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# $\delta$-WAVE FOR 1-D AND 2-D HYPERBOLIC SYSTEMS*[1)]

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**Abstract**

Here a new kind of nonlinear wave, which is called $\delta-$wave, is described by some high resolution difference solutions for Riemann problems of one-dimensional (1-D) and two-dimensional (2-D) nonlinear hyperbolic systems in conservation laws. Some phenomena are numerically shown for the solutions of Riemann problems for 2-D gas dynamics systems.

## 1. Introduction

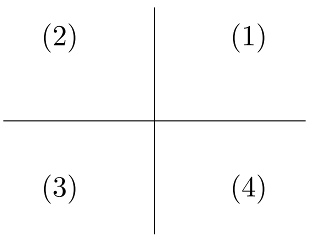
It is well known that hyperbolic systems of conservation laws have studied for a long time. By classical theoretical analyses, one classified the solutions to contain some natural sources such as constant, shock waves, rarefaction waves and contact discontinuities, which coincide with the solutions of practical problems, gas dynamics systems. In the recent years, we have studied for initial value problems of the following 2-D $2 \times 2$ hyperbolic systems[1,2,3,4],

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0 \ , \\ v_t + (uv)_x + (v^2)_y = 0 \end{cases} \tag{1.1}$$

with the 2-D Riemann data

$$(u, v)|_{t=0} = (u_0^i, v_0^i), \quad (i) = 1, 2, 3, 4 \tag{1.2}$$

where (i)-states are described to

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Here (1.1) is called 2-D inviscid Burger's equations.

For some distributions of Riemann initial data, a new kind of phenomenon was firs discovered by numerical simulations, that is, there are a narrow region near shock waves that solutions may produce infinity even though the initial data are bounded[1]. From the 2-D model, we can go back to some 1-D cases, then we consider the 1-D $2 \times 2$ nonlinear hyperbolic systems in conservation laws,

$$\begin{cases} u_t + f(u)_x = 0 \ , \\ v_t + (uv)_x = 0 \end{cases} \tag{1.3}$$

with initial data

$$(u, v)|_{t=0} = (u_0(x), v_0(x)) \ . \tag{1.4}$$

From the structure of (1.3), obviously, we can know that solution $v(x,t)$ may produce infinities if solution $u(x,t)$ has discontinuities.

In the coming sections, we will list some simple systems of (1.3) and give brief analyses and numerical solutions which contain $\delta-$waves for Riemann problems of 2-D $2 \times 2$ nonlinear hyperbolic systems in conservation laws. Finally we will show some singular phenomena for 2-D gas dynamic systems by numerical computations.

## 2. One Dimensional $2 \times 2$ Hyperbolic Systems

From (1.3), here we first choose f(u)=au and g(v)=b, then we have

$$\begin{cases} u_t + au_x = 0 \ . \\ v_t + bu_x = 0 \end{cases} \tag{2.1}$$

where a and b are both constants, and with initial data

$$(u, v)|_{t=0} = (u_0(x), v_0(x)). \tag{2.2}$$

From (2.1), the solutions of (2.1) and (2.2) can easily be obtained in the following forms,

$$\begin{cases} u(x,t) = u_0(x - at), \\ v(x,t) = v_0(x) - b \int_0^t (u_0(x - at))_x dt \ . \end{cases} \tag{2.3}$$

As the solution u(x,t) is obtained by a scalar equation which is the first equation of system (2.1), then obviously, the solution $v(x,t)$ will produce discontinuities if $u(x,t)$ containes discontinuities. However there is no more singular. Numerical description can be seen in [5].

Now we consider the following $2 \times 2$ nonlinear hyperbolic systems in conservation laws

$$\begin{cases} u_t + (u^2)_x = 0, \\ v_t + (uv)_x = 0 \end{cases} \tag{2.4}$$

and with initial data (2.2).

The inconservative form of (2.4) is written to

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 2u & 0 \\ v & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0 \ , \tag{2.5}$$

then the characteristic roots are derived to

$$\lambda_1 = 2u, \quad \lambda_2 = u$$

and from these we know that the model is equations of combinations with one nonlinear hyperbolic equation and one linear degenerate equation. So the solutions contain shock waves, rarefaction waves and contact discontinuities for the model.

As the above description, here if we choose the initial data $u_0(x)$ such that u(x,t) produces shock waves, v(x,t) will perform the phenomena of $\delta-$waves sometimes. The detailed analyses can be seen in [5]. For the numerical computations, we choose the following Riemann initial data,

$$(u_0, v_0) = \begin{cases} (u_l, v_l), & x < 0, \\ (u_r, v_r), & x > 0 \end{cases} \tag{2.6}$$

where $v_l = v_r = 1.0$ and $u_l = 2.0$, $u_r = 0.0$.

In order to understand the phenomena well, here we first give numerical solution by Lax-Friedrichs scheme, then the solution are given by a second order accurate high resolution and nonoscillatory MmB scheme (or upwind TVD scheme) for (2.4) (2.6).

