Journal of Computational Mathematics, Vol.13, No.2, 1995, 172-191.

THE PARTIAL PROJECTION METHOD IN THE FINITE ELEMENT DISCRETIZATION OF THE REISSNER-MINDLIN PLATE MODEL*1)

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Abstract

In the paper a linear combination of both the standard mixed formulation and the displacement one of the Reissner-Mindlin plate theory is used to enhance stability of the former and to remove "locking" of the later. For this new stabilitized formulation, a unified approach to convergence analysis is presented for a wide spectrum of finite element spaces. As long as the rotation space is appropriately enriched, the formulation is convergent for the finite element spaces of sufficiently high order. Optimal-order error estimates with constants independent of the plate thickness are proved for the various lower order methods of this kind.

1. Introduction

It is well known that the standard finite element discretizations of the Reissner-Mindlin plate problem produce poor approximations when the thickness is too small in comparison with the diameter of the region occupied by the midsection of the plate. The root is the so-called "locking" phenomenon which is by now well understood. Among several approaches to avoiding locking is a modification of the standard finite element schemes by interpolating or projecting the discrete transverse shear force into a lower-order finite element space. This kind of method has recently attracted strong research interest due to convenience of implementation and theoretical, experientical evidence.For the details, see [1], [3-5], [9-14] and the papers refered therein.

Both the projection method[1][11][12] and the interpolation method[3-5] are based on the introduction of the shear strain as a new variable. The benefit is that the nonuniform boundedness of the original Reissner-Mindlin variational functional changes into the uniform boundedness of the corresponding Lagrange energy functional with respect to the thickness. As is well known, the nonuniform boundedness of the original formulation leads to locking. But the strong point, i.e. the uniform boundedness of the Lagrange functional is obtained at the expense of the loss of the quadric term of

* Received December 22, 1994.
1) This research was sponsored by State Major Key Project for Basic Research.

primitive shear strain. Therefore it also leads to lack of coerciveness of the resulting discrete formulation which then turns into the difficulty of constructing stable finite element spaces. Based on this analysis, a further modification is considered in the paper. The method here is to divide the discrete shear force into two parts and to project only one of both into a lower-order finite element space. From the point of view of the generalized variational principle, the method can be interpreted as a discretization based on the combination of the original Reissner-Mindlin and the mixed variational principles. To distinguish this from the method in which the whole shear force is projected, the new method is refered to as the partial projection method. For this, a detailed discussion is given in the second section of the paper.

It will be shown that this formulation can be regarded as a reduction of the Hughes-Franca's stabilization technique[11]. More precisely, neglecting one of the two addtional stability terms in the Franca-Hughes's formulation, i.e. removing the least-square residual form of the moment equilibrium equation from their formulation, and expanding the remaining term give essentially the same terms that we get in a equivalent expression of the partial projection formulation. The difference is that the stabilization parameter here appears as a weighted factor independent of the plate thickness and finite element size, but the parameter in [11] is not so. In addition, this reduction leads to a method of Petrov-Galerkin type turning into one of Galerkin type which has foundation of variational principle.

For the case of the deflection interpolations of degree ≥ 2, the partial projection formulation is less versatile than the Hughes-Franca method and the finite element spaces can not be quite arbitrarily chosen, but additional convergence condition can be satisfied much more easily than that required in other mixed methods. As long as the rotation interpolation space is enriched with suitable bubble functions so that a quite simple inf-sup condition holds, this reduced formulation is convergent for any combination of the rotation, deflection and the shear force finite element spaces of sufficiently high order. In particular, the present approach avoids the restrictive condition $\nabla D_h \subset H_h$ required in most mixed methods (where D_h, H_h denote the deflection and the shear force finite element spaces respectively). This condition or its variants lead to rather severe difficulty so that the convergence analysis in the papers [1] [9] [12] can not be extended from the triangular plate elements to equally simple quadrilateral counterparts.

The features mentioned above are confirmed by two general convergence theorems established in section 3 and section 5. In the remaining sections, various applications are discussed. In particular a family of triangular and quadrilateral,conforming and nonconforming Reissner-Mindlin plate elements of order one is constructed, which includes the elements subjected to the discrete Kirchhoff constraint, and for every pair of the triangular and quadrilateral elements, optimal error estimates are achieved in a unified framework with constants independent of the plate thickness.

Following the paper [13], further modification of the methods of order one is considered to cancel the degrees of freedom associated with the bubble functions in the rotation finite element space. For the details, see the section 5 and 6.

2. New Variational Formulation of the Problem

We will use the standard notation for the Sobolev spaces and norms. Let Ω denote the region in R^2 occupied by the midsurface of the plate, and denote by w and β the transverse displacement of Ω and the rotation of fibers normal to Ω respectively. The original Reissner-Mindlin plate model determines w and β as the unique solution to the following variational problem:

Find $(w,\beta)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2$ such that:

$$\begin{aligned} B_1(w,\beta;v,\xi):&=a(\beta,\xi)+\lambda t^{-2}(\nabla w-\beta,\nabla v-\xi)\\ &=(g,v),\qquad \forall(v,\xi)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2 \end{aligned}\tag{2.1}$$

where

$$a(\beta,\xi)=\frac{E}{12(1-v^2)}\int_\Omega\Big\{v\mathrm{div}\beta\mathrm{div}\xi+\frac{1-v}{4}\sum_{i,j=1}^{2}\Big(\frac{\partial\beta_i}{\partial x_j}+\frac{\partial\beta_j}{\partial x_i}\Big)\Big(\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\Big)\Big\}d\Omega,$$

$(\cdot,\cdot)$ is the scalar product either in $L^2(\Omega)$ or $(L^2(\Omega))^2, t$ is the plate thickness, $\lambda=Ek/2(1+v)$ with E the Young's modulus, v the Poisson ratio, and k the shear correction factor, g is the transverse loading function scaled by a constant multiple of the cube of the thickness so that the solution tends to a nonzero limit as t tends to zero. For the details concerned with the original model, see[1][6].

For the problem (2.1), the mixed method is based on the introduction of the shear force:

$$\sigma=\lambda t^{-2}(\nabla w-\beta)$$

as a new variable. In this way, the problem (2.1) takes the equivalent form:

Find $(w,\beta,\sigma)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2\times V$, such that

$$\begin{aligned} B_2(w,\beta,\sigma;v,\xi,\tau):&=a(\beta,\xi)+(\sigma,\nabla v-\xi)-(\tau,\nabla w-\beta)+\lambda^{-1}t^2(\sigma,\tau)\\ &=(g,v),\qquad \forall(v,\xi,\tau)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2\times V \end{aligned}\tag{2.2}$$

where V is a suitable space satisfying

$$\begin{aligned} &H_0(\mathrm{rot};\Omega)\subset V\subset(L^2(\Omega))^2\\ &H_0(\mathrm{rot};\Omega):=\{\tau\in(L^2(\Omega))^2: rot\tau\in L^2(\Omega)\quad and\quad \tau\cdot s=0\quad on\quad \partial\Omega\}\\ &rot\tau:=\frac{\partial\tau_1}{\partial x_2}-\frac{\partial\tau_2}{\partial x_1}, \end{aligned}$$

s is the unit tangent to boundary $\partial\Omega$ of the domain Ω.

For simplicity of notation, we shall henceforth assume $\lambda = 1$. In what follows we denote by $C($ or $C_i)$ a constant independent of t and h (the characteristic parameter of finite element subdivision), but not necessarily the same at each occurrence.

