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THE DIRECT KINEMATIC SOLUTION OF THE PLANAR STEWART PLATFORM WITH COPLANAR GROUND POINTS*[1)]

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Abstract

A procedure of computing the position of the planar Stewart platfrom with coplanar ground points is presented avoiding the computation of Groebner basis by standard algorithm. The polynomial system resulted is triangularized. The number of arithmetic operations needed can be predisely counted.

1. Introduction

The problem for computing the position of a Stewart platfrom has been widely studied for various cases. In this paper we will consider the case for which the ground points are coplanar and the fixations of the legs on the platfrom are coplanar. It is simpler than the coplanar platform in [1] where the ground is not necessary a plane. Due to this simplicity the computation can be carried out directly avoiding the computation of Groebner basis by standard algorithm and then from it to deduce condition for 40 complex solutions is presented.

In section 2 we give the polynomial system we have chosen for the problem. How to transform the system into a simpler one with less unknowns is presented in section 3. The elimination process for solving the simpler system is presented in section 4. In section 5 we relate the result obtained with Groebner basis and characteritic set. The examples and remarks are given in sections 6 and 7 respectively.

The work reported here was done on a SUN SPARC station 2 using MAPLE V release 2. A program has been written in MAPLE functions.

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2. The Polynomial System for the Problem

The coordinates of base points B_i where the fixed legs are on the ground, are given by $(x_i, y_i, 0), i = 1, \cdots, 6$. Since the platform is planar, we denote the coordinates of points M_i of fixation of the legs on it by (p_i, q_i) with respect to any rectangular coordinate system given in the platform plane. Let its origin be M with coordinate (x, y, z) and the direction cosines of its axes MP and MQ be (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. Thus the coordinates of M_i can be expressed as

$$(p_iu_1 + q_iv_1 + x, \quad p_iu_2 + q_iv_2 + y, \quad p_iu_3 + q_iv_3 + z).$$

Let the length of B_iM_i be l_i. We have six equations

$$f_i := (p_iu_1+q_iv_1+x-x_i)^2+(p_iu_2+q_iv_2+y-y_i)^2+(p_iu_3+q_iv_3+z)^2-l_i^2 = 0 \quad i = 1, \cdots, 6.$$

Another three obvious equation are

$$\begin{aligned} f_7 &:= u_1^2 + u_2^2 + u_3^2 - 1 = 0, \\ f_8 &:= v_1^2 + v_2^2 + v_3^2 - 1 = 0, \\ f_9 &:= u_1v_1 + u_2v_2 + u_3v_3 = 0. \end{aligned}$$

These 9 equations in 9 unknowns $u_1, u_2, u_3, v_1, v_2, v_3, x, y$ and z form the fundamental system describing the problem. For any j the total degree of f_j with respect to its unknowns is 2.

Note that this formulation the distances between $M_j's$ and those between $B_j's$ are not used explicitly. And we have not supposed that all $M_j's$ are distinct as well as $B_j's$. It might be more flexible. When some of M_j and/or B_j properly coincide, we get various corresponding special cases.

3. The Transformed System

Using f_7, f_8, f_9 and introducing

$$\begin{aligned} u &:= u_1x + u_2y + u_3z, \\ v &:= v_1x + v_2y + v_3z, \\ w &:= x^2 + y^2 + z^2 \end{aligned}$$

the first six equations can be written as

$$f_{10+i} := p_ix_iu_1 + p_iy_iu_2 - p_iu + q_ix_iv_1 + q_iy_iv_2 - q_iv + x_ix + y_iy - \frac{1}{2}w + m_i = 0$$

where

$$m_i := \frac{1}{2}(l_i^2 - x_i^2 - y_i^2 - p_i^2 - q_i^2).$$

They are linear with respect to $u_1, u_2, u, v_1, v_2, v, x, y$ and w, and can be expressed in matrix form

$$M_{6\times 10}t = 0$$

where the ith row of the matrix $M_{6\times 10}$ is

$$(p_ix_i, \quad p_iy_i, \quad -p_i, \quad q_ix_i, \quad q_iy_i, \quad -q_i, \quad x_i, \quad y_i, \quad -\frac{1}{2}, \quad m_i)$$

and

$$t = (u_1, u_2, u, v_1, v_2, v, x, y, w, 1)^T.$$

Let the submatrix of $M_{6\times 10}$ obtained by deleting its last column be $M_{6\times 9}$, which depends only on the given coordinate data x_i, y_i, p_i and q_i, and does not depend of the lengths l_j. Suppose that the rank of $M_{6\times 9}$ is less than 6, then there exists a row vector $c = (c_1, \cdots, c_6) \neq 0$ such that

$$cM_{6\times 9} = 0.$$

Consequently

$$cM_{6\times 10}t = 0$$

becomes

$$\sum_{i=1}^{6} c_im_i = 0.$$

Therefore the lengths l_j must satisfy this condition and the degree of freedom is less than 6.