Solution by Lax-Friedrichs scheme: See Table 2.1

Table 2.1 M-P=101, $\Delta t/\Delta x = 0.2$, $n$=120

| i=: | u=: | v=: | i=: | u=: | v=: | i=: | u=: | v=: |
|---|---|---|---|---|---|---|---|---|
| 43 | 2.00 | 1.01 | 56 | 2.00 | 1.37 | 69 | 1.99 | 4.68 |
| 44 | 2.00 | 1.01 | 57 | 2.00 | 1.37 | 70 | 1.90 | 5.71 |
| 45 | 2.00 | 1.01 | 58 | 2.00 | 1.58 | 71 | 1.90 | 5.71 |
| 46 | 2.00 | 1.02 | 59 | 2.00 | 1.58 | 72 | 1.40 | 5.45 |
| 47 | 2.00 | 1.02 | 60 | 2.00 | 1.89 | 73 | 1.40 | 5.45 |
| 48 | 2.00 | 1.04 | 61 | 2.00 | 1.89 | 74 | 0.54 | 3.12 |
| 49 | 2.00 | 1.04 | 62 | 2.00 | 2.32 | 75 | 0.54 | 3.12 |
| 50 | 2.00 | 1.08 | 63 | 2.00 | 2.32 | 76 | 0.11 | 1.55 |
| 51 | 2.00 | 1.08 | 64 | 1.99 | 2.90 | 77 | 0.11 | 1.55 |
| 52 | 2.00 | 1.14 | 65 | 1.99 | 2.90 | 78 | 0.02 | 1.11 |
| 53 | 2.00 | 1.14 | 66 | 1.99 | 3.69 | 79 | 0.02 | 1.11 |
| 54 | 2.00 | 1.23 | 67 | 1.99 | 3.69 | 80 | 0.00 | 1.02 |
| 55 | 2.00 | 1.23 | 68 | 1.99 | 4.68 | 81 | 0.00 | 1.02 |

and then MmB solution, see Figure 2.2.

Here M-P is mesh points, n is time steps. From Table 2.1, we also see that v(x,t) is increasing near the region of shock wave u(x,t) with time t even though the solution of Lax-Friedrichs scheme is more smooth in the region, and from Figure 2.2, the region which contains $\delta-$wave is very narrow and the phenomenon of $\delta-$wave is very clear. The general cases can be seen in [5].

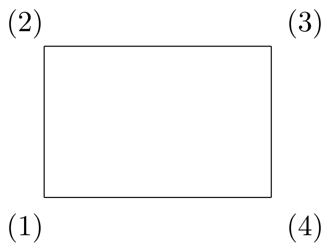
Figure 2.2 $M-P=101, \Delta t/\Delta x=0.1, n=100$

## 3. Two Dimensional $2\times 2$ Hyperbolic Systems

Now we consider the problem (1.1)(1.2) with Riemann initial data

$$(u_0, v_0)=(u_i, v_i), \quad (i)=1,2,3,4 \tag{3.1}$$

The distribution of the Riemann data, which will be calculated, is given as follows,




Distribution of Riemann Data

The data: $u_1=u_2=-2$, $v_1=v_4=-1$, $u_3=u_4=2$, $v_2=v_3=1$.

The solution of (1.1) and (3.1) will produce $\delta-$wave. The Numerical solution is listed in Table 3.1 by a 2-D nonsplitting MmB scheme [6].

From Table 3.1, the $\delta-$waves are numerically shown in the central region of the interactions of the four contact discontinuities. This phenomenon was first discovered by numerical computations, and theoretical analyses were given in the following in [3]. The descriptions in detail are presented in [1].

Table 3.1 M-P=101, $\lambda_x = \lambda_y = 0.1$, $n$=100

| $j$=37 | | | $i$=10 | | | $i$=43 | | |
|---|---|---|---|---|---|---|---|---|
| $i$=: | $u$=: | $v$=: | $j$=: | $u$=: | $v$=: | $j$=: | $u$=: | $v$=: |
| 10 | 2.01 | -2.00 | 31 | 2.00 | 2.00 | 10 | 2.00 | -2.00 |
| 11 | 2.15 | -2.11 | 32 | 2.00 | 2.00 | 11 | 2.00 | -2.00 |
| 12 | 2.12 | -2.14 | 33 | 2.00 | 2.00 | 12 | 2.00 | -2.00 |
| 13 | 2.09 | -2.00 | 34 | 1.99 | 2.00 | 13 | 1.99 | -2.00 |
| 14 | 2.19 | -2.19 | 35 | 1.96 | 2.00 | 14 | 1.99 | -2.00 |
| 15 | 2.45 | -2.45 | 36 | 1.15 | 2.00 | 15 | 1.93 | -2.00 |
| 16 | 2.66 | -2.66 | 37 | -0.19 | 2.00 | 16 | 1.19 | -2.00 |
| 17 | 2.33 | -2.38 | 38 | -0.93 | 2.00 | 17 | -0.15 | -2.00 |
| 18 | 18.46 | -25.2 | 39 | -0.99 | 2.00 | 18 | -0.96 | -2.00 |
| 19 | 15.25 | -18.4 | 40 | -0.99 | 2.00 | 19 | -0.99 | -2.00 |
| 20 | -0.95 | -1.97 | 41 | -1.00 | 2.00 | 20 | -1.00 | -2.00 |
| 21 | -0.99 | -1.99 | 42 | -1.00 | 2.00 | 21 | -1.00 | -2.00 |

