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# $L^\infty$ CONVERGENCE OF QUASI-CONFORMING FINITE ELEMENTS FOR THE BIHARMONIC EQUATION[*1)]

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**Abstract**

In this paper we consider the $L^\infty$ convergence for quasi-conforming finite elements solving the boundary value problems of the biharmonic equation and give the nearly optimal order $L^\infty$ estimates.

## 1. Introduction

The author has considered the $L^\infty$ error estimates of conforming and nonconforming finite elements for the biharmonic equation. This paper will discuss the case of quasi-conforming finite elements.

Let $\Omega$ be a convex polygonal domain. For $p \in [1,\infty]$ and $m \geq 0$, let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ be the usual Sobolev spaces, $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ be the Sobolev norm and semi-norm respectively. When $p=2$, denote them by $H^m(\Omega)$, $H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$. $\alpha=(\alpha_1,\alpha_2)$ is called a multi-index with $|\alpha|=\alpha_1+\alpha_2$ if $\alpha_1$ and $\alpha_2$ are nonnegative integers. Define $0=(0,0), e_1=(1,0), e_2=(0,1)$. For a multi-index $\alpha$, let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$$

be the derivative operator.

Let $M$ be the number of all multi-indexes $\alpha$ with $|\alpha| \leq m$. Define $L^{m,p}(\Omega) = (L^p(\Omega))^M$. For convenience, denote the components of $w \in L^{m,p}(\Omega)$ by $w^\alpha, |\alpha| \leq m$. Then $L^{m,p}(\Omega) = \{w | w = (w^\alpha), w^\alpha \in L^p(\Omega), |\alpha| \leq m\}$. For $w \in L^{m,p}(\Omega)$, define its norm $\|\cdot\|_{m,p,\Omega}$ and semi-norm $|\cdot|_{m,p,\Omega}$ as follows,

$$\|w\|_{m,p,\Omega} = (\sum_{|\alpha|\leq m}\int_\Omega |w^\alpha|^p dxdy)^{1/p}, \quad |w|_{m,p,\Omega} = (\sum_{|\alpha|=m}\int_\Omega |w^\alpha|^p dxdy)^{1/p}, \tag{1.1}$$

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when $p < \infty$, and

$$\|w\|_{m,\infty,\Omega} = \max_{|\alpha|\le m} \operatorname*{esssup}_{(x,y)\in\Omega} |w^\alpha(x,y)|, \quad |w|_{m,\infty,\Omega} = \max_{|\alpha|=m} \operatorname*{esssup}_{(x,y)\in\Omega} |w^\alpha(x,y)|, \tag{1.2}$$

when $p = \infty$. If $p = 2$, $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ can be written as $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively.

Sobolev space $W^{m,p}(\Omega)$ or its subspace, by correspondence $u \in W^{m,p}(\Omega) \to (D^\alpha u) \in L^{m,p}(\Omega)$, is mapped to a subspace of $L^{m,p}(\Omega)$. Because the norm and semi-norm are invariant , it is also denoted by the usual notation.

For $h \in (0, h_0)$ with $h_0 \in (0,1)$, let $\mathsf{T}_h$ be a subdivisions of $\Omega$ by triangles or rectangles. Let $h_T$ =diam $T$ and $\rho_T$ the largest of the diameters of all circles contained in $T$. Assume that there exists a positive constant $\eta$, independent of $h$, such that $\eta h \le \rho_T < h_T \le h$ for all $T \in \mathsf{T}_h$.

For $w \in L^2(\Omega)$ and $w|_T \in H^m(T)$ for all $T \in \mathsf{T}_h$, define

$$|w|_{m,h} = \left( \sum_{T\in\mathsf{T}_h} |w|^2_{m,T} \right)^{1/2} . \tag{1.3}$$

For $w \in L^\infty(\Omega)$ and $w|_T \in W^{m,\infty}(T)$ for all $T \in \mathsf{T}_h$, define

$$|w|_{m,\infty,h} = \max_{T\in\mathsf{T}_h} |w|_{m,\infty,T} . \tag{1.4}$$

The remains of the paper is arranged as follows. In section 2 we give the $L^\infty$ estimates for 9-parameter quasi-conforming element for the biharmoic equation and its properties. In section 3 we present the proof of the $L^\infty$ estimate for the element. In section 4 we consider the case of other quasi-conforming plate elements.

## 2. The 9-Parameter Quasi-Conforming Finite Element

The homogeneous Dirichlet boundary value problem of the biharmonic equation is the following,

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

where $N = (N_x, N_y)$ is the unit normal of $\partial\Omega$.

It is known that for $\forall f \in H^{-1}(\Omega)$, problem (2.1) has unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$, such that

$$\|u\|_{3,\Omega} \le C\|f\|_{-1,\Omega}, \tag{2.2}$$

with $C$ a positive constant.

Let $f \in L^2(\Omega)$, define a bilinear functional $a(\cdot,\cdot)$ and a linear functional $F(\cdot)$ on $L^{2,2}(\Omega)$ by

$$\begin{cases} a(v,w) = \int_\Omega (v^{(2,0)}w^{(2,0)} + 2v^{(1,1)}w^{(1,1)} + v^{(0,2)}w^{(0,2)})dxdy, & \forall v,w \in L^{2,2}(\Omega) \\ F(v) = \int_\Omega f v^0 dxdy, & \forall v \in L^{2,2}(\Omega). \end{cases} \tag{2.3}$$

The variational form of problem (2.1) is to find $u \in H_0^2(\Omega)$, such that,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{2.4}$$

where $(\cdot,\cdot)$ is the $L^2$ product.

For $h \in (0,h_0)$, let $V_h \subset L^{2,2}(\Omega)$ be a finite element space associated with $\mathsf{T}_h$. The finite element approximation to problem (2.1) is to find $u_h \in V_h$, such that,

$$a(u_h,v) = F(v), \qquad \forall v \in V_h. \tag{2.5}$$

From now on, let $\mathsf{T}_h$ be a subdivisions of $\Omega$ by triangles. For each triangle $T$, let $a_0$ be its center point, $a_i$ the vertices of $T, 1 \le i \le 3$. Let $P_3'(T)$ be Zienkiewicz polynomial space on $T$, i.e., $P_3'(T) = \{\, p \mid p \in P_3(T) \text{ and } 6p(a_0)+\sum_{i=1}^3(-2p(a_i)+Dp(a_i)(a_i-a_0)) = 0\}$, where $Dp = (\partial_x p, \partial_y p)$, $Dp(a_i)a_i$ is the $R^2$ inner product of $Dp(a_i)$ and $a_i$. It is known that Zienkiewicz finite element is not convergent for general subdivisions of $\Omega$. Using so called "quasi-conforming" technique, Professor Tang Limin and his colleagues give 9-parameter quasi-conforming finite element in [4,1]. The finite element has been proved to be convergent[8].