The theorem which asserts the problem (2.2) (and consequently also the problem (2.1)) to be well posed was proven in [1][6]. For convenience, it can be written as follows:

Theorem 2.1 *Let Ω be a convex polygon or smoothly bounded domain in the plane. For any $t \in (0,1]$ and any $g \in H^{-1}$, there exists a unique triple $(w,\beta,\sigma) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times V$ solving problem (2.2). Moreover, there exists a constant C independent of t and g, such that:*

$$||w||_1 + ||\beta||_2 + ||\sigma||_0 \le C||g||_{-1}.$$

If $g \in L^2(\Omega)$ and the Helmholtz decomposition to the transverse shear force is $\sigma = \nabla r + curlp$,

$$||\beta||_2 + ||w||_2 + ||r||_2 + ||p||_1 + t||p||_2 \le C||g||_0 \ ,$$

and if $g \in H^1(\Omega)$ and Ω is smoothly bounded,

$$||r||_3 + ||w||_3 \le C||g||_1 \ .$$

Let $H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times V$ be equipped with the norm

$$||(v,\xi,\tau)||_{B_2} := (||(v,\xi)||_{B_1}^2 + t^2||\tau||_0^2)^{\frac{1}{2}}, \qquad \forall (v,\xi,\tau) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times V$$

where

$$||(v,\xi)||_{B_1} := (||\xi||_1^2 + ||\nabla v - \xi||_0^1)^{\frac{1}{2}}.$$

By virtue of the Korn's inequality, we have

$$C||(v,\xi)||_{B_1}^2 \le B_1(v,\xi;v,\xi) \le Ct^{-2}||(v,\xi)||_{B_1}^2$$

and

$$C(||(v,\xi,\tau)||_{B_2}^2 - ||\nabla v - \xi||_0^2 \le a(\xi,\xi) + t^2(\tau,\tau) = B_2(v,\xi,\tau;v,\xi,\tau) \\ \le C||(v,\xi,\tau)||_{B_2}^2.$$

In other words the quadric functional B_1 is coercive but not thickness-independently bounded, $B_2(v,\xi,\tau;v,\xi,\tau)$ is thickness-independently bounded but not coercive. For the "saddle-point" problem (2.2), though a weak coerciveness condition is sufficient, the following inf-sup condition of well-posedness

$$C||\tau||_u \le \mathrm{Sup}_{(v,\xi)\in H} \frac{(\tau,\nabla v - \xi)}{||(v,\xi)||_{B_1}}, \quad H = (H_0^1(\Omega)) \times (H_0^1(\Omega))^2$$

(where U is the dual space of $H_0(rot;\Omega)$), can be derived by using a technique in the paper[6]. However, its discrete analogue can not be generally established due to the discrete version of the Helmholtz decomposition theorem being untrue in general for the practically interesting finite element spaces. On the other hand, in order to fulfil the K-elliplicity condition in the mixed method theory, the assumption$\nabla D_h \subset H_h$

is required in all papers ([1][5][9][10][11]), where D_h and H_h denote the transverse displacement and shear strain finite element subspace respectively. The satisfaction of the condition implies that the number of degree of freedom of H_h is more than enough. But enrichment of the space H_h complicates the implementation and satisfaction of other convergence conditions. The root of these problems is the lack of the coerciveness of B_2 at the discrete level.

As an improvement, we consider another equivalent mixed variational formulation as follows:

Find $(w,\beta,\sigma)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2\times V$ such that:

$$\begin{aligned}B_3(w,\beta,\sigma;v,\xi,\tau):&=\alpha t^2B_1(w,\beta;v,\xi)+(1-\alpha t^2)B_2(w,\beta,\sigma;v,\xi,\tau)\\&=(g,v),\qquad\forall(v,\xi,\tau)\in H_0^1(\Omega)\times(H_0^1(\Omega))^2\times V\end{aligned}\tag{2.3}$$

where α is a weighted factor and $\alpha>0$.

Since

$$\begin{aligned}B_3=&\,a(\beta,\xi)+\alpha\int_\Omega(\nabla w-\beta)(\nabla v-\xi)d\Omega\\&+(1-\alpha t^2)[(\sigma,\nabla v-\xi)-(\tau,\nabla w-\beta)]+(1-\alpha t^2)t^2(\sigma,\tau),\end{aligned}$$

in contrast with B_2 , the coerciveness is enhanced by the additional term

$$\alpha\int_\Omega(\nabla w-\beta)(\nabla v-\xi)d\Omega.$$

This type of the stabilized method was referred to as the mixed method based on the homology family of variational principles in the paper[15]. Clearly the weighted factor α can be considered as a stabilizing parameter. In view of

$$B_2(w,\beta,\sigma;v,\xi,\tau)=B_1(w,\beta;v,\xi)-t^{-2}(t^2\sigma-\nabla w+\beta,-t^2\tau-\nabla v+\xi),$$

we get

$$B_3=B_2+\alpha(t^2\sigma-\nabla w+\beta,-t^2\tau-\nabla v+\xi).$$

Seting $\alpha=\tilde{\alpha}h^{-2}$ and $\tau=-\tilde{\tau}$ in the above expression and relating both the formulation (2.3) and the Hughes-Franca's Petrov-Galerkin method [11] to each other, we can see that the former is essentially a reduction of the later. In other words, a further modification of the Galerkin variational equation (2.3) by including least square residual form of the moment equilibrium equation leads to the Hughes-Franca's Galerkin least squares method [11] in which two stabilizing parameters are used. In addition, the parameter α in [11] seems to be considered as a penalty factor which must take the form of $\tilde{\alpha}h^{-2}$, and the previous analysis shows that as a weighted factor, α in (2.3) may take other forms. In the discussion bellow, we will take $\alpha=\tilde{\alpha}[max(h,t)]^{-2}$ and $\alpha=const.$ respectively for two cases of higher and low order displacement interpolations.

3. Finite Element Approximations and Convergence Analysis

For simplicity we assume henceforth that Ω is a convex polygon. Let $\{\Gamma_h\}_{0<h<1}$ be a regular family [7] of finite element subdivisions of Ω, and R_h, D_h and H_h be finite element spaces associated with Γ_h such that

$$D_h \subset H_0^1(\Omega), \quad R_h \subset (H_0^1(\Omega))^2, \quad H_h \subset (L^2(\Omega))^2.$$

Employing these spaces for the discretization of the problem (2.3), we obtain the following approximation scheme:

Find $(w_h, \beta_h, \sigma_h) \in D_h \times R_h \times H_h$ such that

$$B_3(w_h, \beta_h, \sigma_h; v, \xi, \tau) = (g, v), \qquad \forall (v, \xi, \tau) \in D_h \times R_h \times H_h \tag{3.1}$$

or equivalently

$$\begin{aligned} a(\beta_h, \xi) + \alpha(\nabla w_h - \beta_h, \nabla v - \xi) + (1 - \alpha t^2)(\sigma_h, \nabla v - \xi) &= (g, v), \\ &\forall (v, \xi) \in D_h \times R_h, \\ -(\tau, \nabla w_h - \beta_h) + t^2(\sigma_h, \tau) = 0, \qquad \forall \tau \in H_h & \end{aligned}$$

Let P be the orthogonal projection from $L^2(\Omega_e)$ to $H_h(\Omega_e)$ where $\Omega_e \in \Gamma_h$ is a finite element subdomain. It is easy to show that the last two relations can be jointly rewritten as follows:

$$\begin{aligned} a(\beta_h, \xi) + \alpha(\nabla w_h - \beta_h, \nabla v - \xi) + \frac{(1 - \alpha t^2)}{t^2}(P\nabla w_h - P\beta_h, \nabla v - \xi) \\ = (g, v), \qquad \forall (v, \xi) \in D_h \times R_h. \end{aligned} \tag{3.2}$$

If $\alpha > 0$, this is different from the Arnold & Falk method which is to project also the shear strain of $\alpha(\nabla w_h - \beta_h, \nabla v - \xi)$ into H_h. Based on this, the discretization (3.2) will be refered to as the partial projection method.