We conclude that rank $(M_{6\times 9}) = 6$ is a necessary condition for the Stewart platform with 6 degree of freedom. Checking it is trivial, and we will assume that it holds true in the following.

Let the submatrix of $M_{6\times 10}$ composed of its first 6 columns be $M_{6\times 6}$ and denote

$$a_0 := \det(M_{6\times 6})$$

We assume $a_0 \neq 0$. For the other case, we translate the MPQ coordinate system, replacing p_i and q_i by $p_i + p$ and $q_i + q$ respectively, and choose p and q such that the corresponding $a_0 \neq 0$.

Since $a_0 \neq 0$, solving $M^t_{6\times 10} = 0$ for u_1, u_2, u, v_1, v_2, v, we get the linear expressions of these variables in terms of x, y and w

$$f_{21} := a_0u_1 + a_{11}x + a_{12}y + a_{13}w + a_{14} = 0,$$
$$f_{22} := a_0u_2 + a_{21}x + a_{22}y + a_{23}w + a_{24} = 0,$$
$$f_{23} := a_0u + a_{31}x + a_{32}y + a_{33}w + a_{34} = 0,$$
$$f_{24} := a_0v_1 + a_{41}x + a_{42}y + a_{43}w + a_{44} = 0,$$
$$f_{25} := a_0v_2 + a_{51}x + a_{52}y + a_{53}w + a_{54} = 0,$$
$$f_{26} := a_0v + a_{61}x + a_{62}y + a_{63}w + a_{64} = 0$$

where

$$a_{ij} := \det(c_1, \cdots, c_{i-1}, c_{j+6}, c_{i+1}, \cdots, c_6)$$

and c_j stands for the jth colunm of $M_{6\times 10}$.

By f_7, f_s, f_9 and the definitions of u, v and w, we get the expressions of $u_3^2, v_3^2, uv, u_3 z, v_3 z, z^2$ in terms of x, y and w, and denote them by $f_{31}, \cdots, f_{36}$ respectively

$$\begin{aligned}
f_{31} &:= 1 - u_1^2 - u_2^2 = 1 - a_0^{-2}(A^2 + B^2), \\
f_{32} &:= 1 - v_1^2 - v_2^2 = 1 - a_0^{-2}(F^2 + G^2), \\
f_{33} &:= -u_1 v_1 - u_2 v_2 = -a_0^{-2}(AF + BG), \\
f_{34} &:= u - u_1 x - u_2 y = a_0^{-1}(-C + Ax + By), \\
f_{35} &:= v - v_1 x - v_2 y = a_0^{-1}(-H + Fx + Gy), \\
f_{36} &:= w - x^2 - y^2
\end{aligned}$$

where

$$\begin{aligned}
A &:= a_{11}x + a_{12}y + a_{13}w + a_{14}, \\
B &:= a_{21}x + a_{22}y + a_{23}w + a_{24}, \\
C &:= a_{31}x + a_{32}y + a_{33}w + a_{34}, \\
F &:= a_{41}x + a_{42}y + a_{43}w + a_{44}, \\
G &:= a_{51}x + a_{52}y + a_{53}w + a_{54}, \\
H &:= a_{61}x + a_{62}y + a_{63}w + a_{64}.
\end{aligned}$$

From these expressions the following six equations obviously hold

$$\begin{aligned}
h_1 &:= a_0^2(f_{31}f_{36} - f_{34}^2) = 0, \\
h_2 &:= a_0^2(f_{32}f_{36} - f_{35}^2) = 0, \\
h_3 &:= a_0^2(f_{33}f_{36} - f_{34}f_{35}) = 0, \\
h_4 &:= a_0^3(f_{31}f_{35} - f_{33}f_{34}) = 0, \\
h_5 &:= a_0^3(f_{32}f_{34} - f_{33}f_{35}) = 0, \\
h_6 &:= a_0^4(f_{31}f_{32} - f_{33}^2) = 0.
\end{aligned}$$

For any j the total degree of h_j with respect to x, y and w is 4.

We will be solve this polynomial system in the next section. Once the values of x, y and w are obtained, other unknowns are easily computed.

