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A FINITE DIFFERENCE METHOD FOR THE MODEL OF WHEEZES*)

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Abstract

In this paper, a finite difference scheme for the linear and nonlinear models of wheezes are given. The stability of the finite difference scheme for the linear model is obtained by using of von Neumann method. Moreover, the convergence and stability of the finite difference scheme for the nonlinear model are studied by the energy inequalities method. By some numerical computations, the relationships between angular frequency and wall position, fluid speed and amplitude are discussed. Finally, the author shows that the numerical results are coincided with Grotberg's theoretical results.

1. Introduction

In order to study the pitch of wheezes in patients, J.B.Grotberg and others have given a class of mathematical model of wheezes[1,2]:

$$\begin{cases} u = \Phi_x, \qquad w = \Phi_z, & (1.1)\\ \triangle\Phi = 0, & (1.2)\\ \Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + 2R_f\Phi + P - P_a = 0, & (1.3)\\ MW_{tt} + 2R_wW_t + BW_{xxxx} + 1 + W + \beta(1+W)^3 - TW_{xx} + P - P_e = 0. & (1.4)\end{cases}$$

Where $\triangle$ and ∇ are the Laplace operator and gradient operator, respectively. The Cartesian components (u, w) are the dimensionless axial fluid velocity and dimensionless vertical fluid velocity respectively. $\Phi(x,z,t)$ is the velocity potential function, P is the dimensionless fluid pressure determined from the unsteady Bernoulli equation (1.3), P_a is the steady driving pressure, P_e is the external pressure. M, R_w, B, β and T are wall-to-fluid mass ratio, dimensionless wall damping coefficient, bending stiffness to elastance ratio, nonlinear elastance coefficient and applied longitudinal tension to elastance ratio, respectively. The geometry and physical parameters of the problem are indicated in Fig.1.

Fig. 1

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Generally, $\Phi(x,z,t)$ is a travelling wave like in [1,2]:

$$\Phi(x,z,t) = Ai\frac{\omega - kS}{k\sinh k}\cosh(kz)\mathrm{e}^{\mathrm{i}\theta} \tag{2}$$

where ω is the dimensionless angular frequency, k is the wave number defined by $k = 2\pi b/L$ and L is the dimensionless wavelength, S is the dimensionless fluid speed, A is an arbitrary constant, $\theta = kx - \omega t$. It is very easy to check that (2) satisfies (1.2). Let $P_e = P_a - 2R_f Sx - S^2/2$[1]. In this paper, we shall discuss the periodic problem with the travelling wave $\Phi(x,z,t)$, then equation (1) can be rewritten as the following:

$$\begin{cases} MW_{tt} + 2R_w W_t + BW_{xxxx} + q(1+W) - TW_{xx} = f(x,t), & (x,t)\epsilon R\times I, \quad (3.1)\\ W(x+\lambda,t) = W(x,t), & (x,t)\epsilon R\times I, \quad (3.2)\\ W(x,0) = W_0(0), & x\epsilon R, \quad (3.3)\\ W_t(x,0) = W_t(x), & x\epsilon R. \quad (3.4)\end{cases}$$

Where $f(x,t) = [\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + 2R_f\Phi]\ |_{z=-1}\ -(2R_f Sx + S^2/2)$ and $q(1+W) = 1 + W + \beta(1+W)^3$. W is an unknown function. R is the real line and $I = [0, T_1]$.

J.B.Grotberg and other authors[1,2] discussed the standing wave solutions for system (1). While the solutions are not standing wave, the methods in [1,2] are invalid. Zho Yulin[3] gave the weighted difference schemes, but the weighted coefficient $\alpha islimitedin[\frac{1}{2},1]$. Therefore the difference schemes are implicit. In this paper, we shall give an explicit difference approximation and study the convergence and stability.