The existence and uniqueness of the finite element solution can be easily derived from the coerciveness of $B_3(\cdot, \cdot)$. As the convergence conditions we assume that

H1) there exist the operators:

$$\begin{aligned} \Pi_1 :& \quad H_0^1(\Omega) \to D_h(\Omega) \\ \Pi_2 :& \quad (H_0^1(\Omega))^2 \to R_h(\Omega) \\ \Pi_3 :& \quad (L^2(\Omega))^2 \to H_h \end{aligned}$$

such that for $m > 0$,

$$||(w - \Pi_1 w, \beta - \Pi_2 \beta, \sigma - \Pi_3 \sigma)||_{(1.1)} \leq Ch^m,$$

where

$$\begin{aligned} ||(v, \xi, \tau)||_{(q,r)}^2 :=& (||\xi||_1^2 + h_t^{-2q}||\nabla v - \xi||_0^2 + (t^2 + h^{2r})||\tau||_0^2) \\ h_t :=& \max(h, t). \end{aligned}$$

H2) The space R_h is large enough so that

$$\sup_{\xi \in R_h} \frac{(\tau, \xi)}{||\xi||_0} \geq C||\tau||_0, \quad \forall \tau \in H_h.$$

The following theorem is the main result of this section.

Theorem 3.1. *Let the spaces* $R_h \subset (H_0^1(\Omega))^2$, $D_h \subset H_0^1(\Omega)$ *and* $H_h \subset (L^2(\Omega))^2$ *be such that the hypotheses* $H1$ *and* $H2$ *hold, and* α *in (3.2) takes the form of* $\tilde{\alpha} h t^{-2}$. *If* $0 < \tilde{\alpha} < 1$, *then for any* $h > 0$ *and* $t \geq 0$,

$$||(w - w_h, \beta - \beta_h, \sigma - \sigma_h)||_{(1,0)} \leq C \frac{h^m}{\min(\tilde{\alpha}, 1 - \tilde{\alpha})} .$$

Remark 3.1. Following the idea in [11] [12], the parameter α can be as well assumed to be piecewise constant, i.e. α takes the form of $\tilde{\alpha}(\max(h_e, t))^{-2}$ where h_e denotes the diameter of $\Omega_e \in \Gamma_h$. For this choice, the theorem still remain valid if the nature energy norm $|| \cdot ||_{(q,r)}$ is considered as mesh-dependent norm.

Remark 3.2. The interpolation error conditoin ($H1$) is the same as used in the paper [11]. In comparision with the Hughes-Franca's method [11], the condition ($H2$) is additionally required, but in comparision with other mixed formulations, it is simple and easy to satisfy.

Proof. First we prove that there exists $(\Pi_1 w, \Pi_{2^*}\beta, \Pi_3\sigma) \in D_h \times R_h \times H_h$ such that

$$||(w - \Pi_1 w, \beta - \Pi_{2^*}\beta, \sigma - \Pi_3\sigma||_{(1,1)} \leq C h^m \tag{3.3}$$

and

$$\begin{aligned} &|(\tau, \nabla w - \nabla \Pi_1 w - \beta + \Pi_{2^*}\beta)| \\ &\quad \leq t||\tau||_0 \Big(\frac{||\nabla w - \nabla \Pi_1 w - \beta + \Pi_{2^*}\beta||_0}{h_t} \Big), \quad \forall \tau \in H_h. \end{aligned} \tag{3.4}$$

In order to find $\Pi_{2^*}\beta$, we introduce a linear operator

$$\Pi_4 : (L^2(\Omega))^2 \to R_h,$$

such that for $\tau \in (L^2(\Omega))^2$,

$$(\Pi_4 \tau, \beta) = (\tau, \beta), \qquad \forall \beta \in R_h. \tag{3.5}$$

By virtue of the assumption (H2), we get

$$C_0||\tau||_0 \leq ||\Pi_4\tau||_0 \leq ||\tau||_0, \quad \forall \tau \in H_h.$$

By the Lax-Milgram theorem, H_h−ellipticity:

$$(\Pi_4\tau, \Pi_4\tau) \geq C_0^2||\tau||_0^2, \quad \forall \tau \in H_h$$

implies that there exists a unique $\tau^* \in H_h$ such that

$$(\Pi_4\tau^*, \Pi_4\tau) = (\nabla w - \nabla\Pi_1 w - \beta + \Pi_2\beta, \tau), \quad \forall \tau \in H_h \tag{3.6}$$

and

$$\begin{aligned} ||\Pi_4\tau^*||_0^2 &\leq ||\tau^*||_0 \cdot ||\nabla w - \nabla\Pi_1 w - \beta + \Pi_2\beta||_0 \\ &\leq C_0^1||\Pi_4\tau^*||_0 \cdot ||\nabla w - \nabla\Pi_1 w - \beta + \Pi_2\beta||_0 \end{aligned}$$

hence

$$||\Pi_4\tau^*||_0 \leq C_0^{-1}||\nabla w - \nabla\Pi_1 w - \beta + \Pi_2\beta||_0 . \tag{3.7}$$

Now let us show that

$$\Pi_{2^*}\beta = \begin{cases} \Pi_2\beta - \Pi_4\tau^*, & \text{if } h \geq t, \\ \Pi_2\beta, & \text{if } h < t. \end{cases}$$

In fact, if $h \geq t$, combining the assumption (H1) with the relation (3.7) yields

$$\begin{aligned} ||\Pi_4\tau^*||_0 &\leq C_0^1(||\nabla w - \nabla\Pi_1 w||_1 + ||\beta - \Pi_2\beta||_0) \\ &\leq C(h_t h^m + h^{m+1}) \leq Ch^{m+1}. \end{aligned}$$

Then, by the inverse inequality, we have

$$||\beta - \Pi_{2^*}\beta||_1 \leq ||\beta - \Pi_2\beta||_1 + ||\Pi_4\tau^*||_1 \leq Ch^m.$$

Which together with the assumption (H1) give the estimates (3.3).

By the definition of $\Pi_4, (\Pi_4\tau^*, \Pi_4\tau) = (\Pi_4\tau^*, \tau)$, therefore (3.6) is equivalent to

$$(\nabla w - \nabla\Pi_1 w - \beta + \Pi_2\beta - \Pi_4\tau^*, \tau) = 0, \quad \forall \tau \in H_h$$

i.e., the estimate (3.4) holds.

For the case of $h < t$, owing to $\Pi_{2^*}\beta = \Pi_2\beta$, (3.3) is the very assumption (H1). By the Cauchy-Schwartz inequality and $h_t = t$, we get

$$\begin{aligned} |(\tau, \nabla w - \nabla\Pi_1 w - \beta + \Pi_{2^*}\beta| &\leq ||\tau||||\nabla w - \nabla\Pi_1 w - \beta + \Pi_{2^*}\beta||_0 \\ &\leq t||\tau||\frac{\nabla w - \nabla\Pi_1 w - \beta + \Pi_{2^*}\beta||_0}{h_t} . \end{aligned}$$

Then the proof of (3.3) and (3.4) is completed.

Now we are ready for proving the theorem itself.