Let $N = (N_x, N_y)$ be the exterior unit normal of $\partial T$. For each $p \in P_3'(T)$, define operators, $\Pi_{\partial T}^N p$, $\Pi_T^{(2,0)}p$, $\Pi_T^{(1,1)}p$, $\Pi_T^{(0,2)}p$ as follows. $\Pi_{\partial T}^N p$ is linear on each edge of triangle $T$ and equals to $\partial p/\partial N$ at the endpoints of the edge. $\Pi_T^\alpha p \in P_1(T)$, $|\alpha| = 2$, and are determined by

$$\int_T q \begin{bmatrix} \Pi_T^{(2,0)}p \\ 2\Pi_T^{(1,1)}p \\ \Pi_T^{(0,2)}p \end{bmatrix} dxdy = \int_{\partial T} q \begin{bmatrix} N_x^2 & -N_xN_y \\ 2N_xN_y & N_x^2 - N_y^2 \\ N_y^2 & N_xN_y \end{bmatrix} \begin{bmatrix} \Pi_{\partial T}^N p \\ \partial p/\partial s \end{bmatrix} ds - \int_{\partial T} p \begin{bmatrix} \partial_x q & 0 \\ \partial_y q & \partial_x q \\ 0 & \partial_y q \end{bmatrix} N ds, \quad \forall q \in P_1(T) \ . \tag{2.6}$$

Define $V_T = \{\ (p, \partial_x p, \partial_y p, \Pi_T^{(2,0)}p, \Pi_T^{(1,1)}p, \Pi_T^{(0,2)}p) \mid p \in P_3'(T)\}$. Then $V_T$ is a finite dimensional subspace of $L^{2,2}(T)$. Let $V_h = \{\ v \in L^{2,2}(\Omega) \mid v|_T \in V_T, \forall T \in \mathsf{T}_h$, and $v^0, v^{e_1}$ and $v^{e_2}$ are continuous at the vertices of the triangles and vanish at the vertices on $\partial\Omega\}$.

For each triangle $T$, let $\lambda_1, \lambda_2, \lambda_3$ be its area coordinates. Define, for $v \in C^1(T)$,

$$\begin{cases} \Pi_T^0 v = \sum_{i=1}^3 (\lambda_i^2(3-2\lambda_i) + 2\lambda_1\lambda_2\lambda_3) v(a_i) \\ \qquad\quad + \frac{1}{2}\sum_{\substack{1\le i,j\le 3 \\ i\ne j}} \lambda_i\lambda_j(1+\lambda_i-\lambda_j) Dv(a_i)(a_j-a_i), \\ \Pi_T^{e_1} v = \partial_x \Pi_T^0 v, \qquad \Pi_T^{e_2} v = \partial_y \Pi_T^0 v, \\ \Pi_T^\alpha v = \Pi_T^\alpha \Pi_T^0 v, \quad |\alpha| = 2. \end{cases} \tag{2.7}$$

Then $\Pi_T v = (\Pi_T^0 v, \Pi_T^{e_1} v, \Pi_T^{e_2} v, \Pi_T^{(2,0)} v, \Pi_T^{(1,1)} v, \Pi_T^{(0,2)} v) \in L^{2,2}(T)$. For each $v \in C^1(\bar\Omega)$, define $\Pi_h v = (\Pi_h^0 v, \Pi_h^{e_1} v, \Pi_h^{e_2} v, \Pi_h^{(2,0)} v, \Pi_h^{(1,1)} v, \Pi_h^{(0,2)} v) \in L^{2,2}(\Omega)$ by $\Pi_h^\alpha v|_T = \Pi_T^\alpha v$ with $|\alpha| \le 2$ and $T \in \mathsf{T}_h$.

Let $u_h$ be the solution of (2.5) with $V_h$ the 9-parameter quasi-conforming element space. From [8] and by Nitsche technique, we have the estimate,

$$\sum_{|\alpha|\le 1} \|D^\alpha u - u_h^\alpha\|_{0,\Omega} + h \sum_{|\alpha|=2} \|D^\alpha u - u_h^\alpha\|_{0,\Omega} \le Ch^2 |u|_{3,\Omega} \ . \tag{2.8}$$

From [5,8], the following inequalities are true,

$$\sum_{|\alpha|\le 2} h^{|\alpha|} \|D^\alpha v - \Pi_T^\alpha v\|_{0,T} \le Ch^3 |v|_{3,T}, \quad \forall v \in H^3(T), \quad \forall T \in \mathsf{T}_h, \tag{2.9}$$

$$|v^0|_{2,T} \le C|v|_{2,T}, \qquad \forall v \in V_T,\ \forall T \in \mathsf{T}_h, \tag{2.10}$$

$$\|v_h\|_{2,\Omega} \le C|v_h|_{2,\Omega}, \quad \forall v_h \in V_h, \tag{2.11}$$

$$|v_h|_{2,\Omega} \le Ch^{i-2} \sum_{|\alpha|=i} |v_h^\alpha|_{0,\Omega}, \quad i = 0, 1,\ \forall v_h \in V_h. \tag{2.12}$$

$$|v^0|_{0,\infty,\Omega} \le C|v|_{2,\Omega}, \qquad \forall v \in V_h, \tag{2.13}$$

$$|v^0|_{0,\infty,\Omega} \le C|\ln h|^{1/2} |v|_{1,\Omega}, \qquad \forall v \in V_h, \tag{2.14}$$

$$|v^0|_{1,\infty,\Omega} \le C|\ln h|^{1/2} |v|_{2,\Omega}, \qquad \forall v \in V_h. \tag{2.15}$$

For $L^\infty$ estimates, we have

**Theorem 1.** *Let $V_h$ be the 9-parameter quasi-conforming finite element space, $u$ the solution of problem (2.4) and $u_h$ the one of problem (2.5). Then*

$$|u - u_h^0|_{0,\infty,\Omega} \le Ch^2 |\ln h|^{1/2} |u|_{3,\Omega}, \tag{2.16}$$

*when $f \in H^{-1}(\Omega)$, and*

$$|u - u_h^0|_{1,\infty,\Omega} \le Ch^2 |\ln h| |u|_{3,\infty,\Omega}, \tag{2.17}$$

*when $u \in W^{3,\infty}(\Omega)$.*

The proof of Theorem 1 is left in section 3.

