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# GENERAL INTERPOLATION FORMULAS FOR SPACES OF DISCRETE FUNCTIONS WITH NONUNIFORM MESHES*[1)]

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**Abstract**

The unequal meshsteps are unavoidable in general for scientific and engineering computations especially in large scale computations. The analysis of difference schemes with nonuniform meshes is very rare even by use of fully heuristic methods. For the purpose of the systematic and theoretical study of the finite difference method with nonuniform meshes for the problems of partial differential equations, the general interpolation formulas for the spaces of discrete functions of one index with unequal meshsteps are established in the present work. These formulas give the connected relationships among the norms of various types, such as the sum of powers of discrete values, the discrete maximum modulo, the discrete Hölder and Lipschitz coefficients.

## 1. Introduction

The great number of problems for the large scale scientific and engineering computations concern the numerical solutions of various problems for the partial differential equations and systems in mathematical physics. The finite difference method is the most commonly used in these computations. So the theoretical and numerical studies of the finite difference schemes for the problems of the partial differential equations and systems naturally call people's great attentions.

The imbedding theorems and the interpolation formulas for the functions of Sobolev's spaces are very useful in the linear and nonlinear theory of the partial differential equations [1-4]. It is natural that the analogous extensions of the interpolation formulas for the discrete functional spaces must play the extremely important role in the study of the finite difference approximations to the problems of linear and nonlinear partial differential equations and systems. The discrete interpolation formulas and their consequences can be used in the study of the convergence and stability of the finite difference schemes for the various problems of linear and nonlinear systems of partial differential equations of different types. And they can also be used to construct the

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weak, generalized and classical, local and global solution for the problems of partial differential equations and systems. [5-11]

The finite difference schemes with unequal meshsteps for the problems of partial differential equations are much more complicated than the schemes with equal meshsteps. There are only very few simple results concerning this topic. Establishment of the general interpolation formulas for the spaces of discrete functions with unequal meshsteps obviously gives the possibility and strong apparatus for the systematic studies of the finite difference schemes with unequal meshsteps for the problems of partial differential equations.

The purpose of the present work is to establish a series of general interpolation formulas for the discrete functional spaces of discrete functions with equal and unequal meshsteps. These general interpolation formulas give the connected relationship among the discrete norms as the summations of powers, the maximum modulo and the Lipschitz and Hölder quotients for different discrete functional spaces. Also a series of consequences, derivations and applications for these interpolation formulas are justified. They are very commonly used in the further study for the finite difference approximations to the theory of partial differential equations.

## 2

Let us divide the finite interval $[0,l]$ into the small segments by the grid points $\{x_j|j=0,1,\cdots,J\}$, where $0=x_0<x_1<\cdots<x_{J-1}<x_J=l$, $J$ is an integer and $h_{j+\frac{1}{2}}=x_{j+1}-x_j>0(j=0,1,\cdots,J-1)$ are the equal and unequal meshsteps. The discrete function $u_h=\{u_j|j=0,1,\cdots,J\}$ is defined on the grid points $\{x_j|j=0,1,\cdots,J\}$ with unequal in general meshsteps $h=\{h_{j+\frac{1}{2}}|j=0,1,\cdots,J-1\}$. Let us denote $\triangle_+u_j=u_{j+1}-u_j$ or simplely $\triangle u_j=u_{j+1}-u_j(j=0,1,\cdots,J-1)$ and $\triangle_-u_j=u_j-u_{j-1}(j=0,1,\cdots,J)$.

Now let us introduce some notations of the difference quotients for the discrete function $u_h=\{u_j|j=0,1,\cdots,J\}$. As the discrete functions we take the notation for the difference quotient of first order

$$\delta u_h=\Big\{\delta u_{j+\frac{1}{2}}=\frac{u_{j+1}-u_j}{h_{j+\frac{1}{2}}}\Big|j=0,1,\cdots,J-1\Big\}, \tag{1}$$

which can be regarded as a discrete function defined on the grid points

$$\Big\{x^{(1)}_{j+\frac{1}{2}}=\frac{1}{2}(x_{j+1}+x_j)\Big|j=0,1,\cdots,J-1\Big\}.$$

of the interval $[x^{(1)}_{\frac{1}{2}},x^{(1)}_{J-\frac{1}{2}}]$ of length $x^{(1)}_{J-\frac{1}{2}}-x^{(1)}_{\frac{1}{2}}=l-\frac{1}{2}(h_{\frac{1}{2}}+h_{J-\frac{1}{2}})$ with the unequal in general meshsteps

$$\Big\{h^{(1)}_{j+\frac{1}{2}}=h_{j+\frac{1}{2}}\Big|j=0,1,\cdots,J-1\Big\}.$$

The difference quotient of second order for the discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ is a discrete function

$$\delta^2 u_h = \Big\{\delta^2 u_j = \frac{\delta u_{j+\frac{1}{2}} - \delta u_{j-\frac{1}{2}}}{h_j^{(2)}} \Big| j = 0, 1, \cdots, J-1\Big\}. \tag{2}$$

The grid points of this discrete function are

$$\Big\{x_j^{(2)} = \frac{1}{2}(x_{j+\frac{1}{2}}^{(1)} + x_{j-\frac{1}{2}}^{(1)}) \Big| j = 0, 1, \cdots, J-1\Big\}$$

of the interval $[x_1^{(2)}, x_{J-1}^{(2)}]$ with length $x_{J-1}^{(2)} - x_1^{(2)}$ and the corresponding unequal mesh-steps are

$$\Big\{h_j^{(2)} = \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}) \Big| j = 0, 1, \cdots, J-1\Big\}.$$

For the difference quotients of higher order, we have

$$\delta^3 u_h = \Big\{\delta^3 u_{j+\frac{1}{2}} = \frac{\delta^2 u_{j+1} - \delta^2 u_j}{h_{j+\frac{1}{2}}^{(3)}} \Big| j = 0, 1, \cdots, J-2\Big\},$$

$$\delta^4 u_h = \Big\{\delta^4 u_j = \frac{\delta^3 u_{j+\frac{1}{2}} - \delta^3 u_{j-\frac{1}{2}}}{h_j^{(4)}} \Big| j = 2, \cdots, J-2\Big\}, \tag{3}$$

$$\cdots\cdots$$

$$\delta^{2k+1} u_h = \Big\{\delta^{2k+1} u_{j+\frac{1}{2}} = \frac{\delta^{2k} u_{j+1} - \delta^{2k} u_j}{h_{j+\frac{1}{2}}^{(2k+1)}} \Big| j = k, k+1, \cdots, J-(k+1)\Big\},$$

$$\delta^{2k+2} u_h = \Big\{\delta^{2k+2} u_j = \frac{\delta^{2k+1} u_{j+\frac{1}{2}} - \delta^{2k+1} u_{j-\frac{1}{2}}}{h_j^{(2k+2)}} \Big| j = k+1, \cdots, J-(k+1)\Big\},$$

$$k = 0, 1, \cdots,$$

where

$$h_{j+\frac{1}{2}}^{(1)} = h_{j+\frac{1}{2}};$$

$$h_j^{(2)} = \frac{1}{2}(h_{j+\frac{1}{2}}^{(1)} + h_{j-\frac{1}{2}}^{(1)}) = \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}});$$

$$h_{j+\frac{1}{2}}^{(3)} = \frac{1}{2}(h_{j+1}^{(2)} + h_j^{(2)}) = \frac{1}{4}(h_{j+\frac{3}{2}} + 2h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}});$$

$$h_j^{(4)} = \frac{1}{2}(h_{j+\frac{1}{2}}^{(3)} + h_{j-\frac{1}{2}}^{(3)}) = \frac{1}{8}(h_{j+\frac{3}{2}} + 3h_{j+\frac{1}{2}} + 3h_{j-\frac{1}{2}} + h_{j-\frac{3}{2}});$$

$$\cdots\cdots$$

$$h_{j+\frac{1}{2}}^{(2k+1)} = \frac{1}{2}(h_{j+1}^{(2k)} + h_j^{(2k)}) = \frac{1}{2^{2k}} \sum_{i=0}^{2k} \binom{2k}{i} h_{j+k+\frac{1}{2}-i};$$

$$h_j^{(2k+2)} = \frac{1}{2}(h_{j+\frac{1}{2}}^{(2k+1)} + u_{j-\frac{1}{2}}^{(2k+1)}) = \frac{1}{2^{2k+1}} \sum_{i=0}^{2k+1} \binom{2k+1}{i} h_{j+k+\frac{1}{2}-i},$$
$$k = 0, 1, \cdots. \quad (4)$$

