$

is considered here. If $\max a_{k+1}X > b_{k+1}$, then

$$A_{k+1}X = \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} X > B_{k+1} = \begin{bmatrix} b_1 \\ \vdots \\ b_{k+1} \end{bmatrix} \tag{2.3}$$

is the consistent subsystem of (1.1), otherwise at least there exists one nonreducibly inconsistent subsystem of (1.1) in (2.3).

In order to examine whether

$$\max a_{k+1}X > b_{k+1} \tag{2.4}$$

holds, we use the isometric plane algorithm[2] to solve step by step and to check (2.4) in each step.

We present the theoretical background for selection of nonreducibly inconsistent subsystems in the following.

Theorem 1. *Let $j_1, \cdots, j_k$ be a permutation of $1, \cdots, k$, and Y^* satisfies*

$$\begin{aligned} &a_i Y^* = b_i,\quad (i = j_1, \cdots, j_s,\ \ 1 \le s \le n, \\ &\qquad \sum_{r=1}^{s} \alpha_r a_{j_r}^T \ne 0,\quad \forall \alpha_1, \cdots, \alpha_s \in R^1,\ \ \sum_{r=1}^{s} \alpha_r^2 \ne 0), \\ &a_i Y^* > b_i,\quad (i = j_{s+1}, \cdots, j_k). \end{aligned} \tag{2.5}$$

Assume that

$$a_{k+1}^T = \sum_{i=1}^{s} \lambda_i a_{j_i}^T,\ \ (\lambda_i \le 0,\ \ i = 1, \cdots, s,\ \ 1 \le s \le n,\ \ s \le k) \tag{2.6}$$

and

$$a_{k+1} Y^* \le b_{k+1}. \tag{2.7}$$

Without loss of generality suppose

$$\lambda_i < 0,\quad (i = 1, \cdots, q,\ \ 1 \le q \le s) \tag{2.8}$$

in (2.6). *Then the system of linear inequalities*

$$a_{j_i}X > b_{j_i}, \quad (i = 1, \cdots, q), \tag{2.9}$$

$$a_{k+1}X > b_{k+1}$$

is a nonreducibly inconsistent subsystem of (2.3) *and*

$$\lambda = (-\lambda_1, \cdots, -\lambda_q, 1)^T \tag{2.10}$$

is the informative vector of (2.3)[1].

Proof. Let

$$\bar{A}_q = \begin{bmatrix} a_{j_1} \\ \vdots \\ a_{j_q} \\ a_{k+1} \end{bmatrix}, \quad \bar{B}_q = \begin{bmatrix} b_{j_1} \\ \vdots \\ b_{j_q} \\ b_{k+1} \end{bmatrix}, \tag{2.11}$$

then, by (2.5) and (2.6),

$$rank\bar{A}_q = q$$

and

$$\lambda^T \bar{A}_q = 0, \quad (\lambda > 0).$$

Using (2.5) — (2.7) we obtain that

$$\lambda^T \bar{B}_q \geq \lambda^T \bar{A}_q Y^* = 0. \tag{2.12}$$

So (2.9) is a nonreducible inconsistent subsystem of (2.3) and λ is the informative vector[1].

3. Correction of Right-hand Side Vector of an Inconsistent System

Suppose (1.1) is inconsistent now. First we consider (2.1) is consistent but (2.3) is not. Using the isometric plane algorithm method, we find a nonreducibly inconsistent subsystem in the form (2.9), the informative vector (2.10), an interior point X^i of (2.1), and a boundary point Y^* satisfying (2.5). According to the notations of (2.11), (2.9) is written

$$\bar{A}_q X > \bar{B}_q, \quad (1 \leq q \leq n). \tag{3.1}$$

The basic theorem to correct the right-hand side vector $\bar{B}_q$ of (3.1) is as follows[1]:

Theorem 2. *Assume that* (3.1) *is a nonreducibly inconsistent system of linear inequalities and* λ *is the informative vector. Then system*

$$\bar{A}_q X > \bar{B}_q + \triangle \bar{B}_q$$

is consistent if and only if

$$\lambda^T (\bar{B}_q + \triangle \bar{B}_q) < 0.$$

The vector

$$\triangle\bar{B}_q = (\triangle b_{j_1}, , \cdots, \triangle b_{j_q}, \triangle b_{k+1})^T$$

is called correction vector of (3.1). Let, for instance

$$\triangle\bar{B}_q = (0, \cdots, 0, -\lambda^T \bar{B}_q - \varepsilon)^T = (0, \cdots, 0, -(b_{k+1} - a_{k+1} Y^* - \varepsilon)^T \tag{3.2}$$

where ε is the precision prescribed for numerical calculation. The correction vector (3.2) can make (2.9) and (2.3) consistent simultaneously.