## 3. The proof of Theorem 1

In this section, we will prove Theorem 1. Firstly, let $f \in H^{-1}(\Omega)$, then the solution $u$ of (2.4) is in $H^3(\Omega)$. From (2.14), (2.8) and the interpolation theory, we have

$$\begin{aligned}\|u-u_h^0\|_{0,\infty,\Omega} \leq & \|u-\Pi_h^0 u\|_{0,\infty,\Omega} + \|\Pi_h^0 u - u_h^0\|_{0,\infty,\Omega} \\ \leq & C(h^2|v|_{3,\Omega} + |\ln h|^{1/2}|\Pi_h^0 u - u_h^0|_{1,\Omega}) \\ \leq & Ch^2|\ln h|^{1/2}|u|_{3,\Omega} \quad ,\end{aligned}$$

i.e., (2.16) is true. The remains is to prove (2.17). Assume $u \in W^{3,\infty}(\Omega)$. By the interpolation result[8], we have

$$\begin{aligned}|u-u_h^0|_{1,\infty,\Omega} &\leq |u-\Pi_h^0 u|_{1,\infty,\Omega} + |\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} \\ &\leq Ch^2|u|_{3,\infty,\Omega} + |\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} \quad .\end{aligned} \tag{3.1}$$

So we must estimate $|\Pi_h^0 u - u_h^0|_{1,\infty,\Omega}$. Let $T' \in \mathsf{T}_h$ be the element such that $|\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} = |\Pi_h^0 u - u_h^0|_{1,\infty,T'}$. Not loosing the general, we suppose that

$$|\Pi_h^0 u - u_h^0|_{1,\infty,T'} = |\frac{\partial(\Pi_h^0 u - u_h^0)}{\partial x}|_{0,\infty,T'}.$$

Let $(x_0, y_0) \in T'$ be the point such that

$$|\frac{\partial(\Pi_h^0 u - u_h^0)}{\partial x}|_{0,\infty,T'} = |\frac{\partial(\Pi_h^0 u - u_h^0)}{\partial x}(x_0,y_0)|.$$

To prove (2.17), we need some results about the weight function and the regular Green function. For $(x_0, y_0)$, define the weight function $\rho$ as follows,

$$\rho(x,y) = (x-x_0)^2 + (y-y_0)^2 + h^2 \ ,$$

For integer $\beta$ and $v \in H^m(T)$ and $T \in \mathsf{T}_h$, define

$$|v|_{m,(\beta),T} = \left( \sum_{i+j=m} \int_T \rho^{-\beta} \left| \frac{\partial^m v}{\partial x^i \partial y^j} \right|^2 dxdy \right)^{1/2} . \tag{3.2}$$

When $v \in L^2(\Omega)$ and $v|_T \in H^m(T)$ for all $T \in \mathsf{T}_h$, define

$$|v|_{m,(\beta)} = \left( \sum_{T \in \mathsf{T}_h} |v|^2_{m,(\beta),T} \right)^{1/2} .$$

For the weight function, the following inequalities,

$$|v|_{m,(\gamma)} \leq h^{-(\gamma-\beta)}|v|_{m,(\beta)}, \quad \gamma > \beta \ , \tag{3.3}$$

are true for $v \in L^2(\Omega)$ and $v|_T \in H^m(T), \forall T \in \mathsf{T}_h$. And

$$|v|_{0,(1)} \le C|\ln h|^{1/2}\|v\|_{0,\infty,\Omega}, \qquad \forall v \in L^\infty(\Omega) \tag{3.4}$$

$$\left|\int_\Omega vw\,dxdy\right| \le |v|_{0,(\beta)}\,|v|_{0,(-\beta)}, \quad v,w \in L^2(\Omega) \tag{3.5}$$

$$|v-\Pi_T^0 v|_{k,(\beta),T} + \sum_{|\alpha|=k} |D^\alpha v - \Pi_T^\alpha v|_{0,(\beta),T} \le Ch^{3-k}|v|_{3,(\beta),T}, \quad 0\le k\le 3,\ v\in H^3(T),\ T\in \mathsf{T}_h\ . \tag{3.6}$$

**Lemma 1.** *There exists a constant $C$ such that, for $v,w \in H_0^2(\Omega)\cap H^3(\Omega)$, $\triangle^2 v \in L^2(\Omega)$, and $v_h \in V_h$, the following inequality holds.*

$$|a(v,w-v_h) - (\triangle^2 v, w - v_h^0)| \le Ch^2|v|_{3,(\beta)}|v_h^0|_{3,(-\beta)}. \tag{3.7}$$

*Proof.* Noticing that $v_h^0 \in H_0^1(\Omega)$ and by Green formula, we can derive

$$\begin{aligned} a(v,&w-v_h) - (\triangle^2 v, w - v_h^0) \\ &= -\int_\Omega [\frac{\partial^2 v}{\partial x^2}v_h^{(2,0)} + \frac{\partial^3 v}{\partial x^3}\frac{\partial}{\partial x}v_h^0]dxdy - \int_\Omega [\frac{\partial^2 v}{\partial y^2}v_h^{(0,2)} + \frac{\partial^3 v}{\partial y^3}\frac{\partial}{\partial y}v_h^0]dxdy \\ &\quad - \int_\Omega [2\frac{\partial^2 v}{\partial x\partial y}v_h^{(1,1)} + \frac{\partial^3 v}{\partial x^2\partial y}\frac{\partial}{\partial y}v_h^0 + \frac{\partial^3 v}{\partial x\partial y^2}\frac{\partial}{\partial x}v_h^0]dxdy\ . \end{aligned} \tag{3.8}$$

By Green formula, we have

$$\begin{aligned} \int_\Omega [\frac{\partial^2 v}{\partial x^2}v_h^{(2,0)} + \frac{\partial^3 v}{\partial x^3}\frac{\partial}{\partial x}v_h^0]dxdy &= \sum_{T\in\mathsf{T}_h}\Big\{\int_T \frac{\partial^2 v}{\partial x^2}(v_h^{(2,0)} - \frac{\partial}{\partial x^2}v_h^0)dxdy \\ &\qquad + \int_{\partial T}\frac{\partial^2 v}{\partial x^2}(N_x^2\frac{\partial}{\partial N}v_h^0 - N_xN_y\frac{\partial}{\partial s}v_h^0)ds\Big\} \\ &= \sum_{T\in\mathsf{T}_h}\Big\{\int_T \frac{\partial^2 v}{\partial x^2}(v_h^{(2,0)} - \frac{\partial}{\partial x^2}v_h^0)dxdy \\ &\qquad + \int_{\partial T}\frac{\partial^2 v}{\partial x^2}N_x^2(\frac{\partial}{\partial N}v_h^0 - \Pi_{\partial T}^N v_h^0)ds\Big\} \\ &\qquad + \sum_{T\in\mathsf{T}_h}\int_{\partial T}\frac{\partial^2 v}{\partial x^2}(N_x^2\Pi_{\partial T}^N v_h^0 - N_xN_y\frac{\partial}{\partial s}v_h^0)ds\ . \end{aligned}$$