Let us set $(v, \xi, \tau) = (\Pi_1 w - w_h, \Pi_{2^*}\beta - \beta_h, \Pi_3\sigma - \sigma_h) := (\delta w, \delta\xi, \delta\sigma)$ From the error equation:

$$B_3(w - w_h, \beta - \beta_h, \sigma - \sigma_h; v, \xi, \tau) = 0$$

we obtion

$$B_3(v, \xi, \tau; v, \xi, \tau) = -B_3(w - \Pi_1 w, \beta - \Pi_{2^*}\beta, \sigma - \Pi_3\sigma; v, \xi, \tau)$$

whose right hand side term is equal to

$$\Sigma_1 := a(\delta\beta, \delta\beta) + \alpha||\nabla\delta w - \delta\beta||_0^2 + (1 - \alpha t^2)t^2||\delta\sigma||_0^2.$$

Expanding the left hand side term, we have

$$\begin{aligned} \Sigma_1 &= a(\Pi_{2^*}\beta - \beta, \delta\beta) + (1 - \alpha t^2)t^2(\Pi_3\sigma - \sigma, \delta\sigma) \\ &+ \alpha(\nabla\Pi_1 w - \nabla w - \Pi_{2^*}\beta + \beta, \nabla\delta w - \delta\beta) \\ &+ (1 - \alpha t^2)[(\Pi_3\sigma - \sigma, \nabla\delta w - \delta\beta) - (\delta\sigma, \nabla\Pi_1 w - \nabla w - \Pi_{2^*}\beta - \beta)]. \end{aligned}$$

Recalling $\alpha = \tilde{\alpha}h_t^{-2}$, $t \leq h_t$ and $0 < (1 - \alpha t^2) \leq 1$, the following estimate can be derived by using the generalized Schwartz inequality:

$$\begin{aligned} \Sigma_1 &\leq C||(\Pi_1 w - w, \Pi_{2^*}\beta - \beta, \Pi_3\sigma - \sigma)||_{(1,0)} \cdot ||(\delta w, \delta\beta, \delta\sigma)||_{(1,0)} \\ &+ (1 - \alpha t^2)[h_t||\Pi_3\sigma - \sigma||_0 \cdot ||\nabla\delta w - \delta\beta||_0 \cdot h_t^{-1} \\ &+ \quad |(\delta\sigma, \nabla\Pi_1 w - \nabla w - \Pi_2^*\beta + \beta)|] . \end{aligned} \tag{3.8}$$

Applying (3.4) to the last term (3.8), we have

$$\begin{aligned}\Sigma_1 \leq & C[(||(\Pi_1 w - w, \Pi_{2^*}\beta - \beta, \Pi_3\sigma - \sigma)||^2_{(1,0)} \\ & + h_t^2||\Pi_3\sigma - \sigma||^2_0)]^{\frac{1}{2}} \cdot \sqrt{2}||(\delta w, \delta\beta, \delta\sigma)||_{(1,0)} \\ & + t||\delta\sigma||_0 \cdot \frac{||\nabla\Pi_1 w - \nabla w - \Pi_{2^*}\beta + \beta||_0}{h_t} \\ \leq & C||(\Pi_1 w - w, \Pi_{2^*}\beta - \beta, \Pi_3\sigma - \sigma)||_{(1,1)} \cdot ||(\delta w, \delta\beta, \delta\sigma||_{(1,0)} \\ & + t||\delta\sigma||_0 \cdot \frac{||\nabla\Pi_1 w - \nabla w - \Pi_{2^*}\beta + \beta||_0}{h_t} \\ \leq & C||(\Pi_1 w - w, \Pi_{2^*}\beta - \beta, \Pi_3\sigma - \sigma)||_{(1,1)} \cdot ||(\delta w, \delta\beta, \delta\sigma)||_{(1,0)} \quad . \end{aligned} \tag{3.9}$$

By the Korn inequality, we have

$$||(\delta w, \delta\beta, \delta\sigma)||^2_{(1,0)} \leq C(\min(\tilde{\alpha}, 1-\tilde{\alpha}))^{-1}\sum_1 .$$

Combining this with (3.9) and using the estimate (3.3), we obtain

$$\begin{aligned}||(\delta w, \delta\beta, \delta\sigma)||_{(1,0)} \leq & \frac{C}{\min(\tilde{\alpha}, 1-\tilde{\alpha})} \cdot ||(\Pi_1 w - w, \Pi_{2^*}\beta - \beta, \Pi_3\sigma - \sigma)||_{(1,1)} \\ \leq & \frac{C}{\min(\tilde{\alpha}, 1-\tilde{\alpha})} h^m \ .\end{aligned}$$

By the triangle inquality, this estimate together with (3.3) yields the desired uniform convergence error estimates.

Based on this theorem, checking the inf-sup condition (H2) is thus a important point of constructing various uniform convergent finite elements for the Reissner-Mindlin plate problem. The main task of remaining part of this section is to present a localized criterion for verification of the condition (H2).

By the element patch M (or a macroelement), we now mean the union of a fixed number of adjacent elements along any one of several well defined patterns. As well known, there exists 1-1 correspondence between these patterns and the so-called reference macroelements $\{\hat{M}\}$. Any element patch that is equivalent to $\hat{M}$ through a proper change variables will be classified into a set $\hat{M}_p$ of element patches. Because the subdivision Γ_h is regular, we can assume that for every subdivision $\Gamma_h (0 < h \leq 1)$, there is a fixed number R of the patch sets, $\{\hat{M}_p\} (p = 1, 2, \cdots R_0)$, such that

$$\Omega = \bigcup_{p=1}^{R_0} \bigcup_{\substack{\Omega_e \subseteq \hat{M}_p \\ \Omega_e \in \Gamma_h}} \Omega_e$$

and each $\Omega_e \in \Gamma_h$ is contained in not more than L element patches, L is independent of h.

Theorem 3.2. *Assume that for every set $\hat{M}_p$ and any $M \in \hat{M}_p$,*

$$\{\tau_i|M : \tau \in H_h \quad and \quad (\xi_i, \tau_i)_{(M)} = 0, \quad \forall \xi \in R_h \bigcap (H_0^1(M))^2\} = \{0\}. \tag{3.10}$$

or alternatively, for $\xi \in (H_0^1((M))^2$*, there exists* $\prod \xi \in R_h$ *such that*

$$(\xi - \prod \xi, \tau) = 0, \qquad \forall \tau \in H_h \tag{3.11}$$

and

$$||\prod \xi||_0 \leq C||\xi||_0,$$

then the condition (H2) holds.

This theorem is essentially an application of the general theory [18] [19] of localized criteria for checking the Babuska-Brezzi condition to a particular inequality of inf-sup form. It can be proved through the same steps and by using the same techniques as in [18]. For abbreviation of the space, the proof is omitted.

According to [18], the condition (3.10) can be referred to as rank-nondeficiency criterion, the condition (3.11) as interpolation criterion. (3.10) is a localized criterion but (3.11) is not in general unless H_h has simple construction such as piecewise constant spaces. For other references relative to the localized criteria, see [20] [21] in which divergence stability of inf-sup form was considered for the stokes problem and the identical approach was named macroelement technique.

Based on these theorems, the family of the Reissner-Mindlin plate elements of higher order can be constructed. This problem will be leaved for the forthcoming paper. In the remaining sections, only the elements of lower order are concerned.

Remarck 3.3. It is clear that the theory here can be as well applied to the so-called mixed interpolation methods [3-5] [14] due to permitting $H_h \subset H_0(rot;\Omega)$.

4. The Lower Order Conforming Elements

In this section we present two new pairs of uniformly accurate conforming elements for which $\nabla D_h \subset H_h$ does not hold. For the sake of simplicity , we denote the two pairs of elements by the notations PPM-I and PPM-II.

We use the standard notation for the spaces of polynomials, that is, P_k is the space of polynomials of degree less than or equal to k and Q_k is the space of polynomials of degree less than or equal to k in each variable.