4. Elimination

From these six polynomials h_i, we construct the following nine polynomials h_i.

$$h_7 := Fh_1 - Ah_3 + xh_4,$$
$$h_8 := Gh_1 - Bh_3 + yh_4,$$
$$h_9 := Ah_2 - Fh_3 + xh_5,$$
$$h_{10} := Bh_2 - Gh_3 + yh_5,$$
$$h_{11} := Fh_4 + Ah_5 + xh_6,$$
$$h_{12} := Gh_4 + Bh_5 + yh_6,$$
$$h_{13} := Gh_7 + Bh_9 + yh_{11} \equiv Fh_8 + Ah_{10} + xh_{12},$$
$$h_{14} := \frac{1}{2}(a_0^2h_1 + a_0^2h_2 + Hh_4 + Ch_5 + Fh_7 - Gh_8 + Ah_9 - Bh_{10} + 2xh_{11},$$
$$h_{15} := \frac{1}{2}(a_0^2h_1 + a_0^2h_2 + Hh_4 + Ch_5 - Fh_7 + Gh_8 - Ah_9 + Bh_{10} + 2yh_{12}.$$

It is easy to check that for j from 7 to 15, the total degree of h_j with respect to x, y and w is still 4, not 5.

Now consider them as polynomials in x and y with polynomials coefficients in w. We can write 15 equations $h_j = 0$ in matrix form

$$M_{15\times 15}(m_{ij})T = 0 \qquad (*)$$

where

$$T = (x^4, x^3y, x^2y^2, xy^3, y^4, x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1)^T$$

and the ith row of $M_{15\times 15}$ corresponds to h_i.

For any solution of the system

$$g_1(w) := \det(M_{15\times 15}) = 0. \qquad (*)$$

Let d_{ij} be the degree of m_{ij} with resport to w, and

$$d_j = \max_i(d_{ij}).$$

Obviously the values of d_j are 0,0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4 respectively.

When $g_1(w) \not\equiv 0$, its degree with respect to w is at most 20. In this case, at least one of 15 determinants of 14×14 submatrices of $M_{15\times 15}$ obtained by deleting its last column and one of its row must be not identically to zero. Suppose $g_0(w)$, the determinant of submatrix obtained by deleting i_0th row and last column, is not identically to zero. Then we get

$$g_2(x, w) = g_0(w)x - g_{21}(w)$$

from $h_1, \cdots, h_{i_0-1}, h_{i_0+1}, \cdots h_{15}$, where $g_{21}(w)$ is the determinant of submatrix obtanined by deleting i_0th row and 13th column. Similarly we get

$$g_3(y, w) = g_0(w)y + g_{31}(w)$$

from corresponding equations. And their degrees of $g_0(w), g_{21}(w)$ and $g_{31}(w)$ in w are at most 16, 17, 17 respectively.

The system $g_1(w), g_2(x, w), g_3(y, w)$ is our final result.

The numbers of terms of h_i are 101, 101, 176, 428, 428, 497, 520, 520, 520, 917, 917, 929, 530, 530 respcetively. It is far beyond the capacity of our computer system to expand symbolically these determinants. But we have computed the symbolic expression of the coefficient of w^{20} of $g_1(w)$. The result is g^4, where g is a homogeneous polynomial in $a_{ij}, i = 1, \cdots, 6, j = 1, \cdots, 3$, of total degree 12 with 21360 terms. Note that g depends on the coordinates x_i, y_i, p_i, q_i only.

Once the numerical values of parameters are given , it is very easy to get $g_i's$. Since the determinants with polynomial entries in w can be computed by interpolation method.

5. Interpretation

It is easy to relate the final polynomial system g_1, g_2, g_3 with characteristic set[3] and Groebner basis[2] under reasonable assumptions.

At first any solution of the problem satisfies $h_i = 0, i = 1, \cdots, 6$.

Let H be the ideal spanned by $h_1, \cdots, h_6$. By the construction,

$$h_i \in H \quad i = 7, \cdots, 15,$$

$$g_1(w), g_2(x, w), g_3(y, w) \in H.$$

When $g_1(w)$ is irreducible, reducing g_2, g_3 by g_1 in the case of necessity, the resulting system is a characteristic set with purely lexicographic ordering $w \prec y \prec x$.

From $\mathrm{GCD}(g_1(w), g_0(w)) = 1$, then

$$\hat{g}_2 = x + \hat{g}_{21}(w),$$
$$\hat{g}_3 = y + \hat{g}_{31}(w)$$

can be deduced from g_2 and g_3 respectively, and $\hat{g}_2, \hat{g}_3 \in H$. $g_1, \hat{g}_2$ and $\hat{g}_3$ form the Groebner basis of H with the above ordering.