Let $P_T^0 : L^2(T) \to P_0(T)$ be the orthogonal projection operator. By (2.6), Green formula and

$$\sum_{T\in\mathsf{T}_h}\int_{\partial T}\frac{\partial^2 v}{\partial x^2}(N_x^2\Pi_{\partial T}^N v_h^0 - N_xN_y\frac{\partial}{\partial s}v_h^0)ds = 0,$$

we get

$$\int_\Omega \Big[\frac{\partial^2 v}{\partial x^2} v_h^{(2,0)} + \frac{\partial^3 v}{\partial x^3}\frac{\partial}{\partial x} v_h^0\Big] dxdy$$

$$= \sum_{T\in \mathsf{T}_h} \Big\{ \int_T \Big(\frac{\partial^2 v}{\partial x^2} - P_T^0 \frac{\partial^2 v}{\partial x^2}\Big)\Big(v_h^{(2,0)} - \frac{\partial}{\partial x^2} v_h^0\Big) dxdy + \int_{\partial T} \Big(\frac{\partial^2 v}{\partial x^2} - P_T^0 \frac{\partial^2 v}{\partial x^2}\Big) N_x^2 \Big(\frac{\partial}{\partial N} v_h^0 - \Pi_{\partial T}^N v_h^0\Big) ds\Big\} \quad .$$

From interpolation theory and Schwarz inequality, we have

$$\Big|\int_\Omega \Big[\frac{\partial^2 v}{\partial x^2} v_h^{(2,0)} + \frac{\partial^3 v}{\partial x^3}\frac{\partial}{\partial x} v_h^0\Big] dxdy\Big| \le Ch^2 \sum_{T\in \mathsf{T}_h} |v|_{3,T} |v_h^0|_{3,T}. \tag{3.9}$$

For all $T \in \mathsf{T}_h$,

$$\max_{(x,y)\in T} \rho(x,y) \le C \min_{(x,y)\in T} \rho(x,y) \quad . \tag{3.10}$$

Then

$$\Big|\int_\Omega \Big[\frac{\partial^2 v}{\partial x^2} v_h^{(2,0)} + \frac{\partial^3 v}{\partial x^3}\frac{\partial}{\partial x} v_h^0\Big] dxdy\Big| \le Ch^2 |v|_{3,(\beta)} |v_h^0|_{3,(-\beta)} \quad . \tag{3.11}$$

Similarly, we can show

$$\Big|\int_\Omega \Big[\frac{\partial^2 v}{\partial y^2} v_h^{(0,2)} + \frac{\partial^3 v}{\partial y^3}\frac{\partial}{\partial y} v_h^0\Big] dxdy\Big| \le Ch^2 |v|_{3,(\beta)} |v_h^0|_{3,(-\beta)} \quad , \tag{3.12}$$

$$\Big|\int_\Omega \Big[2\frac{\partial^2 v}{\partial x \partial y} v_h^{(1,1)} + \frac{\partial^3 v}{\partial x^2 \partial y}\frac{\partial}{\partial y} v_h^0 + \frac{\partial^3 v}{\partial x \partial y^2}\frac{\partial}{\partial x} v_h^0\Big] dxdy\Big| \le Ch^2 |v|_{3,(\beta)} |v_h^0|_{3,(-\beta)} \ . \tag{3.13}$$

Inequality (3.7) follows from (3.8) and (3.11) to (3.13).

Now we turn to the regular Green function. Let $q \in P_4(T')$ satisfy

$$\int_{T'} qp\, dxdy = \frac{\partial}{\partial x} p(x_0, y_0), \qquad \forall p \in P_4(T') \quad .$$

Define $\delta_h \in L^2(\Omega)$ such that,

$$\delta_h(x,y) = \begin{cases} q(x,y), & (x,y) \in T' \\ 0, & \text{otherwise} \end{cases} \quad .$$

From Lemma 3 in [6], we have

$$h\|\delta_h\|_{0,\Omega} + \|\delta_h\|_{-1,\Omega} + \|\delta_h\|_{0,(-1)} \le Ch^{-1}. \tag{3.14}$$

Let $g$ be the regular Green function determined by

$$\begin{cases} \triangle^2 g = \delta_h, & \text{in } \Omega \\ g|_{\partial\Omega} = \dfrac{\partial g}{\partial N}|_{\partial\Omega} = 0 \end{cases} \tag{3.15}$$

and $g_h$ be its finite element solution by 9-parameter quasi-conforming element, i.e.,

$$a(g_h, v_h) = (\delta_h, v_h^0). \qquad \forall v_h \in V_h. \tag{3.16}$$

From (2.2), (2.8) and (3.14), we get

$$\|g - g_h^0\|_{1,\Omega} + h|g - g_h|_{2,h} + h^2|g|_{3,\Omega} \leq Ch. \tag{3.17}$$

By the way used in [6], we have

$$\|g\|_{2,\Omega} + |g|_{3,(-1)} \leq C|\ln h|^{1/2}. \tag{3.18}$$

**Lemma 2.**

$$\sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|^2_{0,(-1)} \leq Ch^2|\ln h|. \tag{3.19}$$

*Proof.* Set $p_h = \rho(\Pi_h^0 g - g_h^0)$, and define $\Pi_h p_h \in V_h$ by $\Pi_h^\alpha p_h|_T = \Pi_T^\alpha(p_h|_T)$ for $T \in \mathsf{T}_h$ ,

$$\begin{aligned}
&\sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|^2_{0,(-1)} \\
&\quad\leq \int_\Omega \rho\Big[(\frac{\partial^2 g}{\partial x^2} - g_h^{(2,0)})^2 + 2(\frac{\partial^2 g}{\partial x\partial y} - g_h^{(1,1)})^2 + (\frac{\partial^2 g}{\partial y^2} - g_h^{(0,2)})^2\Big]dxdy \\
&\quad= \int_\Omega \rho\Big[(\frac{\partial^2 g}{\partial x^2} - \Pi_h^{(2,0)} g)(\frac{\partial^2 g}{\partial x^2} - g_h^{(2,0)}) + (\frac{\partial^2 g}{\partial y^2} - \Pi_h^{(0,2)} g)(\frac{\partial^2 g}{\partial y^2} - g_h^{(0,2)}) \\
&\qquad + 2(\frac{\partial^2 g}{\partial x\partial y} - \Pi_h^{(1,1)} g)(\frac{\partial^2 g}{\partial x\partial y} - g_h^{(1,1)})\Big]dxdy + a(g - g_h, \Pi_h p_h) \\
&\qquad + \int_\Omega \Big\{[\rho(\Pi_h^{(2,0)} g - g^{(2,0)}) - \Pi_h^{(2,0)} p_h](\frac{\partial^2 g}{\partial x^2} - g_h^{(2,0)}) \\
&\qquad + [\rho(\Pi_h^{(0,2)} g - g_h^{(0,2)}) - \Pi_h^{(0,2)} p_h](\frac{\partial^2 g}{\partial y^2} - g_h^{(0,2)}) \\
&\qquad + 2[\rho(\Pi_h^{(1,1)} g - g_h^{(1,1)}) - \Pi_h^{(1,1)} p_h](\frac{\partial^2 g}{\partial x\partial y} - g_h^{(1,1)})\Big\}dxdy \quad .
\end{aligned}$$