For the triangular element T of PPM-I, We define the finite element subspaces as follows:

$$H_k = \{\tau \in (L^2(\Omega))^2 : \quad \tau|_T \in \bar{P}_0, \quad \forall T \in \Gamma_h\}, \tag{4.1}$$

$$D_h = \{v \in H_0^1(\Omega) : \quad v|_T \in P_2, \forall T \in \Gamma_h\}, \tag{4.2}$$

$$R_h = \{\xi \in (H_0^1(\Omega))^2 : \xi|_T \in \bar{P}_1 \oplus \bar{P}_0 b_T, \quad \forall T \in \Gamma_h\} \tag{4.3}$$

where b_T is a bubble function of degree 3, namely $b_T \in P_3$ and $b_T = 0$ on ∂T. Another alternative for b_T is a piecewise linear function with respect to three microelements determined by barycentric node of T (see [16]).

For the quadrilateral element Q of PPM-I, we define

$$H_h = \{\tau \in (L^2(\Omega))^2 : \quad \tau|_Q \in \bar{Q}_0, \quad \forall Q \in \Gamma_h\}, \tag{4.4}$$

$$D_h = \{v \in H_0^1(\Omega) : \quad v|_Q \in \bar{Q}_2', \quad \forall Q \in \Gamma_h\}, \tag{4.5}$$

$$R_h = \{\xi \in (H_0^1(\Omega))^2 : \quad \xi|_Q \in \bar{Q}_1' \oplus \bar{Q}_0 b_Q, \quad \forall Q \in \Gamma_h\} \tag{4.6}$$

where Q_1', Q_2' indicate respectively the 4-node isoparametric and 8-node subparametric 'serendipity' set; b_Q is a bubble function on Q, which can be defined as previously.

For this choice of finite element spaces, we have:

$$\sigma_h = t^{-2}(\text{meas}(\Omega_e))^{-1} \int_{\Omega_e} (\nabla w_h - \beta_h) d\Omega, \qquad \text{on} \quad \omega_e \in \Gamma_h \ . \tag{4.7}$$

Inserting this in (3.2) and seting $\alpha = \frac{1}{2h_t^2}$, the partial projection method can be formulated as follows:

Find $(w_h, \beta_h) \in D_h \times R_h$ such that:

$$\begin{aligned} & a(\beta_h, \xi) + (2h_t^2)^{-1}(\nabla w_h - \beta_h, \nabla v - \xi) \\ +(1 - \frac{t^2}{2h_t^2})t^{-2} & \sum_{\Omega_e \in \Gamma_k} (\text{meas}(\Omega_e))^{-1} \int_{\Omega_e} (\nabla w_h - \beta_h) d\Omega \cdot \int_{\Omega_e} (\nabla v - \xi) d\Omega \\ & = (g, v), \qquad \forall (v, \xi) \in D_h \times R_h \ . \end{aligned} \tag{4.8}$$

It is easy to see that by virtue of Theorem 3.2, employing the bubble function for R_h leads to satisfication of the condition (H2). Since the condition (H1) is also satisfied by virtue of the well known interpolation error estimates, by Theorem 3.1 and 2.1, we obtain the following optimal error estimate:

Proposition 4.1. For the element pair PPM-I based on the formulation (4.7) (4.8), we have

$$||\beta - \beta_h||_1 + h_t^{-1}|w - w_h|_1 + t||\sigma - \sigma_h||_0 \le Ch(||g||_0 + ||w||_3)$$

and if $t \le ch$, i.e. the plate is so called " numerical thin ",

$$|w - w_h|_1 \le Ch^2(||g||_0 + ||w||_3) \ .$$

Remark 4.1. In the paper [12] ,the corresponding error estimate for w_h takes the form of $\tilde{\alpha}\frac{1}{h^2}|w-w_h|_1^2 \le Ch^2$. For the case of $t \ge h$, the requirement of $\frac{1}{2}\frac{h^2}{t^2} \le \tilde{\alpha} \le \frac{h^2}{t^2}$ in [12] implies that the result in [12] is essentially the same as Proposition 4.1 .

It is easy to see that the PPM-I elements have degrees of freedom 12 and 16 respectively after the internal degrees of freedom of the rotation interpolations are eliminated by using static condensation. But as far as elements of order one, three degree of freedom at every vertice leads to optimal computational efficiency. Because of this, the remaining discussion of this section is focussed on the elements with discrete Kirchhoff constraint imposed at all the vertices, which will be named the PPM-II elements. The trianguler element of this kind was analyzed in [12] based on the standard mixed formulation. Our aim is to extend the analysis from the triangle to the quadrilateral in the unified framework introduced in the above section.

Let us denote the finite element spaces of PPM-II by $H_{h^*}, D_{h^*}, R_{h^*}$ respectively. For the triangular element, these spaces are the same as those considered in [12]. That is:

$$H_h^* = H_h \tag{4.9}$$

$$D_h^* \times R_h^* = \{(v,\xi) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 : (v,\xi)|_T \in Z(T) \times R_h(T) \quad and \quad \nabla v(a^i) = \xi(a^i) \quad i=1,2,3 \quad \forall T \in \Gamma_h\} \tag{4.10}$$

where H_h, R_h denote the spaces (4.1) (4.3) respectively, a^i denote the vertices of Ω_e. $Z(T)$ is the Zienkiewicz triangle (see [12] or [7]). Then $Z(T) \supset P_2$.

To define its quadrilateral counterpart, we need the following spaces:

$$H_h^* = H_h \tag{4.11}$$

$$D_h^* \times R_h^* = \{(v,\xi) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 : (v,\xi)|_Q \in A(Q) \times R_h(Q) \quad \text{and} \quad \nabla v(a^i) = \xi(a^i) \quad i=1,2,3,4 \quad \forall Q \in \Gamma_h\} \tag{4.12}$$

where H_h, R_h denote the spaces (4.4) and (4.6) respectively,

$$A(Q) := \{F_{\bar{Q}^1}\bar{v} : \quad \forall \bar{v} \in \hat{A}(\hat{Q})\}$$

F_Q is the geometric biliner mapping from the referential square $\hat{Q} = [-1,1] \times [-1,1]$ to the quadriilateral Q and $F_{\bar{Q}^1}$ denotes its inverse. $\hat{A}(\hat{Q})$ denotes the Adini square (see [7]).

Remark 4.2. Instead of $Z(T)$ or $A(Q)$, other C^0-continuous thin plate element can be chosen. For this replacement, the following conclusion still remains valid.