The discrete difference quotients $\delta^{2k+1}u_h$ and $\delta^{2k+2}u_h$ ($k \geq 0$) can be regarded as the discrete functions defined on the grid points $\{x_{j+\frac{1}{2}}^{(2k+1)} | j = k, .., J-(k+1)\}$ and $\{x_j^{(2k+2)} | j = k+1, \cdots, J-(k+1)\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}}^{(2k+1)} | j = k, \cdots, J-(k+1)\}$ and $\{h_j^{(2k+2)} | j = k+1, \cdots, J-(k+1)\}$ of the intervals $[x_{k+\frac{1}{2}}^{(2k+1)}, x_{J-(k+\frac{1}{2})}^{(2k+1)}]$ and $[x_{k+1}^{(2k+2)}, x_{J-(k+1)}^{(2k+2)}]$ with the lengths $\bar{l}_{2k+1} = x_{J-(k+\frac{1}{2})}^{(2k+1)} - x_{k+\frac{1}{2}}^{(2k+1)}$ and $\bar{l}_{2k+2} = x_{J-(k+1)}^{(2k+2)} - x_{k+1}^{(2k+2)}$ respectively, where

$$x_{j+\frac{1}{2}}^{(2k+1)} = \frac{1}{2^{2k}} \sum_{i=0}^{2k+1} \binom{2k+1}{i} x_{j+i-k}, (j = k, \cdots, J-(k+1));$$

$$x_j^{(2k+2)} = \frac{1}{2^{2k+1}} \sum_{i=0}^{2k+2} \binom{2k+2}{i} x_{j+i-(k+1)}, (j = k+1, \cdots, J-(k+1)) \quad (5)$$

and

$$h_{j+\frac{1}{2}}^{(2k+1)} = x_{j+1}^{(2k)} - x_j^{(2k)};$$
$$x_{j+\frac{1}{2}}^{(2k+1)} = \frac{1}{2}(x_{j+1}^{(2k)} + x_j^{(2k)}), \ j = k, \cdots, J-(k+1);$$
$$h_j^{(2k+2)} = x_{j+\frac{1}{2}}^{(2k+1)} - x_{j-\frac{1}{2}}^{(2k+1)},$$
$$x_j^{(2k+2)} = \frac{1}{2}(x_{j+\frac{1}{2}}^{(2k+1)} + x_{j+\frac{1}{2}}^{(2k+1)}), \ j = k+1, \cdots, J-(k+1) \quad (6)$$

with $x_j^0 = x_j (j = 0, 1, \cdots, J)$.

Let us denote

$$h^* = \max_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}, \quad h_* = \min_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}. \quad (7)$$

It is clear that

$$h^* \geq \max_{j=k,\cdots,J-(k+1)} h_{j+\frac{1}{2}}^{(2k+1)}, \quad h^* \leq \max_{j=k+1,\cdots,J-(k+1)} h_j^{(2k+2)},$$

and

$$h_* \leq \min_{j=k,\cdots,J-(k+1)} h_{j+\frac{1}{2}}^{(2k+1)}, \quad h_* \leq \min_{j=k+1,\cdots,J-(k+1)} h_j^{(2k+2)}$$

for $k = 0, 1, \cdots$. And it can be verified that

$$l \geq \bar{l}_{2k+1} \geq l - 2kh^* \geq \frac{1}{2}l, \quad l > \bar{l}_{2k+2} \geq l - (2k+1)h^* \geq \frac{1}{2}l$$

for $k = 0, 1, \cdots$.

## 3

The norms of the discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps are defined as

$$\|u_h\|_p = (\frac{1}{2}|u_0|^p h_{\frac{1}{2}} + \sum_{j=1}^{J-1} |u_j|^p \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}) + \frac{1}{2}|u_J|^p h_{J-\frac{1}{2}})^{\frac{1}{p}} \tag{8}$$

or

$$\|u_h\|_p = (\sum_{j=0}^{J} \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}})|u_j|^p)^{\frac{1}{p}} \tag{9}$$

or

$$\|u_h\|_p = (\sum_{j=0}^{J-1} \frac{1}{2}(|u_j|^p + |u_{j+1}|^p) h_{j+\frac{1}{2}})^{\frac{1}{p}} \tag{10}$$

and

$$\|u_h\|_\infty = \max_{j=0,1,\cdots,J} |u_j|, \tag{11}$$

where $h_{-\frac{1}{2}} = h_{J+\frac{1}{2}} = 0$ and $1 \leq p < \infty$.

The difference quotient of first order is the discrete function $\delta u_h$ has the norm as

$$\|\delta u_h\|_\infty = \max_{j=0,1,\cdots,J-1} |\delta v_{j+\frac{1}{2}}| \tag{12}$$

and

$$\|\delta u_h\|_p = (\sum_{j=0}^{J-1} |\delta u_{j+\frac{1}{2}}|^p h_{j+\frac{1}{2}})^{\frac{1}{p}} \tag{13}$$

where $1 \leq p < \infty$ is a real number. The norms of the difference quotients $\delta^2 u_h$ of second order for the discrete function $u_h$ have the expressions as

$$\|\delta^2 u_h\|_\infty = \max_{j=0,1,\cdots,J-1} |\delta^2 u_j| \tag{14}$$

and

$$\|\delta^2 u_h\|_p = (\sum_{j=1}^{J-1} |\delta^2 u_j|^p h_j^{(2)})^{\frac{1}{p}}, \tag{15}$$

where $1 \leq p < \infty$.

Then for the norms of the difference quotients $\delta^k u_h$ of order $k \geq 1$, we take the notations as follows:

$$\|\delta^{2k+1} u_h\|_p = (\sum_{j=k}^{J-(k+1)} |\delta^{2k+1} u_{j+\frac{1}{2}}|^p h_{j+\frac{1}{2}}^{(2k+1)})^{\frac{1}{p}},$$

$$\|\delta^{2k+2}u_h\|_p = (\sum_{j=k+1}^{J-(k+1)} |\delta^{2k+2}u_j|^p h_j^{(2k+2)})^{\frac{1}{p}}, \tag{16}$$

and

$$\|\delta^{2k+1}u_h\|_\infty = \max_{j=k,\cdots,J-(k+1)} |\delta^{2k+1}u_{j+\frac{1}{2}}|,$$
$$\|\delta^{2k+2}u_h\|_\infty = \max_{j=k+1,\cdots,J-(k+1)} |\delta^{2k+2}u_j|, \tag{17}$$

where $1 \le p < \infty$ and $k = 0, 1, \cdots$.

Denote by

$$M = \max_{j=0,1,\cdots,J-1} \left\{ \frac{h_{j-\frac{1}{2}}}{h_{j+\frac{1}{2}}}, \frac{h_{j+\frac{1}{2}}}{h_{j-\frac{1}{2}}} \right\}$$

the maximum ratio constant of two consecutive unequal meshsteps or simply the ratio constant of meshsteps.

## 4

**Lemma 1.** *For any discrete functions $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for any constants $1 \le q, r \le \infty$ and $q \le p \le \infty$, there is*

$$\|u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta u_h\|_r^\alpha + l^{\frac{1}{p}-\frac{1}{q}} \|u_h\|_q) \tag{18}$$

*with*

$$\frac{1}{p} = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - 1\Big) \tag{19}$$

*and*

$$0 \le \alpha \le \frac{\frac{1}{q}}{1 - \frac{1}{r} + \frac{1}{q}} \le 1, \tag{20}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} > 0 | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$.*

*Proof.* For any $u_h = \{u_j | j = 0, 1, \cdots, J\}$, we have

$$|u_m|^d - |u_s|^d \le |u_m^d - u_s^d| \le d \sum_{j=s}^{m-1} (|u_{j+1}|^{d-1} + |u_j|^{d-1}) |u_{j+1} - u_j|,$$

where $d > 1$ and $0 \le s < m \le J$. Let $1 \le g, r < \infty$ and

$$\frac{1}{g} + \frac{1}{r} = 1.$$

Here we then have

$$|u_m|^d \le d[\sum_{j=s}^{m-1} (|u_{j+1}|^{d-1} + |u_j|^{d-1})^g h_{j+\frac{1}{2}}]^{\frac{1}{g}} [\sum_{j=s}^{m-1} |\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}}|^r h_{j+\frac{1}{2}}]^{\frac{1}{r}} + |u_s|^d$$

$\leq 2d[\sum_{j=s}^{m-1}(|u_{j+1}|^{(d-1)g}+|u_j|^{(d-1)g})h_{j+\frac{1}{2}}]^{\frac{1}{g}}\|\delta u_h\|_r+|u_s|^d$