Thereefore

$$\sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|^2_{0,(-1)}$$

$$\leq C\Big\{|a(g-g_h,\Pi_h p_h)| + \sum_{|\alpha|=2} |D^\alpha g - \Pi_h^\alpha g|_{0,(-1)} |D^\alpha g - g_h^\alpha|_{0,(-1)}$$
$$+ \sum_{|\alpha|=2} |\int_\Omega [\rho(\Pi_h^\alpha g - g_h^\alpha) - \Pi_h^\alpha p_h](D^\alpha g - g_h^\alpha) dxdy|\Big\} \quad . \tag{3.20}$$

From (3.6) and (3.18), we have

$$\sum_{|\alpha|=2} |D^\alpha g - \Pi_h^\alpha g|_{0,(-1)} \leq Ch|g|_{3,(-1)} \leq Ch|\ln h|^{1/2} \quad . \tag{3.21}$$

By (3.7), (3.18) and (3.6), we derive that

$$|a(g-g_h,\Pi_h p_h)| \leq Ch^2 |g|_{3,(-1)} |\Pi_h^0 p_h|_{3,(1)} \leq Ch^2 |\ln h|^{1/2} |p_h|_{3,(1)} \quad . \tag{3.22}$$

Using (3.3), (3.10) and the inverse inequality for polynomial spaces, we get

$$\begin{aligned} |p_h|_{3,(1)} \leq & C(|\Pi_h^0 g - g_h^0|_{3,(-1)} + |\Pi_h^0 g - g_h^0|_{2,h} + |\Pi_h^0 g - g_h^0|_{1,(1)}) \\ \leq & Ch^{-1}(|\Pi_h^0 g - g_h^0|_{2,(-1)} + |\Pi_h^0 g - g_h^0|_{1,h}) \quad . \end{aligned}$$

By (2.10) and (3.10), we get

$$\begin{aligned} |\Pi_h^0 g - g_h^0|^2_{2,(-1)} &= \sum_{T\in \mathsf{T}_h} |\Pi_h^0 g - g_h^0|^2_{2,(-1),T} \leq C \sum_{T\in \mathsf{T}_h} |\rho|_{0,\infty,T} |\Pi_h^0 g - g_h^0|^2_{2,T} \\ &\leq C \sum_{T\in \mathsf{T}_h} |\rho|_{0,\infty,T} \sum_{|\alpha|=2} |\Pi_h^\alpha g - g_h^\alpha|^2_{0,T} = C \sum_{|\alpha|=2} |\Pi_h^\alpha g - g_h^\alpha|^2_{0,(-1)} \quad . \end{aligned}$$

Noticing $\Pi_h^\alpha g - g_h^\alpha = \Pi_h^\alpha g - D^\alpha g + D^\alpha g - g_h^\alpha$ and $\Pi_h^0 g - g_h^0 = \Pi_h^0 g - g + g - g_h^0$, from (3.6), (2.9), (3.18) and (3.17), we get,

$$|p_h|_{3,(1)} \leq C(h^{-1} \sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|_{2,(-1)} + h|\ln h|^{1/2}) \quad .$$

Substituting the above estimate into (3.22), we have

$$|a(g-g_h,\Pi_h p_h)| \leq C(h^2|\ln h| + h|\ln h|^{1/2} \sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|_{0,(-1)}) \quad . \tag{3.23}$$

Now let $\alpha = (2,0)$. For $v \in L^2(\Omega)$, define $P_h^0 v \in L^2(\Omega)$ by $P_h^0 v|_T = P_T^0(v|_T)$, $\forall T \in \mathsf{T}_h$. Then

$$\begin{aligned} &\int_\Omega (\rho(\Pi_h^\alpha g - g_h^\alpha) - \Pi_h^\alpha p_h)(D^\alpha g - g_h^\alpha) dxdy \\ &= \int_\Omega \Big[(\rho - P_h^0\rho)(\Pi_h^\alpha g - g_h^\alpha) - \Pi_h^\alpha((\rho - P_h^0\rho)(\Pi_h^0 g - g_h^0))\Big](D^\alpha g - g_h^\alpha) dxdy \\ &= \int_\Omega (\rho - P_h^0\rho)\Big[(\Pi_h^\alpha g - g_h^\alpha) - D^\alpha(\Pi_h^0 g - g_h^0)\Big](D^\alpha g - g_h^\alpha) dxdy \end{aligned}$$

$$+\sum_{T\in \mathsf{T}_h}\int_T (D^\alpha-\Pi_h^\alpha)[(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)](D^\alpha g-g_h^\alpha)dxdy$$
$$-\int_\Omega \frac{\partial\rho}{\partial x}\frac{\partial}{\partial x}(\Pi_h^0 g-g_h^0)(D^\alpha g-g_h^\alpha)dxdy$$
$$-2\int_\Omega (\Pi_h^0 g-g_h^0)(D^\alpha g-g_h^\alpha)dxdy \quad . \tag{3.24}$$

Applying interpolation theory, considering (3.5), (2.10), (2.9), (3.17), (3.18) and (3.10), we have

$$\Big|\int_\Omega (\rho-P_h^0\rho)\Big[(\Pi_h^\alpha g-g_h^\alpha)-D^\alpha(\Pi_h^0 g-g_h^0)\Big](D^\alpha g-g_h^\alpha)dxdy\Big|$$
$$\le Ch|\ln h|^{1/2}\sum_{|\alpha|=2}|D^\alpha g-g_h^\alpha|_{0,(-1)} \quad . \tag{3.25}$$

By (3.5) and (3.6), we get

$$\Big|\sum_{T\in \mathsf{t}_h}\int_T (D^\alpha-\Pi_h^\alpha)[(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)](D^\alpha g-g_h^\alpha)dxdy\Big|$$
$$\le Ch\sum_{T\in\mathsf{T}_h}|(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)|_{3,(1),T}|D^\alpha g-g_h^\alpha|_{0,(-1),T} \quad . \tag{3.26}$$