Now we check that all the assumptions of Theorem 3.1 are satisfied with the choises of finite element spaces (4.9)-(4.12). Since the rotation and shear force spaces of both PPM-I and PPM-II are the same, it is trival that the condition (H2) holds for PPM-II. For the condition (H1), it is sufficient to prove that there exist a pair of interpolation operators (Π_1, Π_2) such that

$$(\Pi_1 w, \Pi_2 \beta) \in D_h^* \times R_h^*$$

and

$$||\beta - \Pi_2\beta||_1 + h_t^{-1}||\nabla w - \nabla \Pi_1 w||_0 \le Ch(||\varphi||_0 + ||w||_3) \ .$$

In fact, let $\Pi_2\beta \in R_h^*$ be ordinary linear (or bilinear) interpolation of β and $(\Pi_1^* w, \Pi_1 w) \in (D_h^*)^2$ be defined by

$$\Pi_1 w(a^i) = w(a^i) = \Pi_1^* w(a^i), \quad \nabla \Pi_1 w(a^i) = \Pi_2\beta(a^i) = \beta(a^i),$$

and

$$\nabla \Pi_1^* w(a^i) = \nabla w(a^i).$$

It is clear that $||\beta - \Pi_2\beta||_1 \le Ch||\varphi||_0$. Since the relation $||\Pi_2(\nabla w - \beta)||_0 = 0$ implies that $||\nabla(\Pi_1^* - \Pi_1)w||_0 = 0$, it can be proved by using the scaling technique (see [18] [20]) that there exists a constant independent of h such that

$$||\nabla(\Pi_1^* - \Pi_1)w||_{0,\Omega_e} \le C||\Pi_2(\nabla w - \beta)||_{0,\Omega_e}, \qquad \forall \Omega_e \in \Gamma_h.$$

Then we have

$$\begin{aligned}||\nabla w - \nabla\Pi_1 w||_0 &\le ||\nabla w - \nabla\Pi_1^* w||_0 + ||\nabla(\Pi_1^* - \Pi_1)w||_0\\ &\le Ch^2||w||_3 + C||\Pi_2(\nabla w - \beta)||_0\\ &\le Ch^2||w||_3 + C||\nabla w - \beta||_0\\ &\le C(h^2||w||_3 + t^2||\sigma||_0)\end{aligned}$$

which implies that the condition (H1) holds at least for $t \le Ch$, i.e. the "numerical thin" case.

Proposition 4.2. For the PPM-II elements based on the fomulation (4.7) and (4.8), if $t \le Ch$, the following error estimate holds:

$$||\beta - \beta_h||_1 + h^{-1}||\nabla w - \nabla w_h||_0 + t||\sigma - \sigma_h||_0 \le Ch(||\varphi||_0 + ||w||_3).$$

5. Modification of the Lower Order PPM-Methods

The assumption (H1) implies that Theorem 3.1 is not suitable to the plate elements with the displacement interpolations of order < 2. In this section we extend the error analysis of section 3, and a unified approach to convergence analysis of the various nonconforming method including the Arnold & Falk's triangular element [1][9] and its Franca & Stenberg modification [13] is proposed without using the descrete Helmholtz decomposition [1] of the shear strain. Secondly, a low-order quadrilateral element which can couple with the nonconforming triangular element is constructed in section 6.

Let $\nabla_h v \in (L^2(\Omega))^2$ be the piecewise gradient of nonconforming interpolant v whose restriction to each element, T or Q, is equal to ∇v. For the nonconforming element, the PPM-finite element approximation corresponding to the Franca & Stenberg method [13] can be formulated as follows:

Find $(w_h, \beta_h, \sigma_h) \in D_h \times R_h \times H_h$ such that:

$$\begin{aligned}&a(\beta_h, \xi) + \alpha(\nabla_h w_h - \beta_h, \nabla_h v - \xi)\\ &\qquad + (1-\alpha t^2)(t^2 + \alpha^* h^2)^{-1}(P(\nabla_h w_h - \beta_h), \nabla_h v - \xi) = (g, v), \qquad (5.1)\\ &\sigma_h = (t^2 + \alpha^* h^2)^{-1} P(\nabla_h w_h - \beta_h), \qquad \forall (v, \xi) \in D_h \times R_h\end{aligned}$$

or equivalently

$$\begin{aligned}B_3^*(w_h, \beta_h, \sigma_h; v, \xi, \tau) :=& B_3(w_h, \beta_h, \sigma_h; v, \xi, \tau) + (1-\alpha t^2)\alpha^* h^2(\sigma_h, \tau)\\ =&(g, v), \quad \forall (v, \xi, \tau) \in D_h \times R_h \times H_h. \qquad (5.2)\end{aligned}$$

Where $\alpha^* > 0$ is another stabilizing parameter.

Theorem 5.1. *Let the space* $R_h \subset (H_0^1(\Omega))^2, D_h \subset L^2(\Omega)$ *and* $H_h \subset (L^2(\Omega))^2$ *satisfy the assumptions:*

H1′) there exist the interpolation operators:

$$\Pi_1 : H_0^1(\Omega) \to D_h, \quad \Pi_2 : (H_0^1(\Omega))^2 \to R_h, \quad \Pi_4 : (L^2(\Omega))^2 \to H_h$$

such that

$$(\nabla w - \nabla_h \Pi_1 w, \tau) = 0, \quad \forall \tau \in H_h,$$
$$(t+h)||\sigma - \Pi_4\sigma||_0 + ||\beta - \Pi_2\beta||_1 + ||\nabla w - \nabla_h \Pi_1 w||_0 \le Ch||g||_0 \quad .$$

H2′) *there exist a finite element space $E_h \subset H^1(\Omega)$ and another interpolation operator $\Pi_3 : H^1(\Omega) \to E_h$ such that*

$$CurlE_h := \{curlv : \quad \forall v \in E_h\} \subset H_h$$

and

$$||p - \Pi_3 p||_i \le Ch||p||_{i+1} \quad (i = 0, 1) \quad \forall p \in H^{i+1}(\Omega),$$

then there exists a unique solution (w_h, β_h, σ_h) of the problem (5.1) or (5.2) such that for $0 < \alpha < 1$ and $C \ge \alpha^* > 0$,

$$||\beta - \beta_h||_1 + ||\nabla w - \nabla_h w_h||_0 + (t+h)||\sigma - \sigma_h||_0 \le Ch||\varphi||_0.$$

Remark 5.1. If there exists the discrete Holmholtz decomposition $\tau = \nabla_h \bar{r} + curl\bar{p}$ (see [1]) for $\tau \in H_h$, the assumption associated with Π_1 is equivalent to that $\sup_{v \in D_h} \frac{(\tau, \nabla_h v)}{||\nabla_h v||_0} \ge C||\nabla_h \bar{r}||_0$ holds for $\tau \in H_h$.

Proof. Let us set

$$(\delta w, \delta\beta, \delta\sigma) := (\Pi_1 w - w_h, \Pi_2\beta - \beta_h, \Pi_3^*\sigma - \sigma_h) \ ,$$

where $\Pi_3^*\sigma = \Pi_4 \nabla r + Curl\Pi_3 p$ and $\sigma = \nabla r + Curlp$ is the Helmholtz decomposition.

By (H1′) and (H2′), we get

$$||\beta - \Pi_2\beta||_1 + ||\nabla w - \nabla \Pi_1 w||_0 + (t^2 + \alpha^* h^2)^{\frac{1}{2}} ||\sigma - \Pi_3^*\sigma||_0 \le Ch||g||_0 \ . \tag{5.3}$$

Owing to $D_h \not\subset H_0^1(\Omega)$, the error equation now includes consistency terms and it can be written as follows:

$$B_3^*(w - w_h, \beta - \beta_h, \sigma - \sigma_h; v, \xi, \tau)$$
$$= \sum_{\Omega_e \in \Gamma_h} \oint_{\partial\Omega_e} [\alpha(\nabla w - \beta) + (1 - \alpha t^2)\sigma] \cdot nvds + (1 - \alpha t^2)\alpha^* h^2(\sigma, \tau),$$
$$\forall (v, \xi, \tau) \in D_h \times R_h \times H_h$$

where n is the unit vector normal to the element boundary $\partial\Omega_e$.