Consider analogously $h_j's$ as polynomials in y and w with polynomial coefficients in x. We have polynomial

$$\tilde{g}_1(x) \in H$$

of degree in x at most 20. The coefficient of x^{20} of $\tilde{g}_1(x)$ and that of $g_1(w)$ are same. Similarly we have

$$\hat{g}_1(y) \in H$$

with same property. Therefore $g \neq 0$ is a sufficient condition for H being 0-dimensional, and the necessary and sufficient condition for 40 complex solutions.

6. Examples

The outline of the computing procedure is the following. For input data x_j, y_j, p_j, q_j and l_j, at first form the matrix $M_{6\times 10}$ and compute 6×6 determinants a_0 and a_{ij}. Secondly substitute these values into $M_{15\times 15}$. Thirdly obtain $g_j' s$ through corresponding determinants computed by interpolation method with respect to w.

Example 1. For the data given below

j	x_j	y_j	p_j	q_j	l_j
1	9	3	3	1	$\frac{1}{13}\sqrt{36205}$
2	6	8	2	3	$\frac{2}{65}\sqrt{1886308}$
3	0	14	1	5	$\frac{3}{65}\sqrt{101465}$
4	-8	13	-3	4	$\sqrt{237}$
5	-7	-6	-2	2	$\sqrt{462}$
6	-3	-5	-1	-4	$\frac{6}{65}\sqrt{46670}$

$x = 8, y = 9, w = 245$ is a known solution.

The numbers of the decimal digits of the coefficients of w^j of $g_1(w)$ from 0 to 20 are 180, 179, 178, 176, 174, 172, 170, 169, 167, 165, 162, 160, 157, 155, 152, 149, 146, 143, 140, 136, 132 respectively.

The computing times are the following

rationals			$355.500s$
floating − point	60	*digits*	$87.584s$
floating − point	50	*digits*	$78.617s$
floating − point	40	*digits*	$69.217s$

In the latter 3 cases denote the $g_1(w)$'s by $g_1^{(60)}, g_1^{(50)}$ and $g_1^{(40)}$ respectively.

For $g_1(w)$, 245 is its exact solution. Solving $g_1(w)$ with 30 digits, it has 10 real solutions.

Solving $g_1^{(60)}(g_1^{(50)})$ with 30 digits it has 10 real solutions too. The relative errors of these 10 real solutions with respect to those of $g_1(w)$ are less than $10^{-15}(10^{-12})$.

Solving $g_1^{(40)}$ with 40, 60 and 100 digits, we get 4 real solutions for all cases. They are coincident. But $w = 245$ disappears.

The solutions of $g_1(w), g_1^{(60)}, g_1^{(50)}, g_1^{(40)}$, are given below rounded to 12 decimal digits

$g_1(w)$	$g_1^{(60)}$	$g_1^{(50)}$	$g_1^{(40)}$
−76.9293314533	−76.9293314533	−76.9293314533	−76.9293314533
3.48816575791	3.48816575791	3.48816575791	3.48816575791
161.282963821	161.282963821	161.282963821	160.399380256
186.745850849	186.745850849	186.745850792	1883.77074410
204.617059597	204.617059597	204.61711777	
205.056420811	205.056420811	205.056358865	
245.000000000	245.000000000	245.000135810	
302.032827573	302.032827573	302.032813456	
1100.50519039	1100.50519039	1100.50519085	
1313.47312159	1313.47312159	1313.47312051	

Among the 10 solutions of w, we have found that only 2 of them yield real solutions. They are 204.6170595973191926 and 245. The values of other variables corresponding to them are given below for $z > 0$.

w	204.6170595973191926	245
x	−2.1866577467343392775	8
y	10.720329961907161537	9
z	9.2146683610318549129	10
u_1	.043434727364930303410	3/5
u_2	−.82011575750569771624	4/13
u_3	−.57054672792821125508	48/65
v_1	−.033607324523738767635	−4/5
v_2	−.57196187855588785809	3/13
v_3	.81959145750622358226	36/65

The other 8 real roots of $g_1(w) = 0$ yield $z^2 < 0$.