On the other hand, by the interpolation theory and the inverse inequality for polynomials and (3.10), we derive

$$\begin{aligned}
&|(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)|_{3,(1),T}\\
&\quad\le C|\rho^{-1}|_{0,\infty,T}|(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)|_{3,T}\\
&\quad\le C|\rho^{-1}|_{0,\infty,T}\{h|rho|_{0,\infty,T}|\Pi_h^0 g-g_h^0|_{3,T}+|rho|_{0,\infty,T}|\Pi_h^0 g-g_h^0|_{2,T}\\
&\qquad\qquad +|\Pi_h^0 g-g_h^0|_{1,T}\}\\
&\quad\le C\{|\Pi_h^0 g-g_h^0|_{2,T}+h^{-1}|\Pi_h^0 g-g_h^0|_{1,T}\} \quad .
\end{aligned}$$

Hence, from (2.8), (2.9) and (3.17), we get

$$\Big|\sum_{T\in \mathsf{t}_h}\int_T (D^\alpha-\Pi_h^\alpha)[(\rho-P_h^0\rho)(\Pi_h^0 g-g_h^0)](D^\alpha g-g_h^\alpha)dxdy\Big|$$
$$\le Ch\{|\Pi_h^0 g-g_h^0|_{2,\Omega}+h^{-1}|\Pi_h^0 g-g_h^0|_{1,\Omega}\}|D^\alpha g-g_h^\alpha|_{0,(-1)}$$
$$\le Ch|D^\alpha g-g_h^\alpha|_{0,(-1)} \quad . \tag{3.27}$$

From (2.8), (2.9) and (3.17) and

$$|\frac{\partial\rho}{\partial x}|^2+|\frac{\partial\rho}{\partial y}|^2\le C\rho \quad ,$$

we derive that

$$
\begin{aligned}
&\Big|\int_\Omega \frac{\partial\rho}{\partial x}\frac{\partial}{\partial x}(\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \\
&\qquad \le C\int_\Omega \Big|\frac{\partial}{\partial x}(\Pi_h^0 g - g_h^0)\Big|\rho^{1/2}|D^\alpha g - g_h^\alpha|dxdy \\
&\qquad \le C|\Pi_h^0 g - g_h^0|_{1,\Omega}|D^\alpha g - g_h^\alpha|_{0,(-1)} \\
&\qquad \le Ch|D^\alpha g - g_h^\alpha|_{0,(-1)} \quad . \qquad (3.28)
\end{aligned}
$$

For $v \in C(\overline{\Omega})$, let $\tilde{\Pi}_h v$ be the function such that $\tilde{\Pi}_h v|_T \in P_1(T)$ for $\forall T \in \mathsf{T}_h$ and $\tilde{\Pi}_h v$ is equal to $v$ at the vertices of $\mathsf{T}_h$. Then $\tilde{\Pi}_h(\Pi_h^0 g - g_h^0) \in H_0^1(\Omega)$ and

$$
\begin{aligned}
&\Big|\int_\Omega (\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \\
&\le \Big|\int_\Omega \tilde{\Pi}_h(\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \\
&\quad + \Big|\int_\Omega [\Pi_h^0 g - g_h^0 - \tilde{\Pi}_h(\Pi_h^0 g - g_h^0)](D^\alpha g - g_h^\alpha)dxdy\Big| \quad . \qquad (3.29)
\end{aligned}
$$

From the interpolation theory, (3.3), (2.8), (2.9) and (3.17), we have

$$
\begin{aligned}
&\Big|\int_\Omega [\Pi_h^0 g - g_h^0 - \tilde{\Pi}_h(\Pi_h^0 g - g_h^0)](D^\alpha g - g_h^\alpha)dxdy\Big| \\
&\qquad \le Ch^2|\Pi_h^0 g - g_h^0|_{2,(1)}|D^\alpha g - g_h^\alpha|_{0,(-1)} \\
&\qquad \le Ch|\Pi_h^0 g - g_h^0|_{2,h}|D^\alpha g - g_h^\alpha|_{0,(-1)} \\
&\qquad \le Ch|D^\alpha g - g_h^\alpha|_{0,(-1)} \quad . \qquad (3.30)
\end{aligned}
$$

By Green formula and (2.6) and the fact that $\tilde{\Pi}_h(\Pi_h^0 g - g_h^0) \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
&\Big|\int_\Omega \tilde{\Pi}_h(\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \\
&\qquad = \Big|\int_\Omega \frac{\partial}{\partial x}\tilde{\Pi}_h(\Pi_h^0 g - g_h^0)\frac{\partial}{\partial x}(g - g_h^0)dxdy\Big| \\
&\qquad \le C|\tilde{\Pi}_h(\Pi_h^0 g - g_h^0)|_{1,\Omega}|g - g_h^0|_{1,\Omega}.
\end{aligned}
$$

From the interpolation theory, (3.17), (2.8) and (2.9),

$$
\Big|\int_\Omega \tilde{\Pi}_h(\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \le Ch|\Pi_h^0 g - g_h^0|_{1,\Omega} \le Ch^2 \ . \qquad (3.31)
$$

Combining (3.29) to (3.31), we get

$$
\Big|\int_\Omega (\Pi_h^0 g - g_h^0)(D^\alpha g - g_h^\alpha)dxdy\Big| \le Ch(h + |D^\alpha g - g_h^\alpha|_{0,(-1)}) \ . \qquad (3.32)
$$

Estimates (3.24), (3.25), (3.27), (3.28) and (3.32) imply that the following inequality

$$\Big|\int_\Omega [\rho(\Pi_h^\alpha g - g_h^\alpha) - \Pi_h^\alpha p_h](D^\alpha g - g_h^\alpha)dxdy\Big| \le Ch(h + |\ln h|^{1/2}|D^\alpha g - g_h^\alpha|_{0,(-1)}), \quad (3.33)$$

is true for $\alpha = (2,0)$. Similarly, we can show it is also true in the other case of $|\alpha| = 2$.

Combinig (3.20) to (3.23) and (3.33), we get

$$\sum_{|\alpha|=2} |D^\alpha g - g_h^\alpha|_{0,(-1)}^2 \le C\Big\{h^2|\ln h| + h|\ln h|^{1/2}\Big(\sum_{|\alpha|=2}|D^\alpha g - g_h^\alpha|_{0,(-1)}^2\Big)^{1/2}\Big\},$$

Lemma 2 follows.