Proceeding as in theorem 3.1 and using (5.3), we obtain the analogue of (3.8):

$$\sum_1 = a(\delta\beta, \delta\beta) + \alpha||\nabla_h \delta w - \delta\beta||_0^2 + (1 - \alpha t^2)(t^2 + \alpha^* h^2)||\delta\sigma||_0^2$$
$$\le Ch||g||_0||(\delta w, \delta\beta, \delta\sigma)||_{(0,1)}$$
$$+ \alpha\Big| \sum_{\Omega_e \in \Gamma_h} \oint_{\partial\Omega_e} (\nabla w - \beta) \cdot n\delta w ds\Big| + (1 - \alpha t^2)|\sum_2| \tag{5.4}$$

where

$$\sum_2 := (\Pi_3^*\sigma - \sigma, \nabla_h \delta w - \delta\beta) - (\delta\sigma, \nabla_h \Pi_1 w - \nabla w - \Pi_2\beta + \beta) + \sum_{\Omega_e \in \Gamma} \oint_{\partial\Omega_e} \sigma \cdot n \delta w ds$$

and the norm $||\cdot||_{(0.1)}$ should be considered as the nonconforming analogue of $||\cdot||_{(0,1)}$, i.e. $\nabla\delta w$ within is replaced by $\nabla_h \delta w$.

Because $h^{-1}||\nabla w - \beta||_0$ does not appear in the norm $||\cdot||_{(0,1)}$ and $t||\sigma||_1$ is uniformly bounded by theorem 2.1 but $||\sigma||_1$ is not, a treatment different from the proof of theorem 3.1 is needed for estimating $(\Pi_3^*\sigma - \sigma, \nabla_h\delta w - \delta\beta)$ in $\sum_2$.

Since $\quad \Pi_3^*\sigma = \Pi_4 \nabla r + curl \Pi_3 p$
and $\quad (\text{curl}(\Pi_3 p - p), \nabla_h \delta w) = \sum_{\Omega_e \in \Gamma_h} \oint_{\partial\Omega_e} \text{curl}(\Pi_3 p - p) \cdot n \delta w ds$,
we get:

$$\begin{aligned}(\Pi_3^*\sigma - \sigma, \nabla_h\delta w - \delta\beta) = &(\Pi_4\nabla r - \nabla r, \nabla_h \delta w - \delta\beta) \\ &- \sum_{\Omega_e\in\Gamma_h} \oint_{\partial\Omega_e} \text{curlp} \cdot n\delta w ds + (p - \Pi_3 p, \text{rot}\delta\beta) \\ &+ \sum_{\Omega_e\in\Gamma_h} \sum_{l\in\partial\Omega_e} \oint_l \text{curl}\Pi_3 p \cdot n \delta w ds\ . \end{aligned} \tag{5.5}$$

Since $\frac{\partial \Pi_3 p}{\partial s}|_l = (curl\Pi_3 p \cdot n)|_l = const.$ and $\oint_l (\delta w|_R - \delta w|_L) ds = 0$ where $\delta w|_R$ and $\delta w|_L$ denote the right and left hand side value of δw on l respectively, the fourth term of (5.5) is zero and by (H1$'$) (H2$'$), we have:

$$\begin{aligned}\Big|\sum_2\Big| =& \Big|(\Pi_4\nabla r - \nabla r, \nabla_h\delta w - \delta\beta) + (p - \Pi_3 p, \text{rot}\delta\beta) \\ &- (\delta\sigma, -\Pi_2\beta + \beta) + \sum_{\Omega_e\in\Gamma_h} \oint_{\partial\Omega_e} (\sigma - \text{curl}p) \cdot n\delta w ds\Big| \\ \leq& Ch(||r||_2 + ||p||_1 + ||\beta||_2)(||\text{rot}\delta\beta||_0^2 + (t^2+h^2)||\delta\sigma|| + ||\nabla_h\delta w - \delta\beta||_0^2)^{\frac{1}{2}} \\ &+ |\sum_{\Omega_e\in\Gamma_h} \oint_{\partial\Omega_e} \nabla r \cdot n\delta w ds|. \end{aligned} \tag{5.6}$$

In order to estimate the last term on the right side of (5.6) and the second-last term of (5.4), we use the following lemma (for the proof see [8] , or lemma 2.3 in [17]):

Lemma 5.1. *Let $\psi \in (H^1(\Omega))^2$ and $v \in D_h$ then:*

$$|\sum_{\Omega_e\in\Gamma_h} \oint_{\partial\Omega_e} (\psi \cdot n) v ds| \leq Ch||\psi||_1 ||\nabla_h v||_0\ .$$

It is easy to see that a direct application of both this lemma and Theorem 2.1 give an estimate for $\sum_2$ and consequently for $\sum_1$, namely

$$\sum_1 \leq Ch||g||_0||(\delta w, \delta\beta, \delta\sigma)||_{(0.1)}\ .$$

The same argument as in the proof of theorem 3.1 yields the desired conclusion. The theorem is proved.

Since the natural energy norm now includes $(t+h)||\sigma||_0$, by the argument used in Theorem 3.1 (see (3.8) and (3.9)), it is easy to see that the following conclusion holds:

Theorem 5.2. *If there exists* $(\Pi_1 w, \Pi_2\beta, \Pi_3\sigma) \in D_h \times R_h \times H_h \subset H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times V$ *such that*

$$h^{-1}||\nabla w - \nabla\Pi_1 w||_0 + ||\beta - \Pi_2\beta||_1 + (t+h)||_\sigma - \Pi_3\sigma||_0 \le Ch||\varphi||_0,$$

then for the solution (w_h, β_h, σ_h) *of the problem (5.1) with* $\alpha = \tilde{\alpha}h^{-2}(0 < \tilde{\alpha} < 1)$ *and* $0 < \alpha^* \le C$,

$$h^{-1}||\nabla w - \nabla w_h||_0 + ||\beta - \beta_h||_l + (t+h)||\sigma - \sigma_h||_0 \le \frac{Ch||\varphi||_0}{\min(\tilde{\alpha}, (1-\tilde{\alpha}t^2), \alpha^*)}\ .$$

Based on this theorem, it can be proved that a pair of elements, corresponding to PPM-I without the 'bubble' degree of freedom in R_h, is uniformly convergent.

6. The Lower-order Nonconforming Elements

Now we check that the assumptions of Theorem 5.1 are satisfied by a pair of the nonconforming elements. The triangular element of this pair was considered in [13]. As a modification of the Arnold & Falk element [1], the element is possessed of the simplest construction due to H_h, R_h being piecewise constant and C^0-linear interpolation space respectively as well as

$$\begin{aligned} D_h =&\{v \in L^2(\Omega) : v|_T \in P_1, \forall T \in \Gamma_h, \text{and v is continuous at} \\ &\text{midpoints of element edges and vanishes at midpoints} \\ &\text{of boundary edges}\}. \end{aligned} \tag{6.1}$$

Corresponding to this, for the quadrilateral element, we define

$$\begin{aligned} D_h =&\{v \in L^2(\Omega) : v|_Q \in Q_1^*, \forall Q \in \Gamma_h,\ \text{and}\ \oint_l (v_R - v_L)ds = 0 \\ &\text{for common edge l of any element pair}(Q_R, Q_L)\text{adjacent} \\ &\text{to each other, where} v_R = v|_{Q_R}, v_L = v|_{Q_L}\ \text{and}\ \oint_l dds = 0 \\ &\text{for any boundary edge l}\}. \end{aligned} \tag{6.2}$$

where $Q_1^* = \hat{Q}_1^*(F_Q^{-1}(x_1, x_2)),$ and $\hat{Q}_1^* = \text{Span}\{1, y_1, y_2, y_1^2 - y_2^2\}$.

$$R_h = \{\xi \in (H_0^1(\Omega))^2 : \xi|_Q \in \hat{Q}_1', \forall Q \in \Gamma_h\}$$
$$H_h = \{\tau \in (L^2(\Omega))^2 : \tau|_Q \in \hat{Q}_0^*, \forall Q \in \Gamma_h\}$$

where

$$Q_0^* = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \oplus c_3 \text{curl}\psi_0 : \quad \forall c_1, c_2, c_3 \in R \right\},$$

$$\psi_0 = \hat{\psi}_0(F_Q^{-1}(x_1, x_2)), \quad \hat{\psi}_0(y_1, y_2) = 4^{-1}(1 + y_1)(1 + y_2), X = (x_1, x_2)^T = F_Q(y_1, y_2)$$

is the geometric bilinear mapping from the referential square $[-1, 1] \times [-1, 1]$ to the quadrilateral Q and F_Q^{-1} denotes its inverse, M^T denotes the transposed matrix of M.