$\leq 2d2^{\frac{1}{g}}\|u_h\|_{(d-1)g}^{d-1}\|\delta u_h\|_r+|u_s|^d.Take$(d-1)g=q(q$\geq 1$), then

$$\frac{1}{d}=\frac{\frac{1}{q}}{1-\frac{1}{r}+\frac{1}{q}}.$$

If for $j=0,1,\cdots,J$, $|u_j|\geq a>0$, then

$$\|u_h\|_q\geq al^{\frac{1}{q}}.$$

Thus there always exists such a $u_s$, that

$$\|u_h\|_q\geq |u_s|l^{\frac{1}{q}}.$$

Taking this special $u_s$, we get for any $m=0,1,\cdots,J$,

$$|u_m|^d\leq 4d\|u_h\|_q^{d-1}\|\delta u_h\|_r+l^{-\frac{d}{q}}\|u_h\|_q^d.$$

Hence we have

$$\|u_h\|_\infty\leq 4\|u_h\|_q^{1-\frac{1}{d}}\|\delta u_h\|_r^{\frac{1}{d}}+l^{-\frac{1}{q}}\|u_h\|_q,$$

where $(4d)^{\frac{1}{d}}\leq 4$ for $d\geq 1$.

For any $1\leq q\leq p<\infty$, there is

$$\|u_h\|_p^p\leq \|u_h\|_\infty^{p-q}\|u_h\|_q^q.$$

Therefore, we have

$$\|u_h\|_p\leq 4\|u_h\|_q^{1-\alpha}\|\delta u_h\|_r^\alpha+l^{\frac{1}{p}-\frac{1}{q}}\|u_h\|_q,$$

where

$$\alpha=\frac{\frac{1}{q}-\frac{1}{p}}{1-\frac{1}{r}+\frac{1}{q}}$$

for any $1\leq q\leq p<\infty$.

By means of Hölder inequality, we have

$$\frac{\|u_h\|_q}{l^{\frac{1}{q}}}\leq\frac{\|u_h\|_p}{l^{\frac{1}{p}}}\leq\|u_h\|_\infty,p\geq q$$

and then also

$$\frac{\|\delta u_h\|_r}{l^{\frac{1}{r}}}\leq\|\delta u_h\|_\infty.$$

These show that

$$\lim_{q\to\infty}\|u_h\|_q=\|u_h\|_\infty,\quad \lim_{r\to\infty}\|\delta u_h\|_r=\|\delta u_h\|_\infty. \tag{21}$$

Hence the obtained estimate is valid also for $r = \infty$ and $q = \infty$. Then the lemma is proved.

**Lemma 2.** *For every discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for any constants $1 \leq q, r \leq \infty$ and $q \leq p \leq \infty$, there is*

$$\|\delta^k u_h\|_p \leq C(\|\delta^k u_h\|_q^{1-\alpha} \|\delta^{k+1} u_h\|_r^{\alpha} + l^{\frac{1}{p}-\frac{1}{q}} \|\delta^k u_h\|_q) \tag{22}$$

*with*

$$\frac{1}{p} = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - 1\Big) \tag{23}$$

*and*

$$0 \leq \alpha \leq \frac{\frac{1}{q}}{1 - \frac{1}{r} + \frac{1}{q}} \leq 1, \tag{24}$$

*where $k \geq 1$ and $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} > 0 | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$ and dependent on the ratio constant $M$ of meshsteps.*

*Proof.* For the sake of brevity, let us consider the case of $k$ being odd integer, $k = 2k' + 1$, $k' = 0, 1, \cdots$. Then let $v_h = \delta^k u_h$ or

$$v_h = \{v_j = \delta^k u_{j+\frac{1}{2}} | j = k', \cdots, J - (k'+1)\}.$$

This discrete function $v_h = \delta^k u_h$ is defined on the grid points

$$\{y_j = x^{(k)}_{j+\frac{1}{2}} | j = k', \cdots, J - (k'+1)\}$$

with the meshsteps

$$\{\tau_{j+\frac{1}{2}} = y_{j+1} - y_j = x^{(k)}_{j+\frac{3}{2}} - x^{(k)}_{j+\frac{1}{2}} = h^{(k+1)}_{j+1} | j = k', \cdots, J - k' - 2\}$$

on the interval $[y_{k'}, y_{J-(k'+1)}] \equiv [x^{(k)}_{k'+\frac{1}{2}}, x^{(k)}_{J-k'-\frac{1}{2}}]$ of length $\bar{l}_k = y_{J-k'-1} - y_{k'} = x^{(k)}_{J-k'-\frac{1}{2}} - x^{(k)}_{k'+\frac{1}{2}} \geq \frac{1}{2} l$. Here we also have $\delta v_h = \delta^{k+1} u_h$, in fact

$$\delta v_{j+\frac{1}{2}} = \frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}} = \frac{\delta^k u_{j+\frac{3}{2}} - \delta^k u_{j+\frac{1}{2}}}{h^{(k+1)}_{j+1}} = \delta^{k+1} u_{j+1}$$

for $j = k', \cdots, J - k' - 2$.

By the same way as the begin of the proof of Lemma 1, we have for $d > 0$, $k' \leq s < m \leq J - k' - 1$ and $1 \leq q, r < \infty$ with $\frac{1}{g} + \frac{1}{r} = 1$, the estimate

$$|v_m|^d \leq 2d \Big[ \sum_{j=s}^{m-1} (|v_{j+1}|^q + |v_j|^q) \tau_{j+\frac{1}{2}} \Big]^{\frac{1}{g}} \Big[ \sum_{j=s}^{m-1} \Big| \frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}} \Big|^r \tau_{j+\frac{1}{2}} \Big]^{\frac{1}{r}} + |v_s|^d$$

or

$$|\delta^k u_{m+\frac12}|^d \le 2d\Big[\sum_{j=s}^{m-1}(|\delta^k u_{j+\frac32}|^q+|\delta^k u_{j+\frac12}|^q)h_{j+1}^{(k+1)}\Big]^{\frac1g}$$

$\times\Big[\sum_{j=s}^{m-1}|\delta^{k+1}u_{j+1}|^r h_{j+1}^{(k+1)}\Big]^{\frac1r}+|\delta^k u_{s+\frac12}|^d, where$q=(d-1)g$\ge 1$. Then we have

$$|\delta^k u_{m+\frac12}|^d \le 2dW^{\frac1g}\|\delta^{k+1}u_h\|_r+|\delta^k u_{s+\frac12}|^d,$$

where

$$W=\sum_{j=k'}^{J-k'-2}(|\delta^k u_{j+\frac32}|^q+|\delta^k u_{j+\frac12}|^q)h_{j+1}^{(k+1)}.$$

Here we have

$$W=|\delta^k u_{k'+\frac12}|^q h_{k'+1}^{(k+1)}+\sum_{j=k'+1}^{(J-k'-2)}|\delta^k u_{j+\frac12}|^q(h_j^{(k+1)}+h_{j+1}^{(k+1)})+|\delta^k u_{J-k'-\frac12}|^q h_{J-k'-1}^{(k+1)}$$

$=$—$\delta^k u_{k'+\frac12}|^q h_{k'+\frac12}^{(k)}\Big(\frac{h_{k'+1}^{(k+1)}}{h_{k'+\frac12}^{(k)}}\Big)+\sum_{j=k'+1}^{J-k'-2}|\delta^k u_{j+\frac12}|^q h_{j+\frac12}^{(k)}\Big(\frac{h_j^{(k+1)}+h_{j+1}^{(k+1)}}{h_{j+\frac12}^{(k)}}\Big)$

$+$—$\delta^k u_{J-k'-\frac12}|^q h_{J-k'-\frac12}^{(k)}\Big(\frac{h_{J-k'-1}^{(k+1)}}{h_{J-k'-\frac12}^{(k)}}\Big)$

$\le 2\|\delta^k u_h\|_q^q\Big\{\max_{j=k'+1,\cdots,J-k'-1}\frac{h_j^{(k+1)}}{h_{j+\frac12}^{(k)}},\max_{j=k',\cdots,J-k'-2}\frac{h_{j+1}^{(k+1)}}{h_{j+\frac12}^{(k)}}\Big\}.Since\frac{h_j^{(k+1)}}{h_{j+\frac12}^{(k)}}=\frac{h_j^{(2k'+2)}}{h_{j+\frac12}^{(2k'+1)}}=$ $\frac{\frac{1}{2^{2k'+1}}\sum_{i=0}^{2k'+1}\binom{2k'+1}{i}h_{j+k'+\frac12-i}}{\frac{1}{2^{2k'}}\sum_{i=0}^{2k'}\binom{2k'}{i}h_{j+k'+\frac12-i}}$