2. Finite Difference Scheme and It's Stability for the Linear Equation

Let τ, h be time-step and space-step lengths, respectively. $J = [\lambda/h]$, $N = [T_1/\tau]$. $x_j = jh$, $t_n = n\tau$. $0 \le j \le J$, $0 \le n \le N$. Where $[y]$ denotes the larger integer which is not greater than y. Notation W_j^n is the approximation of $W(x_j,\ t_n)$. A finite difference scheme is given as follows:

For $0 \le j \le J$

$$\begin{cases} MW^n_{jt\bar{t}} + 2R_w W^n_{j\hat{t}} + BW^n_{jxx\bar{x}\bar{x}} + q(1+W^n_j) - TW^n_{jx\bar{x}} = f^n_j, & 0 < n < N, \quad (4.1)\\ W^n_{j+rj} = W^n_j, & 0 \le n \le N, \quad (4.2)\\ W^0_j = W_0(x_j), & (4.3)\\ W^0_t = W_t(x_j). & (4.4)\end{cases}$$

Where $W^n_{jt} = (W^{n+1}_j - W^n_j)/\tau$, $W^n_{j\bar{t}} = (W^n_j - W^{n-1}_j)/\tau$ and $W^n_{j\hat{t}} = (W^{n+1}_j - W^{n-1}_j)/(2\tau)$. Similarly, we can define W^n_{jx} and $W^n_{j\bar{x}}$.

In the following we consider the von Neumann stability of the finite difference scheme for the linear equation of (3), *i.e.* $q(1+W) = 1+W$, $\beta = 0$. Say $e^n_j = W^n_j - \hat{W}^n_j$, where W^n_j and $\hat{W}^n_j$ are the solution of the finite difference scheme for the linear equation and the solution of the finite difference scheme for the linear equation with some perturbation. Let $e_j^n = G^n \mathrm{e}^{\mathrm{i}j\eta}$. Where i is the imaginary unit. Then e_j^n satisfies

$$M G^n_{t\bar{t}} \mathrm{e}^{ij\eta} + 2R_w G^n_{\hat{t}} \mathrm{e}^{ij\eta} + BG^n \mathrm{e}^{ij\eta}_{xx\bar{x}\bar{x}} - TG^n \mathrm{e}^{ij\eta}_{x\bar{x}} = 0. \tag{5}$$

We rewrite (5) as follows

$$(M + \tau R_w)G^2 + [\tau^2 - 2M + 4Tc^2 \sin^2 \frac{\eta}{2} + 16Bs^2 \sin^4 \frac{\eta}{2}]G + (M - \tau R_w) = 0, \tag{6}$$

where $c = \tau/h,\ s = \tau/h^2$.

For convenience sake, we use a, b and c to denote the coefficients of G in (6), so G can be written as $G = (-b \pm \sqrt{\triangle})/(2a)$, here

$$\triangle = b^2 - 4ac = 4[M - 2Tc^2 \sin^2 \frac{\eta}{2} - 8Bs^2 \sin^4 \frac{\eta}{2} - \tau^2/2]^2 - 4(M^2 - \tau^2 R_w^2).$$

In physical view[1,2], $R_w \ll \min(B,\ M)$. Therefore it is reasonable to assume that $R_w^2 < M$. If $\tau < \min(2\sqrt{M - R_w^2},\ 1)$ and s and c satisfy that $2Tc^2 + 8Bs^2 + \frac{1}{2} < M$, then we have

$$M - 2Tc^2 \sin^2 \frac{\eta}{2} - 8Bs^2 \sin^4 - \frac{\tau^2}{2} > M - 2Tc^2 - 8Bs^2 - \frac{1}{2} > 0.$$

By our assumption we obtain that

$$\triangle < 4(M - \frac{\tau^2}{2})^2 - 4(M^2 - \tau^2 R_w^2) = 4\tau^2(\frac{\tau^2}{4} - M + R_w^2) \le 0.$$

Therefore $G = (-b \pm i\sqrt{-\triangle})/(2a)$, we have

$$|G|^2 = (b^2 + 4ac - b^2)/(4a^2) = c/a = \frac{M - \tau R_w}{M + R_w} < 1. \tag{7}$$

In the linear case, it is very easy to check that (4) is consistent with (3). Therefore, we have :

Theorem 1. *For the linear case, assume the coefficients of the equation* (3) *satisfy* $R_w^2 < M$. *Then if* $\tau < \min(2\sqrt{M - R_w^2}, 1)$ *and* s *and* c *satisfy that* $2Tc^2 + 8Bs^2 + \frac{1}{2} < M$, *the finite difference solution of* (4) *is stable and convergent. Moreover, its convergent order is* $O(h^2 + \tau^2)$.