From (3.16), (3.5) and (3.7), we get

$$\begin{aligned}|\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} &= |(\delta_h, \Pi_h^0 u - u_h^0)| \\ &= |a(g_h, \Pi_h u - u_h)| \\ &= |a(g - g_h, u - \Pi_h u) + [a(u, g_h - g) - (\triangle^2 u, g_h^0 - g)] \\ &\quad + [a(g, \Pi_h u - u) - (\triangle^2 g, \Pi_h^0 u - u)] + (\triangle^2 g, \Pi_h^0 u - u)| \\ &\le C\Big\{\sum_{|\alpha|=2}|D^\alpha g - g_h^\alpha|_{0,(-1)}|D^\alpha u - \Pi_h^\alpha u|_{0,(1)} + h^2|u|_{3,(1)}|g_h^0|_{3,(-1)} \\ &\quad + h^2|g|_{3,(-1)}|\Pi_h^0 u|_{3,(1)} + |\delta_h|_{0,(-1)}|u - \Pi_h^0 u|_{0,(1)}\Big\}.\end{aligned}$$

By (3.4), (3.6), (3.14), (3.18) and (3.19), we derive that

$$|\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} \le Ch^2|u|_{3,\infty,\Omega}(|\ln h| + |\ln h|^{1/2}|g_h^0|_{3,(-1)}).$$

On the other hand,

$$\begin{aligned}|g_h^0|_{3,(-1)} &\le |g_h^0 - \Pi_h^0 g|_{3,(-1)} + |\Pi_h^0 g|_{3,(-1)} \\ &\le C(|g|_{3,(-1)} + h^{-1}|g_h^0 - \Pi_h^0 g|_{2,(-1)}) \\ &\le C(|\ln h|^{1/2} + h^{-1}\sum_{|\alpha|=2}|g_h^\alpha - \Pi_h^\alpha g|_{0,(-1)}) \\ &\le C[|\ln h|^{1/2} + h^{-1}\sum_{|\alpha|=2}(|D^\alpha g - g_h^\alpha|_{0,(-1)} + |D^\alpha g - \Pi_h^\alpha g|_{0,(-1)})] \\ &\le C|\ln h|^{1/2},\end{aligned}$$

by (3.18), (3.19), (3.6), (2.10) and the inverse inequality of polynomials. Hence

$$|\Pi_h^0 u - u_h^0|_{1,\infty,\Omega} \le Ch^2|\ln h||u|_{3,\infty,\Omega}. \quad (3.34)$$

Estimate (2.17) follows from (3.1) and (3.34). Theorem 1 is proved.

## 4. Other Quasi-conforming Finite Elements

The principle which led to the $L^\infty$ estimates for 9-parameter quasi-conforming element can be also applied to some other finite elements, such as, 12-parameter quasi-conforming triangle element[4,1], 15-parameter quasi-conforming element [2] and 12-parameter quasi-conforming rectangle element[3]. The results will be listed without proofs.

Let $\mathsf{T}_h$ be the subdivision of $\Omega$ described in section 2. For each triangle $T$, let $a_i, 1 \le i \le 3$ be its vertices numbered counter clockwise, $a_{ij} = \frac{1}{2}(a_i + a_j), 1 \le i < j \le 3$.

**15-Parameter Quasi-Conforming Element.** For $\forall p \in P_4(T)$, $p$ is uniquely determined by its values at $a_i (i = 1, 2, 3)$ and $a_{jk}, 1 \le j < k \le 3$, the values of $Dp$ at $a_i (i = 1, 2, 3)$, and its outer normal derivative at $a_{jk}, 1 \le j < k \le 3$. For all $p \in P_4(T)$, $\Pi^N_{\partial T} p$, $\Pi^{(2,0)}_T p$, $\Pi^{(1,1)}_T p$ and $\Pi^{(0,2)}_T p$ are defined as follows. $\Pi^N_{\partial T} p$ is quadratic on each edge of $T$ and equals to $\partial p/\partial N$ at the two endpoints and the midpoint of the edge. $\Pi^\alpha_T p \in P_2(K), |\alpha| = 2$, and are determined by

$$\int_T q \begin{bmatrix} \Pi^{(2,0)}_T p \\ 2\Pi^{(1,1)}_T p \\ \Pi^{(0,2)}_T p \end{bmatrix} dxdy = \int_{\partial T} q \begin{bmatrix} N_x^2 & -N_x N_y \\ 2N_x N_y & N_x^2 - N_y^2 \\ N_y^2 & N_x N_y \end{bmatrix} \begin{bmatrix} \Pi^N_{\partial T} p \\ \partial p/\partial s \end{bmatrix} ds$$
$$- \int_T \begin{bmatrix} \partial_x q & 0 \\ \partial_y q & \partial_x q \\ 0 & \partial_y q \end{bmatrix} \begin{bmatrix} \partial p/\partial x \\ \partial p/\partial y \end{bmatrix} dxdy, \quad \forall q \in P_2(T). \tag{4.1}$$

Define $V_T = \{(p, \partial_x p, \partial_y p, \Pi^{(2,0)}_T p, \Pi^{(1,1)}_T p, \Pi^{(0,2)}_T p) \mid p \in P_4(K)\}$. Define $V_h = \{ v \in L^{2,2}(\Omega) \mid v|_T \in V_T, \forall T \in \mathsf{T}_h$, and $v^0, v^{e_1}$ and $v^{e_2}$ are continuous at the vertices and midpoints of edges of the triangles in $\mathsf{T}_h$ and vanish at the vertices and midpoints on $\partial\Omega\}$.

**Theorem 2.** *Let $V_h$ be the 15-parameter quasi-conforming finite element space, $u$ the solution of problem (2.4) and $u_h$ the one of problem (2.5). Then*

$$|u - u_h^0|_{0,\infty,\Omega} \le Ch^3 |\ln h|^{1/2} |u|_{4,\Omega} \ , \tag{4.2}$$

*when $u \in H^2_0(\Omega) \cap H^4(\Omega)$, and*

$$|u - u_h^0|_{1,\infty,\Omega} \le Ch^3 |\ln h| |u|_{4,\infty,\Omega} \ , \tag{4.3}$$

*when $u \in W^{4,\infty}(\Omega)$.*

**12-Parameter Quasi-Conforming Triangle Element**. For $T \in \mathsf{T}_h$, let $P_4'(T)$ be the subspace of $P_4(T)$, such that, for $p \in P_4'(T)$, $p(a_{ij}) = \frac{1}{2}(p(a_i) + p(a_j)) + \frac{1}{8}(Dp(a_i) -$

$Dp(a_j))(a_j - a_i)$, $1 \le i < j \le 3$. It is easily to show that $P_3(T) \subset P_4'(T)$ and for $p \in P_4'(T)$, $p$ is uniquely determined by the values of $p$ at $a_i (1 \le i \le 3)$, the values of $Dp$ at $a_i (1 \le i \le 3)$ and the values of $\partial p/\partial N$ at $a_{ij} (1 \le i < j \le 3)$, and that $p$ is cubic on each edge of $T$.