$\le\ \frac12(1+M)M^{k-1},j\ =\ k'+1,\cdot\cdot\cdot,J-k'-1and\frac{h_{j+1}^{(k+1)}}{h_{j+\frac12}^{(k)}}\ \le\ \frac12(1+M)M^{k-1},j\ =$ $k',\cdot\cdot\cdot,J-k'-2,thenwehaveW\le(1+M)M^{k-1}\|\delta^k u_h\|_q^q.$

This shows that

$$|\delta^k u_{m+\frac12}|^d \le 2d(1+M)^{\frac1g}M^{\frac{k-1}{g}}\|\delta^k u_h\|_q^{d-1}\|\delta^{k+1}u_h\|_r+|\delta^k u_{s+\frac12}|^d.$$

Similarly to the proof of Lemma 1, we get the estimate

$$\|\delta^k u_h\|_p\le C(M)(\|\delta^k u_h\|_q^{1-\alpha}\|\delta^{k+1}u_h\|_r^{\alpha}+l^{\frac1p-\frac1q}\|\delta^k u_h\|_q)$$

with

$$\frac1p=\frac{1-\alpha}{q}+\alpha\Big(\frac1r-1\Big),$$

where $C(M)$ depends on the maximum ratio constant $M$ of two consecutive unequal meshsteps. This gives the result of the lemma for the case $k=2k'+1$ being odd.

For the case of $k$ being even, $k = 2k'$, $k' = 1, 2, \cdots$, let us denote $v_h = \delta^k u_h$ or

$$v_h = \{v_j = \delta^k u_j | j = k', \cdots, J - k'\}.$$

This discrete function $v_h$ is defined on the grid points

$$\{y_j = x_j^{(k)} | j = k', \cdots, J - k'\}$$

with the meshsteps

$$\{\tau_{j+\frac{1}{2}} = y_{j+1} - y_j = x_{j+1}^{(k)} - x_j^{(k)} = h_{j+\frac{1}{2}}^{(k+1)} | j = k', \cdots, J - k' - 1\}$$

on the interval $[y_{k'}, y_{J-k'}] \equiv [x_{k'}^{(k)}, x_{J-k'}^{(k)}]$ of length $\bar{l}_k = y_{J-k'} - y_{k'} = x_{J-k'}^{(k)} - x_{k'}^{(k)} \geq \frac{1}{2} l$. Here we have $\delta v_h = \delta^{k+1} u_h$, in fact

$$\delta v_{j+\frac{1}{2}} = \frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}} = \frac{\delta^k u_{j+1} - \delta^k u_j}{h_{j+\frac{1}{2}}^{(k+1)}} = \delta^{k+1} u_{j+\frac{1}{2}}$$

for $j = k', \cdots, J - k' - 1$.

Similarly we have for $d > 0$, $k' \leq s < m \leq J - k'$ and $1 \leq q, r < \infty$ with $\frac{1}{g} + \frac{1}{r} = 1$, the estimate

$$|v_m|^d \leq 2d \Big[ \sum_{j=s}^{m-1} (|v_{j+1}|^q + |v_j|^q) \tau_{j+\frac{1}{2}} \Big]^{\frac{1}{g}} \Big[ \sum_{j=s}^{m-1} \Big| \frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}} \Big|^r \tau_{j+\frac{1}{2}} \Big]^{\frac{1}{r}} + |\delta^k v_s|^d$$

or

$$|\delta^k u_m|^d \leq 2d \Big[ \sum_{j=s}^{m-1} (|\delta^k u_{j+1}|^q + |\delta^k u_j|^q) h_{j+\frac{1}{2}}^{(k+1)} \Big]^{\frac{1}{g}} \Big[ \sum_{j=s}^{m-1} \Big| \delta^{k+1} u_{j+\frac{1}{2}} \Big|^r h_{j+\frac{1}{2}}^{(k+1)} \Big]^{\frac{1}{r}} + |\delta^k u_s|^d,$$

where $q = (d-1)g \geq 1$. Then we have also

$$|\delta^k u_m|^d \leq 2d W^{\frac{1}{g}} \|\delta^{k+1} u_h\|_r + |\delta^k u_s|^d,$$

where

$$W = \sum_{j=k'}^{J-k'-1} (|\delta^k u_{j+1}|^q + |\delta^k u_j|^q) h_{j+\frac{1}{2}}^{(k+1)}.$$

Similarly, we can prove that

$$W \leq (1 + M) M^{k-1} \|\delta^k u_h\|_q^q.$$

Hence we have also the estimate

$$|\delta^k u_m|^d \leq 2d(1 + M)^{\frac{1}{g}} M^{\frac{k-1}{g}} \|\delta^k u_h\|_q^{d-1} \|\delta^{k+1} u_h\|_r + |\delta^k u_s|^d.$$

By the method of the proof of Lemma 1, we also obtain the result of present lemma for the case of $k = 2k'$ being even, $k' = 1, 2, \cdots$.

Thus the lemma is proved.

## 5

**Lemma 3.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for the constants $1 \le q, r \le \infty$ and $1 \le p \le \infty$, there is*

$$\|\delta u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta^2 u_h\|_r^\alpha + l^{\frac{1}{p} - \frac{1}{q} - 1} \|u_h\|_q) \tag{25}$$

*with*

$$\frac{1}{p} - 1 = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - 2\Big) \tag{26}$$

*and*

$$\frac{1}{2} \le \alpha \le \frac{1 + \frac{1}{q}}{2 - \frac{1}{r} + \frac{1}{q}} \le 1, \tag{27}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} > 0 | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$ and $C$ depends on the ratio constant $M$ of the meshsteps.*

*Proof.* (1) For any $u_h = \{u_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$, we have

$$\|\delta u_h\|_2^2 = \sum_{j=0}^{J-1} \Big| \frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}} \Big|^2 h_{j+\frac{1}{2}} = u_J \frac{\triangle_- u_J}{h_{J-\frac{1}{2}}} - u_0 \frac{\triangle_+ u_0}{h_{\frac{1}{2}}} - \sum_{j=1}^{J-1} u_j \left( \frac{\frac{\triangle_+ u_j}{h_{j+\frac{1}{2}}} - \frac{\triangle_- u_j}{h_{j-\frac{1}{2}}}}{h_j^{(2)}} \right) h_j^{(2)}$$

$\le 2\|u_h\|_\infty \|\delta u_h\|_\infty + \|u_h\|_{q_0} \|\delta^2 u_h\|_r$, *where* $h_j^{(2)} = \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}})$ for $j = 0, 1, \cdots, J-1$ and

$$\frac{1}{q_0} + \frac{1}{r} = 1.$$

From Lemmas 1 and 2, there are

$$\|u_h\|_\infty \le C(\|u_h\|_{q_0}^{1-\lambda} \|\delta u_h\|_2^\lambda + l^{-\frac{1}{q_0}} \|u_h\|_{q_0}),$$

$\|\delta u_h\|_\infty \le C(\|\delta u_h\|_2^{1-\mu} \|\delta^2 u_h\|_r^\mu + l^{-\frac{1}{2}} \|\delta u_h\|_2)$, 28*where* $0 = \frac{1-\lambda}{q_0} - \frac{\lambda}{2}$, $\quad 0 = \frac{1-\mu}{2} + \mu\Big(\frac{1}{r} - 1\Big)$, (29)*andthesecondconstant*C*dependsontheratioconstant*M*of meshsteps.Thus*1-$\lambda = \mu = \frac{r}{3r-2}$.