Example 2. For the coordinates of B_j and M_j given below

j	x_j	y_j	p_j	q_j
1	r_b	0	r_m	0
2	$r_b\cos(\frac{\pi}{3}+\theta)$	$r_b\sin(\frac{\pi}{3}+\theta)$	$r_m\cos(\frac{\pi}{3}+\phi)$	$r_m\sin(\frac{\pi}{3}+\phi)$
3	$r_b\cos(\frac{2\pi}{3})$	$r_b\sin(\frac{2\pi}{3})$	$r_m\cos(\frac{2\pi}{3})$	$r_m\sin(\frac{2\pi}{3})$
4	$r_b\cos(\pi+\theta)$	$r_b\sin(\pi+\theta)$	$r_m\cos(\pi+\phi)$	$r_m\sin(\pi+\phi)$
5	$r_b\cos(\frac{4\pi}{3})$	$r_b\sin(\frac{4\pi}{3})$	$r_m\cos(\frac{4\pi}{3})$	$r_m\sin(\frac{4\pi}{3})$
6	$r_b\cos(\frac{5\pi}{3}+\theta)$	$r_b\sin(\frac{5\pi}{3}+\theta)$	$r_m\cos(\frac{5\pi}{3}+\phi)$	$r_m\sin(\frac{5\pi}{3}+\phi)$

where r_0 and r_m are radii of the circles on which the $B_j's$ and $M_j's$ are located respectively, $0 \le \theta, \phi \le \pi/3$, we have studied in [4]. The results are the following

(1) rank $(M_{6\times 9}) = 6$ iff $\theta \ne \phi$.

(2) for $\theta \neq \phi$, the degree of $g_1(w)$ is 14 with leading coefficient

$$d_0 d_1^6 d_2^6 d_3^6$$

where d_0 is nonzero, not depending on the lengths l_j

$$\begin{aligned} d_1 &= 1 - 2k_1, \\ d_2 &= 2 - k_1 - 4k_1k_2, \\ d_3 &= r_m^2/r_b^2 - k_1^2 - 4k_1^2k_2^2, \\ k_1 &= \cos\frac{\theta-\phi}{2}\cos\frac{2\theta+\phi}{2}/\cos\frac{\theta+2\phi}{2}, \\ k_2 &= \frac{1}{2}\tan\frac{\theta-\phi}{2}. \end{aligned}$$

(3) $d_1 = 0$ iff $\theta = \pm\frac{\pi}{3}$, and for $\theta = \pm\frac{\pi}{3}, d_2d_3 \neq 0$, the degree of $g_1(w)$ is 8 with nonzero leading coefficient depending only on r_b, r_m and ϕ.

(4) $d_2 = 0$ iff $\phi = \pm\frac{\pi}{3}$, and for $\phi = \pm\frac{\pi}{3}, d_1d_3 \neq 0$, the degree of $g_1(w)$ is 8 with nonzero leadng coefficient depending only on r_b, r_m and θ.

Recently we have studied the case $d_1d_2 \neq 0$ and $d_3 = 0$, the degree of $g_1(w)$ readuces to 11, and the leading coefficient d_{11} depends on the length l_j. When $d_{11} = 0$, the degree of $g_1(w)$ is 10 and the leading coefficients d_{10} depends on l_j. When $d_{10} = 0$, the degree of $g_1(w)$ is 9 and the leading coefficients d_9 depends on l_j. When $d_9 = 0$ them $g_1(w)$ is identically to zero. In this case the lengths satisfy the following conditions:

$$\begin{bmatrix} l_2^2 \\ l_4^2 \\ l_6^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} l_1^2 \\ l_3^2 \\ l_5^2 \end{bmatrix}$$

where $a = 1 - \cos(\theta+\phi) - \sqrt{3}\sin(\theta+\phi), b = 1 - \cos(\theta+\phi) + \sqrt{3}\sin(\theta+\phi)$ and $c = 3 - a - b$.

And the polymial system has solution manifold which has been computed and all isolated solutions as well. All the explicit formulas are omitted here due to the lack of space.

7. Remarks

The number of the arithmetic operations needed to produce the g_i can be counted. The computing time depends on the lengths of the input data, the type of the arithmetic operations used and their implementation. Note that, in example 1 we run a symbolic program with floating point arithmetics. Thus the computing time is longer.

When the coefficient of w^{20} of $g_1(w)$ is zero. It is not known to us the condition under which ideal H is still 0-dimensional.

The problem for choosing proper number of digits used in the floating-point operation has not been studied in detail. Example 1 shows that g_1, g_2 and g_3 should be calculated more accurately. But the accumulation of round off errors can be controlled to some extent.

In the polynomial system describing the problem in [5], the lengths between $M_j's$ are used. The determinants needed to compute are of order 21. And there is no proof for at most 40 complex solutions in the 0-dimensional case.

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