The definition (6.2) implies that $|l|^{-1} \oint_l v ds$, instead of the value on the midpoint of the edge, is employed as a degree of freedom for the interpolated polynomial. For any linear function v, $|l|^{-1} \oint_l v ds = v$ on the midpoint of l.

Thus the same sort of degree of freedom is actually used for both the triangular and the quadrilateral elements and there is no problem for their coupling. This pair of nonconforming elements will be named the PPM-II.

Why D_h in (6.2) can be constructed by means of Q_1^* is answered by the following lemma:

Lemma 6.1. *Any nonconforming function v in the space D_h in (6.2) is uniquely determined by all possible edge integrals $|l|^{-1} \oint_l v ds$. Moreover, for $w \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists $\Pi_1 w \in D_h$ such that*

$$\oint_l (w - \Pi_1 w) ds = 0$$

and

$$||w - \Pi_1 w||_i \leq C h^{2-i} ||w||_2, \quad i = 0, 1.$$

Proof. For $v \in D_h, \hat{v} = v(F_Q(y_1, y_2)) \in Q_1^*$ implies that there exist four constants (a, b, d, e) such that:

$$\hat{v} = a + b y_1 + d y_2 + e(y_1^2 - y_2^2) \quad .$$

Since

$$ds_x = \Big|\frac{\partial F_Q}{\partial L}\Big| ds_y = 2^{-1}|l| ds_y$$

and hence

$$|l|^{-1} \oint_l v ds_x = 2^{-1} \oint_{\hat{l}} \hat{v} ds_y$$

where $\hat{l}$ indicates the inverse image of l through the one-to-one mapping $X = F_Q(Y)$, it is easy to prove that $a = b = d = e = 0$ is sufficient and necessary to ensure $\oint_{\hat{l}} \hat{v} ds_y = 0$ for any edge $\hat{l}$ of the square $[-1, 1] \times [-1, 1]$. The first conclusion is proved.

By virtue of the general theory on interpolation approximation [7], it is well known that the second conclusion is a straightforword consequence of the first.

As shown in [10], it is clear that the triangular element satisfies the assumptions (H1′) (H2′).

For the quadrilateral counterpart, since

$$(\tau, \nabla v - \nabla \Pi_1 v) = \sum_Q \Big[\oint_{\partial Q} (v - \Pi_1 v)(\tau \cdot n) ds - \int_Q (v - \Pi_1 v)(\text{div}\tau) d\Omega \Big]$$

and for $\tau \in H_h$, $\quad \tau \cdot n = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cdot n + c_3 \frac{\partial \psi_0}{\partial s}$ is constant on every edge of $Q \in \Gamma_h$ as well as $(div\tau)|_Q = 0$, by Lemma 6.1 we get:

$$\sum_Q \int_Q \tau \cdot (\nabla v - \nabla \Pi_1 v) d\Omega = 0, \qquad \forall \tau \in H_h \ .$$

Then the quadrilateral element satisfies H1$'$.

The hypothesis H2$'$ is guaranteed by the following lemma:

Lemma 6.2. *Let* $E_h = \{v \in H_0^1(\Omega) : \quad v|_Q \in Q_0', \quad \forall Q \in \Gamma_h\}$, *then*

$$\mathrm{curl} E_h \subset H_h.$$

Proof. According to the definition of $v \in E_h$, we have

$$v(x_1, x_2) = \hat{v}(F_Q^{-1}(x_1, x_2)) \quad \text{on} \quad Q \in \Gamma_h$$

where the bilinear function

$$\hat{v}(y_1, y_2) = \sum_{(i,j)} v_{(i,j)} \psi_{(i,j)}, \quad \psi_{(i,j)} = 4^{-1}(1 + iy_1)(1 + jy_2),$$
$$(i,j) = (1,1), (-1,1), (-1,-1), (1,-1), \quad v_{(i,j)} = v|_{X(i,j)}$$

and

$$X = F_Q(y_1, y_2) = \sum_{(i,j)} X_{(i,j)} \psi_{(i,j)} = [X_{(i,j)}]_{(i,j)}^T \cdot [\psi_{(i,j)}]_{(i,j)}^T,$$

$X_{(i,j)}$ are the geometric coordinate vector of the vertices of the quadrilateral Q corresponding to vertices of the referential square.

Let us introduce the matrix-vector notations:

$$G_c = \begin{bmatrix} x_1^{(1,1)} & \cdots & x_1^{(1,-1)} \\ x_2^{(1,1)} & \cdots & x_2^{(1,-1)} \end{bmatrix}, \qquad G_j = \nabla_y [\psi_{(i,j)}]_{(i,j)} := \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{bmatrix} [\psi_{(i,j)}]_{(i,j)}$$

$$q_v^T = (v_{(1,1)}, v_{(-1,1)}, v_{(-1,-1)}, v_{(1,-1)}), \qquad I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} := \begin{bmatrix} I_1 \\ I_0 \end{bmatrix}.$$

By some simple calculations, we get:

$$curl v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Big[\sum_{t=1}^{2} \frac{\partial \hat{v}}{\partial y_t} \frac{\partial y_t}{\partial x_i} \Big]_{i=1,2}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (G_f \cdot G_c^T)^{-1} \nabla_y \hat{v}$$

in which $[\frac{\partial y_t}{\partial x_i}]_{i,j} = [\frac{F_Q^{(i)}}{\partial y_t}]_{i,t}^{-1} = (G_c \cdot G_f^T)^{-1}$ is used.

Since the 4×4 matrix $[G_c^T, I_2^T]$ is rank-full due to the partition Γ_h being regular, there exists a constant vector $q_c = (c_1, c_2, c_3, c_0)^T$ such that

$$q_v = G_c^T \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_3 I_1^T + c_0 I_0^T,$$

then we get

$$\mathrm{curl} v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (G_f \cdot G_c^T)^{-1} \nabla_y (\hat{v} - c_3)$$

and

$$\nabla_y(\hat{v} - c_3) = \nabla_y \sum_{(i,j)} (v_{(i,j)} - c_3) \frac{(1 + iy_1)(1 + jy_2)}{4}$$

$$= G_f(q_v - c_3 I_1^T) = G_f\Big(G_c^T \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 I_0^T\Big)$$

which implies that

$$\mathrm{curl} v = \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} + c_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (G_f G_c^T)^{-1} G_f I_0^T$$

$$= \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} + c_0 \mathrm{curl} \psi_0 \quad \in H_h(\Omega) \ .$$

The lemma is proven.

Sumarizing these results, we have the following conclusion:

Proposition 6.1. For the PPM-III elements based on the formulation (5.1) with the stabilizing parameters: $0 < \alpha \leq 1$ and $0 < \alpha^* \leq C$,

$$||\beta - \beta_h||_1 + ||\nabla w - \nabla_h w_h||_0 + (t + h)||\sigma - \sigma_h^{(\alpha)}||_0 \leq \frac{Ch||g||_0}{\min(\alpha, 1 - \alpha t^2, \alpha^*)}.$$

Acknowledgment. The author is very grateful to the anonymous referee for valuable suggestions leading to an improved presentation of this paper.

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