Substituting two above inequalities into the previous one, then we have

$$\|\delta u_h\|_2^2 \le C\{(\|u_h\|_{q_0} \|\delta^2 u_h\|_r)^\mu \|\delta u_h\|_2^{2-2\mu} + (l^{-\frac{1}{2\mu}} \|u_h\|_{q_0})^\mu \|\delta u_h\|_2^{2-\mu}$$

$$+(\mathrm{l}^{-\frac{1}{2\mu}} \|u_h\|_{q_0}) \|\delta u_h\|_2^{1-\mu} (\|u_h\|_{q_0} \|\delta^2 u_h\|_r)^\mu$$

$+(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})\|\delta u_h\|_2 + \|u_h\|_{q_0}\|\delta^2 u_h\|_r\}.$

For every terms in the right hand side of the above inequality, we have

$$C(\|u_h\|_{q_0}\|\delta^2 u_h\|_r)^{\mu}\|\delta u_h\|_2^{2-2\mu} \le \varepsilon\|\delta u_h\|_2^2 + \frac{\varepsilon\mu}{1-\mu}\Big(C\frac{1-\mu}{\varepsilon}\Big)^{\frac{1}{\mu}}\|u_h\|_{q_0}\|\delta^2 u_h\|_r,$$

$\mathrm{C}(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})^{\mu}\|\delta u_h\|_r)_2^{2-\mu} \le \varepsilon\|\delta u_h\|_2^2 + \frac{\varepsilon\mu}{2-\mu}\Big(C\frac{2-\mu}{2\varepsilon}\Big)^{\frac{2}{\mu}}(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})^2,$

$\mathrm{C}(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})\|\delta u_h\|)_2^{1-\mu}(\|u_h\|_{q_0}\|\delta^2 u_h\|_r)^{\mu}$

$\le \varepsilon\|\delta u_h\|_2^2 + \frac{2\varepsilon}{1-\mu}\Big(C\frac{1-\mu}{2\varepsilon}\Big)^{\frac{2}{1+\mu}}(l^{-\frac{1}{2\mu}}\|u_h\|_q)^2$

$+2\varepsilon\mu\frac{}{1-\mu\Big(C\frac{1-\mu}{2\varepsilon}\Big)^{\frac{2}{1+\mu}}(\|u_h\|_{q_0}\|\delta^2 u_h\|_r,}$

$\mathrm{C}(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})\|\delta u_h\|_2 \le \varepsilon\|\delta u_h\|_2^2 + \frac{C}{4\varepsilon}(l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})^2.$

Thus we get

$$(1-4\varepsilon)\|\delta u_h\|_2^2 \le \Big[\frac{\varepsilon\mu}{1-\mu}\Big(C\frac{1-\mu}{\varepsilon}\Big)^{\frac{1}{\mu}} + \frac{2\varepsilon\mu}{1-\mu}\Big(C\frac{1-\mu}{2\varepsilon}\Big)^{\frac{2}{1+\mu}}\Big](\|u_h\|_{q_0}\|\delta^2 u_h\|_r)$$

$+\Big[\varepsilon\mu\frac{}{2-\mu\Big(C\frac{2-\mu}{2\varepsilon}\Big)^{\frac{2}{\mu}}+\frac{2\varepsilon}{1-\mu}\Big(C\frac{1-\mu}{2\varepsilon}\Big)^{\frac{2}{1+\mu}}+\frac{C}{4\varepsilon}\Big](l^{-\frac{1}{2\mu}}\|u_h\|_{q_0})^2.} Taking \varepsilon = \frac{1}{8}$ and regarding $1 \le r < \infty$ and $\frac{1}{3} < \mu \le 1$, we see that the coefficients on the right hand side of the above inequality are bounded. Hence we obtain

$$\|\delta u_h\|_2 \le C(\|u_h\|_{q_0}^{\frac{1}{2}}\|\delta^2 u_h\|_r^{\frac{1}{2}} + l^{-\frac{1}{2}-\frac{1}{q_0}}\|u_h\|_{q_0}), \tag{30}$$

where $C$ depends on the ratio constant $M$ of meshsteps and is independent of the constants $1 \le q_0, r < \infty$ and $\frac{1}{q_0} + \frac{1}{r} = 1$.

(2) Let $1 \le q \le q_0$. From Lemma 1, we have

$$\|u_h\|_{q_0} \le C(\|u_h\|_q^{1-\beta}\|\delta u_h\|_2^{\beta} + l^{\frac{1}{q_0}-\frac{1}{q}}\|u_h\|_q),$$

where

$$\frac{1}{q_0} = \frac{1-\beta}{q} - \frac{\beta}{2}.$$

Substituting this relation into the inequality obtained in the previous section, we get the following inequality

$$\|\delta u_h\|_2 \le C\{\|u_h\|_q^{\frac{1-\beta}{2}}\|\delta u_h\|_2^{\frac{\beta}{2}}\|\delta^2 u_h\|_r^{\frac{1}{2}} + l^{\frac{1}{2q_0}-\frac{1}{2q}}\|u_h\|_q^{\frac{1}{2}}\|\delta^2 u_h\|_r^{\frac{1}{2}}$$

$+l^{-\frac{1}{2}-\frac{1}{q_0}}\|u_h\|_q^{1-\beta}\|\delta u_h\|_2^{\beta}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q\}. This inequality can be rewritten in the following form \|\delta u_h\|_2 \le C\{(\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma})^{\frac{\beta}{2}}\|\delta u_h\|_2^{\frac{\beta}{2}} + (l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q)^{\frac{\beta}{2}}(\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma})^{\frac{1}{2\sigma}}$

$+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q)^{1-\beta}\|\delta u_h\|_2^{\beta}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q\}.where\sigma=\frac{1}{2-\beta}$. This implies the estimate

$$\|\delta u_h\|_2 \leq C(\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q),$$

where

$$-\frac{1}{2}=\frac{1-\sigma}{q}+\sigma\Big(\frac{1}{r}-2\Big).$$

Again let us suppose that $q_0 \leq q < \infty$. Substituting

$$\|u_h\|_{q_0} \leq l^{\frac{1}{q_0}-\frac{1}{q}}\|u_h\|_q$$

into the right hand part of the inequality (30) obtained in the previous section, we get

$$\|\delta u_h\|_2 \leq C\{l^{\frac{1}{2q_0}-\frac{1}{2q}}\|u_h\|_q^{\frac{1}{2}}\|\delta^2 u_h\|_r^{\frac{1}{2}}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q\}$$

$=C\{(l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q)^{\frac{\beta}{2}}(\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma})^{\frac{1}{2\sigma}}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q\},where\sigma=\frac{1}{2-\beta}$. Then there is

$$\|\delta u_h\|_2 \leq C(\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma}+l^{-\frac{1}{2}-\frac{1}{q}}\|u_h\|_q), \tag{31}$$

where

$$-\frac{1}{2}=\frac{1-\sigma}{q}+\sigma\Big(\frac{1}{r}-2\Big).$$

Hence we then obtain the inequality in the present lemma for the case of $p=2$ and $1 \leq q, r < \infty$.

(3) For the case $p=\infty$, let us substitute the estimate (31) for $p=2$ into the estimate (28) for the case of $p=\infty$, $q=2$ of Lemma 2. we obtain

$$\|\delta u_h\|_\infty \leq C\{\|u_h\|_q^{(1-\mu)(1-\sigma)}\|\delta^2 u_h\|_r^{\mu+\sigma(1-\mu)}+l^{-(\frac{1}{2}+\frac{1}{q})(1-\mu)}\|u_h\|_q^{1-\mu}\|\delta^2 u_h\|_r^{\mu}$$

$+l^{-\frac{1}{2}}\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma}+l^{-1-\frac{1}{q}}\|u_h\|_q\},where\mu=\frac{\frac{1}{2}}{\frac{3}{2}-\frac{1}{r}}$ and $\sigma=\frac{\frac{1}{q}+\frac{1}{2}}{2+\frac{1}{q}-\frac{1}{r}}.Thisinequalitycanberewriteninthefollo$
$C\{\|u_h\|_q^{1-\bar\alpha}\|\delta^2 u_h\|_r^{\bar\alpha}+(l^{-1-\frac{1}{q}}\|u_h\|_q)^{1-\frac{\mu}{\bar\alpha}}(\|u_h\|_q^{1-\bar\alpha}\|\delta^2 u_h\|_r^{\bar\alpha})^{\frac{\mu}{\bar\alpha}}$

$+(l^{-1-\frac{1}{q}}\|u_h\|_q)^{1-\frac{\sigma}{\bar\alpha}}(\|u_h\|_q^{1-\bar\alpha}\|\delta^2 u_h\|_r^{\bar\alpha})^{\frac{\sigma}{\bar\alpha}}+l^{-1-\frac{1}{q}}\|u_h\|_q,where\bar\alpha=\mu+\sigma(1-\mu)=\frac{1+\frac{1}{q}}{2+\frac{1}{q}-\frac{1}{r}}$. This implies

$$\|\delta u_h\|_\infty \leq C(\|u_h\|_q^{1-\bar\alpha}\|\delta^2 u_h\|_r^{\bar\alpha}++l^{-1-\frac{1}{q}}\|u_h\|_q), \tag{32}$$

where

$$-1=\frac{1-\bar\alpha}{q}+\bar\alpha\Big(\frac{1}{r}-2\Big).$$

(4) Let us now consider the case of $1 \leq p < \infty$. At first let $p \geq 2$, then

$$\|\delta u_h\|_p^p \leq \|\delta u_h\|_\infty^{p-2}\|\delta u_h\|_2^2 \leq C\{\|u_h\|_q^{(1-\bar\alpha)(p-2)+2(1-\sigma)}\|\delta^2 u_h\|_r^{\bar\alpha(p-2)+2\sigma}$$