## 4. 2-D Gas Dynamics Systems

Now we want to present the numerical results for 2-D Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2)_y + p_y = 0, \\ (\rho(e + \frac{u^2+v^2}{2}))_t + (\rho u(h + \frac{u^2+v^2}{2}))_x + (\rho v(h + \frac{u^2+v^2}{2}))_y = 0 \end{cases} \tag{4.1}$$

where

$$e = \frac{p}{(\gamma-1)\rho}, \quad h = e + \frac{p}{\rho}$$

and with Riemann initial data

$$(\rho, u, v, p)|_{t=0} = \begin{cases} (\rho_1, u_1, v_1, p_1), & x > 0, y > 0, \\ (\rho_2, u_2, v_2, p_2), & x < 0, y > 0, \\ (\rho_3, u_3, v_3, p_3), & x > 0, y < 0, \\ (\rho_4, u_4, v_4, p_4), & x < 0, y < 0\ . \end{cases} \tag{4.2}$$

We all know that fundamental waves, such as shock waves, rarefaction waves and contact disconuities, will exist if there is a discontinuous in one direction for the problem of (4.1)(4.2). Here we give the discontinuous condition of initial data which only contain contact discontinuity in one direction. See the following figure where $\rho_1 \neq \rho_2$, $u_1 = u_2$

and $v_1 \neq v_2$, then we give one distribution of initial data (4.2) which only contains four contact discontinuities

$$\begin{aligned}
&\rho_1 = 0.5,\ p_1 = 1.0,\ u_1 = -2.0,\ v_1 = -1.0, \\
&\rho_2 = 1.5,\ p_2 = 1.0,\ u_2 = -2.0,\ v_2 = 1.0, \\
&\rho_3 = 0.5,\ p_3 = 1.0,\ u_3 = 2.0,\quad v_3 = 1.0, \\
&\rho_4 = 1.5,\ p_4 = 1.0,\ u_4 = 2.0,\quad v_4 = -1.0
\end{aligned}$$

In this case, from numerical solutions by using MmB schemes [6], we find that density $\rho$ has some mountain values in the subsonic region described by psudo-steady form of (4.1) in [7]. See Figures 4.2 and 4.3.

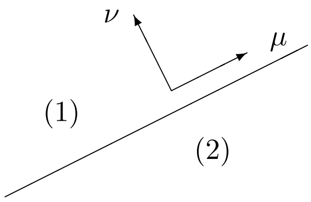



Figure 4.1

Figure 4.2 Density contour lines

As the limitation of theoretical analysis, it is very difficult to derive the exact solution for Riemann problem of 2-D gas dynamic systems.

In this paper, the phenomena of $\delta-$ waves are described by numerical computations for both 1-D and 2-D $2 \times 2$ nonlinear hyperbolic systems in conservation laws. We think these phenomena could make ones to deeply study for the structure of nonlinear hyperbolic systems even though the two models have no practical senses. However we expect that the phenomena of mountain values, which are like $\delta-$waves, can be solved by theoretical analyses for the two dimensional gas dynamic systems in the future.

Figure 4.3 Density surface

## References


[1] S.L. Yang and T. Zhang, The MmB difference solutions of Riemann problems for two dimensional $2\times 2$ nonlinear conservation laws, *IMPACT of Computing in Science and Engineering*, 3(1991),146-180.

[2] S.L. Yang, Characteristic Petrov-Galerkin schemes for 2-D conservation laws, *IMPACT of Computing in Science and Engineering*, 5(1993), 379-807.

[3] D.C. Tan and T. Zhang, 2-D Riemann problems for $2\times 2$ nonlinear hyperbolic systems in conservation laws, I. only contact discontinuities, *Journal of Differential Equations*, 111(1994), 203-254.

[4] D.C. Tan and T. Zhang, 2-D Riemann problems for $2 \times 2$ nonlinear hyperbolic systems in conservation laws, II Contain contact disconuities and rarefaction waves, *Journal of Differential Equations*, 111(1994), 255-282.

[5] S.L. Yang, $\delta-$waves for one-dimensional $2 \times 2$ hyperbolic systems, to appear in *J. Comput. Math.*.

[6] H.M. Wu and S.L. Yang, MmB $-$ A new class of accurate high resolution schemes for conservation laws in two dimensions. *IMPACT of Computing in Science and Engineering*, 1(1989), 217-259.

[7] T. Chang, G.Q. Chen and S.L. Yang, Riemann problems for 2-D gas dynamics systems, II. Interactions of contact discontinuities, to appear.