For all $v \in P_4'(T)$, let $\Pi_{\partial T}^N p$ be defined as for 15-parameter quasi-conforming element, $\Pi_T^{(2,0)} p$, $\Pi_T^{(1,1)} p$, $\Pi_T^{(0,2)} p$ be determined by (4.1).

Let $V_T = \{(p, \partial_x p, \partial_y p, \Pi_T^{(2,0)} p, \Pi_T^{(1,1)} p, \Pi_T^{(0,2)} p) \mid p \in P_4'(K)\}$.

Define $V_h = \{\ v \in L^{2,2}(\Omega) \mid v|_T \in V_T, \forall T \in \mathsf{T}_h$, and $v^0, v^{e_1}$ and $v^{e_2}$ are continuous at the vertices and the midpoints of the edges of the triangles in $\mathsf{T}_h$ and vanish at the vertices and midpoints on $\partial\Omega\}$.

**Theorem 3.** *Let $V_h$ be the 12-parameter quasi-conforming triangle finite element space, $u$ the solution of problem (2.4) and $u_h$ the one of problem (2.5). Then*

$$|u - u_h^0|_{0,\infty,\Omega} \le Ch^2 |\ln h|^{1/2} |u|_{3,\Omega}, \tag{4.4}$$

*when $f \in H^{-1}(\Omega)$, and*

$$|u - u_h^0|_{1,\infty,\Omega} \le Ch^2 |\ln h| |u|_{3,\infty,\Omega}, \tag{4.5}$$

*when $u \in W^{3,\infty}(\Omega)$.*

**12-Parameter Quasi-Conforming Rectangle Element**. Now let $\Omega$ be a rectangle in $R^2$ which edges parallel to some coordinate axis. Let $\mathsf{T}_h$ be a family of rectangle subdivisions of $\Omega$. For each rectangle $T$, let $a_0 = (x_0, y_0)$ be its center point and $P_3''(T) = P_3(T) \oplus$ span $\{(x-x_0)^3(y-y_0), (x-x_0)(y-y_0)^3\}$. For $p \in P_3''(T)$, $p$ is determined uniquely by the values of $p$ and the values of $Dp$ at four vertices of $T$.

For each rectangle $T$ with center $(x_0, y_0)$ and $p \in P_3''(T)$, define $\Pi_{\partial T}^N p$, $\Pi_T^{(2,0)} p$, $\Pi_T^{(1,1)} p$ and $\Pi_T^{(0,2)} p$ as follows. $\Pi_{\partial T}^N p$ is linear on each edge of $T$ and equals to $\partial p/\partial N$ at the endpoints. Let $P^{(2,0)} = P^{(0,2)} =$span$\{1, x-x_0, y-y_0, (x-x_0)(y-y_0)\}$, $P^{(1,1)} =$span$\{1, x-x_0, y-y_0, (x-x_0)^2, (y-y_0)^2\}$, then $\Pi_T^\alpha p \in P^\alpha$ for $|\alpha| = 2$, and are determined by

$$\left\{\begin{aligned} \int_T q \Pi_T^{(2,0)} p \, dxdy &= \int_{\partial T} q(N_x^2 \Pi_{\partial T}^N p - N_x N_y \tfrac{\partial p}{\partial s}) ds \\ &\quad - \int_T \partial_x q \partial_x p \, dxdy, \qquad \forall q \in P^{(2,0)}, \\ 2\int_T q \Pi_T^{(1,1)} p \, dxdy &= \int_{\partial T} q(2 N_x N_y \Pi_{\partial T}^N p + (N_x^2 - N_y^2) \tfrac{\partial p}{\partial s}) ds \\ &\quad - \int_T (\partial_x q \partial_y p + \partial_y q \partial_x p) \, dxdy, \qquad \forall q \in P^{(1,1)}, \\ \int_T q \Pi_T^{(0,2)} p \, dxdy &= \int_{\partial T} q(N_y^2 \Pi_{\partial T}^N p + N_x N_y \tfrac{\partial p}{\partial s}) ds \\ &\quad - \int_T \partial_y q \partial_y p \, dxdy, \qquad \forall q \in P^{(0,2)}. \end{aligned}\right. \tag{4.6}$$

Let $V_T = \{(p, \partial_x p, \partial_y p, \Pi_T^{(2,0)} p, \Pi_T^{(1,1)} p, \Pi_T^{(0,2)} p) \mid p \in P_3''(T)\}$. Define $V_h = \{\ v \in L^{2,2}(\Omega)|\ v|_T \in V_T$ for $\forall T \in \mathsf{T}_h$, and $v^0, v^{e_1}$ and $v^{e_2}$ are continuous at the vertices of the rectangles in $\mathsf{T}_h$ and vanish at the vertices on $\partial\Omega\}$.

**Theorem 4.** *Let $V_h$ be the 12-parameter quasi-conforming rectangle finite element space, $u$ the solution of problem (2.4) and $u_h$ the one of problem (2.5). Then*

$$|u - u_h^0|_{0,\infty,\Omega} \le Ch^2 |\ln\, h|^{1/2} |u|_{3,\Omega}, \tag{4.7}$$

*when $f \in H^{-1}(\Omega)$, and*

$$|u - u_h^0|_{1,\infty,\Omega} \le Ch^2 |\ln\, h| |u|_{3,\infty,\Omega}, \tag{4.8}$$

*when $u \in W^{3,\infty}(\Omega)$.*

## References


[1] Chen Wanji, Liu Yingxi and Tang Limin, The formulation of quasi-conforming elements, *Journal of Dalian Institute of Technology*, **19** : 2 (1980), 37–49.

[2] Jiang Heyang, Derivation of higher precision trianglar plate element by quasi-conforming element method, *Journal of Dalian Institute of Technology*, **20**, Suppl. 2 (1981), 21–28.

[3] Shi Guangyu, 12-parameter rectangle quasi-conforming element for plate bending, Master Thesis, Dalian Institute of Technology, 1980.

[4] Tang Limin, Chen Wanji and Liu Yingxi, Quasi-conforming elements for finite element analysis, *Journal of Dalian Institute of Technology*, **19** : 2 (1980), 19–35.

[5] Wang Ming, On the inequalities for the maximum norm of nonconforming finite element spaces, *Mathematica Numerica Sinica*, **12** : 1 (1990), in Chinese.

[6] Wang Ming, $L^\infty$ error estimates of nonconforming finite elements for the biharmonic equation, Research report, 11 (1992), Institute of Mathematics and Department of Mathematics, Peking Unversity.

[7] Wang Ming, $L^\infty$ convergnece of conforming finite elements for the biharmonic equation.

[8] Zhang Hongqing and Wang Ming, *The Mathematical Theory of Finite Element Methods*, Science Press, Beijing, 1991, in chinese.