$+l^{-(1+\frac{1}{q})(p-2)}\|u_h\|_q^{p-2\sigma}\|\delta^2 u_h\|_r^{2\sigma}+l^{-(1+\frac{2}{q})}\|u_h\|_q^{p-\bar{\alpha}(p-2)}\|\delta^2 u_h\|_2^{\bar{\alpha}(p-2)}$

$+l^{-(1+\frac{1}{q})(p-2)-(1+\frac{2}{q})}\|u_h\|_q^p\} or \|\delta u_h\|_p^p \le C\{(\|u_h\|_q^{1-\alpha}\|\delta^2 u_h\|_r^{\alpha})^p+(l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q)^p$

$+(l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q)^{p-\frac{2\sigma}{\alpha}}(\|u_h\|_q^{1-\alpha}\|\delta^2 u_h\|_r^{\alpha})^{\frac{2\sigma}{\alpha}}$

$+(l^{-1-\frac{1}{p}-\frac{1}{q}}\|u_h\|_q)^{p-\frac{\bar{\alpha}(p-2)}{\alpha}}(\|u_h\|_q^{1-\alpha}\|\delta^2 u_h\|_r^{\alpha})^{\frac{\bar{\alpha}(p-2)}{\alpha}}\} where \alpha=\frac{1-\frac{1}{p}+\frac{1}{q}}{2-\frac{1}{r}+\frac{1}{q}}. Hence we have \|\delta u_h\|_p <$ $C(\|u_h\|_q^{1-\alpha}\|\delta^2 u_h\|_r^{\alpha}+l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q), (33) where \frac{1}{p}-1=\frac{1-\alpha}{q}+\alpha\Big(\frac{1}{r}-2\Big). Then let 1\le p<$ $2$, we have

$$\|\delta u_h\|_p \le l^{\frac{1}{p}-\frac{1}{2}}\|\delta u_h\|_2.$$

Hence

$$\|\delta u_h\|_p \le C\{l^{\frac{1}{p}-\frac{1}{2}}\|u_h\|_q^{1-\sigma}\|\delta^2 u_h\|_r^{\sigma}+l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q\}$$
$$\le C\{(l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q)^{1-\frac{\sigma}{\alpha}}(\|u_h\|_q^{1-\alpha}\|\delta^2 u_h\|_r^{\alpha})^{\frac{\sigma}{\alpha}}+l^{-1+\frac{1}{p}-\frac{1}{q}}\|u_h\|_q\}.$$

This shows the above estimate (33) for $1\le p<2$.

(5) We have proved till now the estimates of the present lemma for the cases of $1\le p\le\infty$ and $1\le q,r<\infty$. The estimate of the present lemma is also valid for $q=\infty$ and $r=\infty$.

In fact for any interval $[0,l]$ of finite length $l<\infty$, the set $\{\|\delta^k u_h\|_p | 1\le p<\infty\}$ is uniformly bounded $(k=0,1,\cdots)$, that is

$$\frac{\|\delta^k u_k\|_q}{\bar{l}^{\frac{1}{q}}}\le\frac{\|\delta^k u_k\|_p}{\bar{l}^{\frac{1}{q}}}\le\|\delta^k u_h\|_\infty \tag{34}$$

for $1\le q\le p<\infty$, $k=0,1,\cdots$, where

$$\bar{l}=\sum_{j=[\frac{k}{2}]}^{J-[\frac{k+1}{2}]}h_{j+\frac{1}{2}}\ge l-(k-1)h^*\ge\frac{l}{2},$$

as $J$ large and $h^*$ is small. Then we have also

$$\lim_{p\to\infty}\|\delta^k u_h\|_p=\|\delta^k u_h\|_\infty \tag{35}$$

for $k=0,1,\cdots$ and for the finite length $l<\infty$.

Therefore the lemma is proved completely for any constants $1\le p,q,r\le\infty$ and for any interval $[0,l]$ of finite length $l<\infty$.

**Lemma 4.** *For any discrete function $u_h=\{u_j|j=0,1,\cdots,J\}$ defined on the grid points $\{x_j|j=0,1,\cdots,J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}}|j=0,1,\cdots,J-1\}$ of the interval $[0,l]$ with finite length $l<\infty$ and for constants $1\le p,q,r\le\infty$, there is*

$$\|\delta^k u_h\|_p\le C(\|\delta^{k-1}u_h\|_q^{1-\alpha}\|\delta^{k+1}u_h\|_r^{\alpha}+l^{\frac{1}{p}-\frac{1}{q}-1}\|\delta^{k-1}u_h\|_q) \tag{36}$$

*with*

$$\frac{1}{p}-1=\frac{1-\alpha}{q}+\alpha\Big(\frac{1}{r}-2\Big), \tag{37}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$ and $C$ depends on the ratio constant $M$ of meshsteps.*

The proof of this lemma is similar to that of Lemma 2.

## 6

**Lemma 5.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ with finite length $l < \infty$ and the constants $1 \leq q, r \leq \infty$ and $q \leq p \leq \infty$, there is the estimate*

$$\|u_h\|_p \leq C(\|u_h\|_q^{1-\alpha} \|\delta^2 u_h\|_r^{\alpha} + l^{\frac{1}{p}-\frac{1}{q}} \|u_h\|_q) \tag{38}$$

*with*

$$\frac{1}{p} = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - 2\Big) \tag{39}$$

*and*

$$0 \leq \alpha \leq \frac{\frac{1}{q}}{2 - \frac{1}{r} + \frac{1}{q}}, \tag{40}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$ and $C$ depends on the ratio constant $M$ of the meshsteps.*

*Proof.* From Lemma 1 and 3, we have

$$\|u_h\|_p \leq C(\|u_h\|_q^{1-\lambda} \|\delta u_h\|_s^{\lambda} + l^{\frac{1}{p}-\frac{1}{q}} \|u_h\|_q),$$

$\|\delta u_h\|_s \leq C(\|u_h\|_q^{1-\mu} \|\delta^2 u_h\|_r^{\mu} + l^{\frac{1}{s}-\frac{1}{q}-1} \|u_h\|_q)$,*where*$1 \leq q, r \leq \infty$, $q \leq p \leq \infty$ and $1 \leq s \leq \infty$,

$$\frac{1}{p} = \frac{1-\lambda}{q} + \lambda(\frac{1}{s} - 1), \quad \frac{1}{s} - 1 = \frac{1-\mu}{q} + \mu(\frac{1}{r} - 2).$$

By direct substitution and similar as before but much more simpler calculation, we obtain

$$\|u_h\|_p \leq C(\|u_h\|_q^{1-\lambda\mu} \|\delta^2 u_h\|_r^{\lambda\mu} + l^{\frac{1}{p}-\frac{1}{q}} \|u_h\|_q),$$

where

$$\frac{1}{p} = \frac{1-\lambda\mu}{q} + \lambda\mu(\frac{1}{s} - 2),$$

and the constant $C$ depends on the ratio constant $M$ of meshsteps. This shows the conclusion of the lemma.

## 7

**Theorem 1.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for any constants $1 \le q, r \le \infty$ and $0 \le k < n$, there is the estimate*

$$\|\delta^k u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta^n u_h\|_r^{\alpha} + l^{\frac{1}{p} - \frac{1}{q} - k} \|u_h\|_q) \tag{41}$$

*with*

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha(\frac{1}{r} - n) \tag{42}$$

*and*

$$\frac{k}{n} \le \alpha \le \frac{k + \frac{1}{q}}{n - \frac{1}{r} + \frac{1}{q}} \le 1, \tag{43}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$ and $C$ depends on the ratio constant $M$ of the meshsteps.*

*Proof.* Lemma 1 is the case of the present theorem for $n = 1$ and $k = 0$. Lemmas 3 and 5 are the cases of $k = 1$ and $k = 0$ respectively for $n = 2$ of (41).

Let us now prove the theorem by mathematical induction. Then suppose that the relation (41) is valid for $n = m \ge 2$.