Remark 1. von Neumann stability and the linear stability of [1,2] are coincided. To study the former is to study the increasing ratio of the finite difference solution with time increase. While the later is the increasing ratio of the solution of the system with the time increase. But we concern the linear system with the travelling wave solution in both cases.

Remark 2. In the numerical experiment in [1], $M = 3193$, we can calculate $R_w = 0.026528687$. In many examples in [1,2], $R_w = 0$. So, the assumption in the theorem 1 for the coefficients of M, R_w is reasonable.

3. Theoretical Analysis for the Difference Approximation

Lemma 1. *For all lattice function u^n, the following identities hold:*

$$2\text{Re}(u^n_{t\bar{t}},\, u^n) = \|u^n_t\|^2_{\bar{t}}, \tag{8.1}$$
$$2\text{Re}(u^n,\, v^n_{\hat{t}}) = \|u^n\|^2_{\hat{t}} - \frac{\tau^2}{2}\|u^n_t\|^2_{\bar{t}}. \tag{8.2}$$

Where $(u^n,\, v^n) = h\sum_{j=1}^{J} u^n_j \overline{v^n_j}$, $\|u^n\|^2 = (u^n,\, u^n)$, Re *is the real part of the complex number.*

Lemma 2. *For all lattice function u^n, v^n, we have:*

$$(u^n_{x\bar{x}},\, v^n) = -(\nabla u^n,\, \nabla v^n). \tag{9}$$

The periodic Sobolev space is defined by: $H^m_p(I) = \{u\epsilon H^m_{loc}(\mathbf{R}) \mid u(x+\lambda) = u(x)\}$. Say $|u^n|^2_j = \|u^n_{x^j}\|^2$, $\|u^n\|^2_m = \sum_{j=0}^{m} |u^n|^2_j$. Where $u^n_{x^j}$ denotes the j'th order difference quotient of u^n with respect to x.

Lemma 3. *Let $r = \tau/h^{\alpha-\beta}$, $\alpha \geq \beta+1$, $C > 0$. For arbitrary $\epsilon > 0$, when u^n_j satisfies* (4.2), *it follows:*

$$C\frac{\tau^2}{2}|u^n_t|^2_\alpha \leq \epsilon(|u^n|^2_\alpha + |u^{n+1}|^2_\alpha) + \frac{C^2\tau^2}{2\epsilon}4^{\alpha-\beta-1}r^2|u^n_t|^2_\beta \tag{10}$$

Proof. By the ϵ inequality, we have:

$$C\frac{\tau^2}{2}|u^n_t|^2_\alpha \leq \frac{\epsilon\tau^2}{2}|u^n_t|^2_\alpha + \frac{C^2\tau^2}{8\epsilon}|u^n_t|^2_\alpha \tag{11}$$

The definition of the difference quotient implies

$$\frac{\epsilon\tau^2}{2}|u^n_t|^2_\alpha \leq \epsilon(|u^n|^2_\alpha + |u^{n+1}|^2_\alpha). \tag{12}$$

By using the definition of norm, we get:

$$\frac{C^2\tau^2}{8\epsilon}|u^n_t|^2_\alpha \leq \frac{r^2C^2}{8\epsilon}\cdot(1+1)^{2(\alpha-\beta)}|u^n_t|^2_\beta = \frac{C^2}{2\epsilon}4^{\alpha-\beta-1}\rho^2|u^n_t|^2_\beta \tag{13}$$

Formulas (11)—(13) implies that (10) holds.