4. Numerical Experiments

The system of linear equations formed by Hilbert matrix

$$AX = B, \ \ a_{ij} = 1/(i+j), \ \ b_i = \sum_{j=1}^{n} a_{ij} \tag{4.1}$$

has the exact solution

$$e_n = (1, \cdots, 1)^T.$$

In the experiments the right-hand side vector of (4.1) is perturbed with the perturbation quantity δ[3], hence (4.1) is replaced by

$$B - \delta e_n < AX < B + \delta e_n. \tag{4.2}$$

The experiments correcting (4.2) are completed on Micro VAX II by double-precision operation for different n and δ . The precision prescribed for numerical calculation and correction

$$\varepsilon \geq \varepsilon_0 \approx 10^{-16},$$

here ε_0 depends on the double-length of floating-point number of the computer. Clearly δ should also be no less than ε_0 for practical consistence of (4.2). In the program δ is first set with zero and (4.2) is corrected according to the formula (3.2) with $\varepsilon = 10^{-8}$. Since computation is affected by roundoff error, the objective point X^i of the isometric plane algorithm is probably moved outside the polyhedron Ω^K. If this phenomenon once occurs to some $k < 2n, \delta$ is automatically modified into some multiple of ε and the iteration begins again at $k = 1$. The experimental results show that the perturbation quantity δ needs to modify repeatedly. We thus obtain an estimate value of δ which makes (4.2) solvable.

The experimental results about correction of inconsistency and estimation of δ are listed in Table 1, where

i: order of the system (4.1),

$\|\triangle B\|_*$: maximum norm of the correction vector when $\delta = 0$ in (4.2),

δ_m : estimate value of the perturbation quantity

t_m : time (sec.) added up by the basic internal clocking function.

Sometimes the phenomenon of wrongly moving objective point occurs earlier than the correction of inconsistency, therefore it is possible that $\|\triangle B\|_* = 0$.

Table 1

n	10	20	30	40	50
$\|\triangle B\|_*$	10^{-8}	0	10^{-8}	10^{-8}	10^{-8}
δ_m	0.8×10^{-7}	0.128×10^{-5}	0.256×10^{-5}	0.64×10^{-6}	0.64×10^{-6}
t_m	3.02	16.92	48.98	67.04	133.57

n	60	70	80	90	100
$\|\triangle B\|_*$	10^{-8}	0	10^{-8}	10^{-8}	10^{-8}
δ	0.128×10^{-5}	0.64×10^{-6}	0.1024×10^{-4}	0.2048×10^{-4}	0.128×10^{-5}
t_m	182.53	272.75	496.63	755.04	770.93

In order to solve (4.2) we choose a proper δ with the aid of Table 1 and then again use the isometric plane algorithm to solve the following problem[2]

$$\begin{aligned}&-\xi\to\max,\\&AX+(B-\delta e_n-AX^t+\mu e_n)\xi>B-\delta e_n,\\&-AX+(-B-\delta e_n+AX^t+\mu e_n)\xi>-B-\delta e_n,\\&\xi\ge 0,\end{aligned}\tag{4.3}$$

where X^t is a given point in R^n and

$$\mu=\max_i|\sum_{j=1}^n a_{ij}X^t-b_i|+\delta+\theta_3,\quad(\theta_3>0).$$

If (4.3) has solution and the solution is $\begin{bmatrix}X^*\\0\end{bmatrix}$, then X^* is clearly a solution of (4.2). For the ill-conditioned matrix A, the isometric plane algorithm probably comes to the conclusion that the constraint polyhedron of (4.3) is not closed in the hyperplane $\xi=0$. Even if this mistake occurs, the isometric plane algorithm can usually record a better solution before the mistaken conclusion.

The experiments for solving (4.3) are also completed on Micro VAX II with different δ and

$$X^t=0,\quad\theta_3=1.$$

The results are listed in Table 2, where

n : order of the system (4.1),

δ : perturbation quantity of right-hand side vector,

ε : error bound solving (4.3)

$\|X^*-e_n\|_*$: maximum norm of the error vector between perturbation solution and exact solution,

nc : the number of iterative circles,

nr : the total number of iterations,

t_m : time (sec.) added up by the basic internal clocking function.

Table 2

n	10	20	30	40	50
δ	0.8×10^{-7}	0.64×10^{-6}	0.64×10^{-6}	0.64×10^{-6}	0.64×10^{-6}
ε	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$\|x^* - e_n\|_*$	0.002095	0.008634	0.007226	0.009641	0.003814
nc	15	13	13	15	9
nr	56	50	51	66	36
t_m	3.54	7.63	14.77	34.46	24.98

n	60	70	80	90	100
δ	0.64×10^{-6}	0.64×10^{-6}	0.64×10^{-6}	0.64×10^{-6}	0.64×10^{-6}
ε	0.5×10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$\|x^* - e_n\|_*$	0.000950	0.004476	0.007408	0.000917	0.000773
nc	14	5	9	10	5
nr	61	25	43	39	22
t_m	57.79	36.44	72.76	78.39	59.39

Acknowledgment. The authors thank professor Y.I. Shokin (Computing Center, Russian Academy of Sciences, Siberian Division) for his enthusiasm to submit the copy of the reference [1].

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