From Lemma 4 for $k = m$, there is the inequality relation

$$\|\delta^m u_h\|_p \le C(\|\delta^{m-1} u_h\|_s^{1-\lambda} \|\delta^{m+1} u_h\|_r^{\lambda} + l^{\frac{1}{p} - \frac{1}{s} - 1} \|\delta^{m-1} u_h\|_s), \tag{44}$$

where $1 \le s, p, r \le \infty$, and

$$\frac{1}{p} - 1 = \frac{1-\lambda}{s} + \lambda(\frac{1}{r} - 2)$$

and

$$\frac{1}{2} \le \lambda \le \frac{1 + \frac{1}{s}}{2 - \frac{1}{r} + \frac{1}{s}}.$$

By the induction hypothesis for $n = m$ and $k = m + 1$, we have

$$\|\delta^{m-1} u_h\|_s \le C(\|u_h\|_q^{1-\mu} \|\delta^m u_h\|_p^{\mu} + l^{\frac{1}{s} - \frac{1}{q} - (m-1)} \|u_h\|_q), \tag{45}$$

where

$$\frac{1}{s} - (m-1) = \frac{1-\mu}{q} + \mu(\frac{1}{p} - m)$$

and

$$\frac{m-1}{m} \le \mu \le \frac{m - 1 + \frac{1}{q}}{m - \frac{1}{p} + \frac{1}{q}}.$$

Substituting (45) into (44), we get

$$\|\delta^m u_h\|_p \le C\{\|u_h\|_q^{(1-\mu)(1-\lambda)}\|\delta^{m+1}u_h\|_r^{\lambda}\|\delta^m u_h\|_p^{\mu(1-\lambda)}$$

$+1^{(\frac{1}{s}-\frac{1}{q}-m+1)(1-\lambda)}\|\delta^{m+1}u_h\|_r^{\lambda}\|u_h\|_q^{1-\lambda}$

$+1^{\frac{1}{s}-\frac{1}{q}-1}\|u_h\|_q^{1-\mu}\|\delta^m u_h\|_p^{\mu}+l^{\frac{1}{p}-\frac{1}{q}-m}\|u_h\|_q\}.Bysimilarderivationasbefore, weobtain\|\delta^m u_h\|_p \le C(\|u_h\|_q^{1-\beta}\|\delta^{m+1}u_h\|_r^{\beta}+l^{\frac{1}{p}-\frac{1}{q}-m}\|u_h\|_q), (46)where\beta = \frac{\lambda}{1-(1-\lambda)\mu}and\frac{1}{p}-m = \frac{1-\beta}{q}+\beta\Big[\frac{1}{r}-(m+1)\Big].Since\beta$ increases as the parameters $\lambda$ and $\mu$ increases respectively, then

$$\frac{m}{m+1} \le \beta \le \frac{m+\frac{1}{q}}{(m+1)-\frac{1}{r}+\frac{1}{q}}.$$

This gives the inequality (41) for the case $n = m+1$ and $k = m$.

For $0 \le k < m$ and $1 \le p, s \le \infty$, by induction assumption, we have

$$\|\delta^k u_h\|_p \le (\|u_h\|_q^{1-\lambda}\|\delta^m u_h\|_s^{\lambda} + l^{\frac{1}{p}-\frac{1}{q}-k}\|u_h\|_q), \tag{47}$$

where

$$\frac{1}{p} - k = \frac{1-\lambda}{q} + \lambda(\frac{1}{s} - m)$$

and

$$\frac{k}{m} \le \lambda \le \frac{k+\frac{1}{q}}{m-\frac{1}{s}+\frac{1}{q}}.$$

From (46) we have

$$\|\delta^m u_h\|_s \le C(\|u_h\|_q^{1-\mu}\|\delta^{m+1}u_h\|_r^{\mu} + l^{\frac{1}{s}-\frac{1}{q}-k}\|u_h\|_q),$$

with

$$\frac{1}{s} - m = \frac{1-\mu}{q} + \mu\Big(\frac{1}{r} - (m+1)\Big).$$

Hence substituting this inequality into (47), we obtain immediately by derivation as before

$$\|\delta^k u_h\|_p \le C(\|u_h\|_q^{1-\lambda\mu}\|\delta^{m+1}u_h\|_r^{\lambda\mu} + l^{\frac{1}{p}-\frac{1}{q}-k}\|u_h\|_q),$$

where

$$\frac{1}{p} - k = \frac{1-\lambda\mu}{q} + \lambda\mu\Big(\frac{1}{r} - (m+1)\Big)$$

and also

$$\frac{k}{m+1} \le \lambda\mu \le \frac{k+\frac{1}{q}}{(m+1)-\frac{1}{r}+\frac{1}{q}}.$$

Thus (41) is valid for $n = m+1$.

Therefore (41) is valid for any $n$. This completes the proof of the theorem.

## 8

In this section we are going to consider the interpolation formulas for the norms of discrete functions with unequal meshsteps of negative index.

Let $p < 0$ be a negative number, denote $s = [\frac{1}{|p|}]$ and $\lambda = \{\frac{1}{|p|}\}$ the integer and the decimal parts of the positive real number $\frac{1}{|p|}$ respectively.

For the discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$, let us now define the norms $\|\delta^k u_h\|_p$, $k = 0, 1, \cdots$ of the negative index $p < 0$ as follows: for the case of $0 < \lambda < 1$, then $\frac{1}{p} = -(s + \lambda)$, when $k + s$ is odd, there is

$$\|\delta^k u_h\|_p = \max_{r>m; r,m=[k+s],\cdots,J-[k+s+1]} \frac{|\delta^{k+s} u_{r+\frac{1}{2}} - \delta^{k+s} u_{m+\frac{1}{2}}|}{|x^{(k+s)}_{r+\frac{1}{2}} - x^{(k+s)}_{m+\frac{1}{2}}|^\lambda} \tag{48}$$

and when $k + s$ is even, there is

$$\|\delta^k u_h\|_p = \max_{r>m; r,m=[k+s],\cdots,J-[k+s]} \frac{|\delta^{k+s} u_r - \delta^{k+s} u_m|}{|x^{(k+s)}_r - x^{(k+s)}_m|^\lambda} \tag{49}$$

for the case of $\lambda = 0$, then $\frac{1}{p} = -s$, there is

$$\|\delta^k u_h\|_p = \max_{j=[k+s],\cdots,J-[k+s+1]} |\delta^{k+s} u_{j+\frac{1}{2}}| \tag{50}$$

for odd $k + s$ and

$$\|\delta^k u_h\|_p = \max_{j=[k+s],\cdots,J-[k+s+1]} |\delta^{k+s} u_j| \tag{51}$$

for even $k + s$. These norms for the discrete functions correspond to the Hölder and Lipschitz coefficients of the derivatives for the differentiable functions.

For these kinds of norms for the discrete functions of unequal meshsteps, we can also have the following simple notations as

$$U_h^{k+s,\lambda} = \|\delta^{k+s} u_h\|_p \tag{52}$$

for $0 < \lambda < 1$ and

$$U_h^{k+s} = \|\delta^{k+s} u_h\|_p \tag{53}$$

for $\lambda = 0$, where $k = 0, 1, \cdots$, $s = [\frac{1}{|p|}]$ and $\lambda = \{\frac{1}{|p|}\}$ for the negative index $p < 0$.

When $0 < \lambda < 1$, for the case for example that $k + s$ is odd, from the definition of norm with negative index, we then have

$$\|\delta^k u_h\|_p = \max_{r>m; r,m=[k+s],\cdots,J-[k+s+1]} \frac{|\sum_{j=m+1}^{r} (\delta^{k+s} u_{j+\frac{1}{2}} - \delta^{k+s} u_{j-\frac{1}{2}})|}{|x^{(k+s)}_{r+\frac{1}{2}} - x^{(k+s)}_{m+\frac{1}{2}}|^\lambda}$$

$\leq \max_{r>m;r,m=[k+s],\cdots,J-[k+s+1]}\Big(\sum_{j=m+1}^{r}|\delta^{k+s+1}u_j|^{\frac{1}{1-\lambda}}h_j^{(k+s+1)}\Big)^{1-\lambda}$

$\cdot\frac{|x_r^{(k+s+1)}-x_{m+1}^{(k+s+1)}|^\lambda}{|x_{r+\frac12}^{(k+s)}-x_{m+\frac12}^{(k+s)}|^\lambda} \leq M^\lambda\|\delta^{k+s+1}u_h\|_{\frac{1}{1-\lambda}}.Wehavethesimilarestimateforthecasethat$k+s$iseven.Hencewehave|$ $M^\lambda\|\delta^{k+s+1}u_h\|_{\frac{1}{1-\lambda}},wherep$¡0,0¡$\lambda<1$ and $1<\frac{1}{1-\lambda}<\infty$.