Theorem 2. *Assume that $q\epsilon\mathrm{C}^1(\mathbf{C})$, this means for arbitrary $\epsilon > 0$, there exists $\sigma_1 > 0$. If $M - T^2c^2/(T-2\epsilon) - 4B^2s^2/(B-2\epsilon) - \epsilon/2 \geq \sigma_1$ and $M, R_w, T, B > 0$, W and $\hat{W}$ are solution of* (4) *and* (4)*with the perturbations $\tilde{f}$, $\hat{W}_0$ and $\hat{W}_t$, then there exists a positive constant C, independent of h, τ, such that*

$$\sup_{0\leq n\leq N}(\|W^n_t - \hat{W}^n_t\| + \|W^n - \hat{W}^n\|_2) \leq C(\|W^0_t - \hat{W}^0_t\| + \|W^0 - \hat{W}^0\|_2 + \max_{0\leq n\leq N}\|\tilde{f}\|). \tag{14}$$

Where $s = \frac{\tau}{h^2}$, $c = \frac{\tau}{h}$.

Proof. Set $e^n = W^n - \hat{W^n}$, by (4.1) we have that e^n satisfies

$$Me^n_{t\bar{t}} + 2R_w e^n_{\hat{t}} + Be^n_{xx\overline{xx}} + Q(e^n) - Te^n_{x\bar{x}} = \tilde{f^n}, \tag{15}$$

where $Q(e^n) = q(1 + W^n) - q(1 + \hat{W^n})$.

Taking the inner product between (15) and $2e^n_{\hat{t}}$, we have

$$2M(e^n_{t\bar{t}},\ e^n_{\hat{t}}) + 4R_w(e^n_{\hat{t}},\ e^n_{\hat{t}}) + 2B(e^n_{xx\overline{xx}},\ e^n_{\hat{t}}) - 2T(e^n_{x\bar{x}},\ e^n_{\hat{t}}) = 2(\tilde{f^n} - Q(e^n),\ e^n_{\hat{t}}). \tag{16}$$

Taking the real part for (16), Lemma 1 and Lemma 2 implies

$$M\|e^n_t\|^2_{\bar{t}} + 4R_w\|e^n_{\hat{t}}\|^2 + B(|e^n|^2_{2\bar{t}} - \tau^2/2|e^n_t|^2_{2\bar{t}}) + T(|e^n|^2_{1\bar{t}} - \tau^2/2|e^n_t|^2_{1\bar{t}}) = 2\mathrm{Re}(\tilde{f^n} - Q(e^n),\ e^n_{\hat{t}}). \tag{17}$$

Summation of (17) with respect to n from 1 to m $(m < N)$, yields

$$\rho^m + 4\tau R_w \sum_{n=1}^{m} \|e^n_{\hat{t}}\|^2 = \rho^0 + 2\tau \sum_{n=1}^{m} \mathrm{Re}(\tilde{f^n} - Q(e^n),\ e^n_{\hat{t}}), \tag{18}$$

where

$$\rho^m = M\|e^m_t\|^2 + \frac{B}{2}(|e^{m+1}|^2_2 + |e^m|^2_2 - \frac{\tau^2}{2}|e^m_t|^2_2) + \frac{T}{2}(|e^{m+1}|^2_1 + |e^m|^2_1 - \frac{\tau^2}{2}|e^m_t|^2_1).$$

We first assume that $q\epsilon C^1_b(\mathbf{C})$(the definition of $C^1_b(\mathbf{C})$ can be seen in [4]), then by the ϵ inequality, we have:

$$\mathrm{Re}(\tilde{f^n} - Q(e^n),\ e^n_{\hat{t}}) \le \epsilon_1(\|e^n_t\|^2 + \|e^{n-1}_t\|^2) + (1 + C_1^2)/\epsilon_1(\|e^n\|^2 + \|\tilde{f^n}\|^2). \tag{19}$$

By Lemma 3 and the assumption of the theorem we get

$$\eta^m \le \rho^m \le C_2^2(\|e^m_t\|^2 + \|e^{m+1}\|^2_2 + \|e^m\|^2_2), \tag{20}$$

where $\eta^m = \sigma_1\|e^m_t\|^2 + \epsilon(|e^{m+1}|^2_2 + |e^m|^2_2 + |e^{m+1}|^2_1 + |e^m|^2_1)$.