By means of Theorem 1, we have the following estimate formula

$$\|\delta^{k+s+1}u_h\|_{\frac{1}{1-\lambda}} \leq C(\|u_h\|_q^{1-\alpha}\|\delta^n u_h\|_r^\alpha + l^{-\lambda-\frac1q-k-s}\|u_h\|_q)$$

for $1\leq q,r\leq\infty$ and $0<\lambda<1$, where

$$(1-\lambda)-(k+s+1)=\frac{1-\alpha}{q}+\alpha\Big(\frac1r-n\Big)$$

since $\frac1p=-(s+\lambda)$, there is

$$\frac1p-k=\frac{1-\alpha}{q}+\alpha\Big(\frac1r-n\Big)$$

for $p<0$ and $0<\lambda<1$, where

$$0\leq\alpha=\frac{k-\frac1p+\frac1q}{n-\frac1r+\frac1q}\leq 1.$$

When $\lambda=0$, then $\frac1p=-s$. From definition, there is

$$\|\delta^k u_h\|_p=\|\delta^{k+s}u_h\|_\infty.$$

Then the interpolation formulas of Theorem 1 are also valid for this case.

Thus we obtain the following theorem of interpolation formulas for the norms of nagetive index for the discrete functions with unequal meshsteps.

**Theorem 2.** *For any discrete function $u_h=\{u_j|j=0,1,\cdots,J\}$ defined on the grid points $\{x_j|j=0,1,\cdots,J\}$ with unequal meshsteps $\{h_{j+\frac12}=x_{j+1}-x_j>0|j=0,1,\cdots,J-1\}$ of the interval $[0,l]$ of finite length $l<\infty$ and for the constants $1\leq q,r\leq\infty$ and $-(n-k-\frac1r)\leq\frac1p\leq 0$, there is the estimate relation*

$$\|\delta^k u_h\|_p\leq C(\|u_h\|_q^{1-\alpha}\|\delta^n u_h\|_r^\alpha+l^{\frac1p-\frac1q-k}\|u_h\|_q) \tag{54}$$

*with*

$$\frac1p-k=\frac{1-\alpha}{q}+\alpha\Big(\frac1r-n\Big) \tag{55}$$

*and*

$$\frac{k+\frac1q}{n-\frac1r+\frac1q}\leq\alpha\leq 1, \tag{56}$$

*where $0\leq k<n$ and $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l<\infty$, the meshsteps $\{h_{j+\frac12}|j=0,1,\cdots,J-1\}$ and the discrete functions, but the constant $C$ depends on the ratio constant $M$ of the unequal meshsteps.*

## 9

Combining Theorem 1 for positive index $1 \le p \le \infty$ and Theorem 2 for negative index $-\infty \le p \le -\frac{1}{n-k-\frac{1}{r}} \le 0$, we get the following theorem for general interpolation formulas with both positive and negative index $-(n-k-\frac{1}{r}) \le \frac{1}{p} \le 1$ for the discrete functions with unequal meshsteps.

**Theorem 3.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for the constants $-(n-k-\frac{1}{r}) \le \frac{1}{p} \le 1$ and $1 \le q, r \le \infty$, and $0 \le k < n$, there is the estimate*

$$\|\delta^k u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta^n u_h\|_r^{\alpha} + l^{\frac{1}{p}-\frac{1}{q}-k} \|u_h\|_q) \tag{57}$$

*with*

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - n\Big), \quad 0 \le \alpha \le 1, \tag{58}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function, but $C$ depends on the ratio constant $M$ of the unequal meshsteps.*

As a consequence of the above general theorem, we have the interpolation relations among the maximum modulo and the Hölder coefficients for the discrete functions $u_h = \{u_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps.

**Theorem 4.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$, there are the interpolation formulas*

$$U_h^k \le C((U_h^0)^{1-\frac{k}{n}} (U_k^n)^{\frac{k}{n}} + l^{-k} U_h^0), U_h^{k,\lambda} \le C((U_h^0)^{1-\frac{k+\lambda}{n}} (U_h^n)^{\frac{k+\lambda}{n}} + l^{-k-\lambda} U_h^0), \tag{59}$$

*where $0 \le k < n$, $0 \le \lambda < 1$ and $C$ is a constant independent of the finite length $l < \infty$, the unequal meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function, but $C$ depends on the ratio constant $M$ of the unequal meshsteps.*

## 10

As an immediate consequence of the previous theorem for general interpolation formulas, we have the following theorem for so-called Sobolev's inequality for the discrete functions with unequal meshsteps.

**Theorem 5.** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j | j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for the constants $1 \le q, r \le \infty$ and $-(n-k-\frac{1}{r}) \le \frac{1}{p} \le 1$, there exists a constant $C(\varepsilon)$ depending on $\varepsilon > 0$, such that*

$$\|\delta^k u_h\|_p \le \varepsilon \|\delta^n u_h\|_r + C(\varepsilon) \|u_h\|_q, \tag{60}$$

*where*

$$0 < \frac{k + \frac{1}{q} - \frac{1}{p}}{n - \frac{1}{r} + \frac{1}{q}} < 1, \tag{61}$$

*$C(\varepsilon)$ is independent of the constants $p$, $q$, $r$, the finite length $l < \infty$, the unequal meshsteps $\{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and the discrete function $u_h$, but $C(\varepsilon)$ depends on the ratio constant $M$ of the unequal meshsteps.*

## 11

For the discrete function $u_h = \{u_j | j = 0, \underline{+}1, \underline{+}2, \cdots\}$ defined on the grid points $\{x_j | j = 0, \underline{+}1, \underline{+}2, \cdots\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, \underline{+}1, \underline{+}2, \cdots\}$ of the real line $R = (-\infty, \infty)$ and the discrete function $u_h = \{u_j | j = 0, 1, 2, \cdots\}$ defined on the grid points $\{x_j | j = 0, 1, 2, \cdots\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, 2, \cdots\}$ of the half real line $R_+ = [0, \infty)$, the norms of these discrete functions and their corresponding difference quotients are defined to be the limit of the convergent infinite sums of the powers of discrete values on the corresponding grid points. Similar to the theorem for the discrete functions on the functions on the finite interval, we have the following theorems of interpolation formulas for the discrete functions on infinite interval.

**Theorem 6.** *For the discrete function $u_h = \{u_j | j = 0, \underline{+}1, \underline{+}2, \cdots\}$ defined on the grid points $\{x_j | j = 0, \underline{+}1, \underline{+}2, \cdots\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, 2, \cdots\}$ on the real line $R = (-\infty, \infty)$ and the discrete function $u_h = \{u_j | j = 0, 1, 2, \cdots\}$ defined on the grid points $\{x_j | j = 0, 1, 2, \cdots\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0, 1, 2, \cdots\}$ on the half real line $R = [0, \infty)$ and for the constants $-(n - k - \frac{1}{r}) \le \frac{1}{p} \le 1$, $1 \le q, r \le \infty$ and $0 \le k < n$, there is the estimate relation*

$$\|\delta^k u_h\|_p \le C \|u_h\|_q^{1-\alpha} \|\delta^n u_h\|_r^{\alpha}, \tag{62}$$

*with*

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha(\frac{1}{r} - n) \tag{63}$$

*where $C$ is a constant independent of the constants $p$, $q$, $r$, the unequal meshsteps $\{h_{j+\frac{1}{2}}\}$ and the discrete function $u_h$ defined on the grid points of real line $R_+$ and $C$ depends on $M$.*

**Theorem 7.** *For any discrete function $u_h$ defined on the grid points of the real linr $R$ and the half real line $R_+$ with unequal meshsteps and for any constants $1 \le q, r \le \infty$, $-(n - k - \frac{1}{r}) \le \frac{1}{p} \le 1$, and $0 \le k < n$, then for any positive small constant $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ depending on $\varepsilon$, such that*

$$\|\delta^k u_h\|_p \le \varepsilon \|\delta^n u_h\|_r + C(\varepsilon) \|u_h\|_q^{\alpha}, \tag{64}$$

*where*

$$0 < \frac{k - \frac{1}{p} + \frac{1}{q}}{n - \frac{1}{r} + \frac{1}{q}} < 1 \tag{65}$$

*and $C(\varepsilon)$ is independent of the constants $p$, $q$, $r$, the unequal meshsteps $\{h_{j+\frac{1}{2}}\}$ and the discrete function $u_h$ defined on the real line $R$ and the half real line $R_+$, but $C(\varepsilon)$ depends on the ratio constant $M$ of the unequal meshsteps.*

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