$e^1 = e^0 + \tau e^0_t$ implies:

$$\|e^1\|^2 \le 2(\|e^0\|^2 + \tau^2\|e^0_t\|^2). \tag{21}$$

By(18)—(21), we have:

$$\eta^m \le C_3(\|e^0\|^2_2 + \|e^0_t\|^2 + \max_{1\le n\le N}\|\tilde{f^n}\|^2) + \tau C_4 \sum_{n=1}^{m} \|e^n_t\|^2. \tag{22}$$

Application of Gronwall's inequality for (22), implies

$$\eta^m \le C_3 \mathrm{e}^{C_4 T_1}(\|e^0\|^2_2 + \|e^0_t\|^2 + \max_{1\le n\le N}\|\tilde{f^n}\|^2).$$

Because that $\|e^{n+1}\| \le \|e^n\| + \tau\|e_t^n\|$, then

$$\|e^m\| \le \|e^0\| + \tau \sum_{n=1}^{m} \|e_t^{n-1}\| \le \|e^0\| + T_1 \max_{1\le n\le N} \|e_t^n\|. \tag{23}$$

By (22) and (23), the theorem holds for $q\epsilon \mathrm{C}_\mathrm{b}^1(\mathbf{C})$.

Finally, similarly to the standard argument in [4,5,6], we can remove the hypothesis that q and its first derivative are bounded. This completes the proof of the theorem.

Similarly to the proof of theorem 2, we have:

Theorem 3. *Assume that the solution of the equation* (1) *is in* $\mathrm{C}^4(0, \mathrm{T}_1;\ \mathrm{H}_\mathrm{p}^6(\mathrm{I}))$. *Let* W_j^n *be the solution of the finite difference equation* (4). *For arbitrary* $\epsilon > 0$, *there exists a positive constant* $\sigma_1 > 0$. *If* $M - \frac{T^2c^2}{T-2\epsilon} - \frac{4B^2s^2}{B-2\epsilon} - \frac{\epsilon}{2} \ge \sigma_1$ *and* M, R_w, T, $B > 0$. *Then there exists a positive constant* C *independent of* h *and* τ, *such that the following error estimate holds:*

$$\sup_{0\le n\le N} (\|W_t^n - W_t(\cdot,\ t_n)\| + \|W^n - W(\cdot,\ t_n)\|_2) \le C(h^2 + \tau^2). \tag{24}$$

Remark 3: In the case that the mass is larger than the fluid mass, the condition of convergence and stability is not important. It is almost unconditionally stable and convergent. While in the case of thin tube (M is very small), it is conditionally stable and convergent, the condition is near $Tc^2 + 4Bs^2 \le M$, *i.e.* $(Th^2 + 4B)\tau^2 \le Mh^4$.

4. Numerical Analysis

We consider the equation with $W_0(x) = A\mathrm{e}^{ikx}$ and $W_t(x) = -i\omega A\mathrm{e}^{ikx}$ [1]. The parameter values chosen for this example are taken from [1]. We can find that $M = 3193$, $B = 529$, $T = 0$, $G = 40$, $R_w = 0.0265$, $R_f = 0.0003$ and $k = 0.1$. Then the wave length is $2\pi/k = 62.8$. We take S and ω satisfing the stable conditions of [1]:

$$S(k) = [\frac{(1 + Tk^2 + Bk^4)\tanh k}{(Ma^2k^2 \tanh k + k(1-a)^2}]^{1/2}$$

$$\omega = akS.$$

Where $a = G/(G + Mk\tanh k)$.

Now given $h = 0.62$, $\tau = 0.001$ we compute the equation (4) on Philips/486DX-33 by Turbo Pascal. In the linear case, the numerical results are coincided with the theoretical results of [1]. The amplitude of W is like-periodic. When $S > S_0 = 0.3$, the amplitude of W is increasing in each periodic with the increase of time. When $S = S_0$, the system is absolutely stable. When $S < S_0$, the amplitude of W is decreasing in each periodic with the increase of time. See Fig 2. We take $S = 10.3$, $S = 5.3$, $S = 0.3$, $S = 0.2$ and $S = 0.1$ as examples.

For the nonlinear system, we obtain the results: when $S = S(k) + g$, $\omega = ak(S(k) + g)$, the system is stable. Moreover, the nonlinear elastance coefficient plays a role of the damping. We take $S = 0.5$, then $\omega = ak(0.5 + 0.2) = -0.1737473$. The amplitude of W is a 106.5-periodic function with respect to t.

Fig. 2

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