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ON COMPUTING ZEROS OF A BIVARIATE BERNSTEIN POLYNOMIAL*

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Abstract

In this paper, the problem of computing zeros of a general degree bivariate Bernstein polynomial is considered. An efficient and robust algorithm is presented that takes into full account particular properties of the function considered. The algorithm works for rectangular as well as triangular domains. The outlined procedure can also be applied for the computation of the intersection of a Bézier patch and a plane as well as in the determination of an algebraic curve restricted to a compact domain. In particular, singular points of the algebraic curve are reliably detected.

1. Introduction

In [6] and [4], the problem of finding the intersection of a cubic Bézier patch and a plane was considered. [6] considered a rectangular, and [4] a triangular patch. Since the Bernstein operator $B_n : f \mapsto B_n(f)$ preserves linear functions, the problem was simplified to the computation of zeros of a bivariate Bernstein polynomial $B_n(f)$. Both papers produced simple and efficient computational algorithms. It is based upon the following idea: determine the points where inside the support the topology of zeros of $B_n(f)$ changes. This was done by restricting the bivariate polynomial to a particular line direction, and determine these points from the fact that this restriction is a cubic polynomial. The zero branches were then separately computed between each pair of exceptional points.

A similar problem can be traced to [7] in a slightly different context, this time for general n. Let

$$p(x,y) = 0, \quad p(x,y) := \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{ij} x^i y^j \tag{1.1}$$

be the equation that defines a given planar algebraic curve. Suppose that one is interested in computing set of points $\{(x,y)\}$ that satisfies (1.1) in a given triangle $T \in \mathbb{R}^2$. The recipe in [7] suggests to rewrite p as a Bézier patch over triangle, and look for

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its zeros. Though no special algorithm for this particular problem was suggested, some help was given by the theorem 1 ([7]): if the corresponding Bézier net is strictly increasing along mesh lines in one direction, any line in this direction will cross the algebraic curve at most once.

In this paper, we extend the algorithm for computing zeros of the Bernstein polynomial presented in [4] to the general degree case. As it turns out rather unexpectedly, the general algorithm is as simple as its cubic counterpart though general degree algebraic equations admit no radical solutions. As already pointed out, the outlined procedure solves also the Bézier patch-plane intersection problem as well as the problem of computation of algebraic curves restricted to compact domains $\Omega \in \mathbb{R}^2$. It should also be mentioned that the given algorithm is computationally superior to the methods for computing algebraic functions that one encounters in standard mathematical packages. In particular, the comparison with the ImplicitPlot procedure used in the Mathematica package ([1 , p. 127]) was tested.

For the sake of simplicity, we shall consider only the triangle case. It is easy to see that the basic steps work out for both the rectangular and triangular support. Thus it is obvious that the algorithm can be simply transformed to handle the rectangular case. We shall demonstrate this just by computational examples.

Let us recall some notation and basic facts. Let T be a given nondegenerate triangle. The most natural way to express the Bernstein polynomial on a triangle is to write it in the barycentric form. The Bernstein basic functions B^n_{ijk} in this case are defined as

$$B^n_{ijk}(\beta_T(x,y)) := B^n_{ijk}(u,v,w) := \frac{n!}{i!j!k!}u^i v^j w^k$$

with

$$\beta := \beta_T : \mathbb{R}^3 \to \mathbb{R}^2$$

being the (invertible) barycentric map. The Bernstein polynomial of a function $f : \mathbb{R}^2 \to \mathbb{R}$ reads as

$$B_n(f) := \sum_{i+j+k=n} f_{ijk} B^n_{ijk}$$

where $f_{ijk} := f(\beta_T^{-1}(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}))$ are given coefficients.

The key step of the algorithm is to reduce two-dimensional problem to one dimension. Let T_1, T_2, T_3 denote the vertices of T. Choose fixed s, $0 \le s \le 1$, and somewhat arbitrary $T_4 = (1-s)T_2 + sT_3$. Then by [3]

$$Q_s := B_n(f)|_{T_1 T_4}$$

is a Bernstein polynomial of one variable, with coefficients being polynomials in s. As already computed in [4]

$$Q_s(t) = \sum_{i=0}^{n} a_i(s) B^n_i(t),$$

$$a_i(s) := \sum_{j=0}^{i} f_{n-i,i-j,j} B^i_j(s) \tag{1.2}$$

with

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}. \tag{1.3}$$

If now the intersections of the algebraic curve branches $B_n(f) = 0$ and the line T_1T_4 are known for some s, i.e.

$$Q_s(t_i) = 0, \quad i = 1, 2, \cdots, r,$$

they can be efficiently computed also for $s \pm \Delta s$ for small Δs as long as the intersection changes continuously. Thus it is crucial to determine in advance all the (exceptional) values

$$E : \{0 < s_1 < s_2 < \cdots < s_m < 1\} \tag{1.4}$$

for which at least at some (t_i, s_i) the line-algebraic curve branch intersection changes topology. On each of the subintervals (s_i, s_{i+1}) separately it is then simple and stable to compute all the branches of the algebraic curve since their number as well as topological structure does not change.

A search for (1.4) will be done with the help of the following proposition.

Proposition. If s is an exceptional value then one of the following alternatives must hold

- $Q_s(1) = 0$,
- $\exists t \in [0, 1)$ such that $Q_s^{(r)}(t) = 0, \quad r = 0, 1, \cdots, k, \quad k \geq 1.$

Proof. The first condition detects the algebraic curve branch that leaves the support. Assume now that $Q_s(1) \neq 0$. By definition, if a point (t, s), $t \in (0, 1)$ is exceptional then at least one branch of the algebraic curve $B_n(f) = 0$ does not continue in at least one direction $(t, s) \to (t, s \pm 0)$. But $B_n(f)$ is a continuous function, thus if (t, s) is an isolated singular point of the algebraic curve,

$$Q_{s+0}(t+0),\ Q_{s+0}(t-0),\ Q_{s-0}(t-0),\ Q_{s-0}(t+0)$$

are nonvanishing, and of the same sign. This implies that order of the zero of $Q_s(t)$ as a function of t must be even. Assume now that a given branch continues, say as $(t, s) \to (t, s + 0)$. The zero (t, s) cannot be simple. Since $Q_s(t)$ does not vanish identically, a simple zero would contradict the fact that

$$Q_s(t+0),\ Q_{s-0}(t),\ Q_s(t-0)$$

are nonvanishing, and of the same sign. By continuity, this discussion holds for $s = 0$ too.

Thus one has only look for zeros of $B_n(f)$ at the boundary, and for zeros of a particular direction derivative of order at least 2. In [4] the condition that determined an exception point was necessary and sufficient due to the cubic case. In the general case the condition given can be only necessary, but not sufficient. This implies that some computed s_i might be actually extraneous without really influencing efficiency of the algorithm.

Figure 1. Different types of line-algebraic curve branch intersections

Figure 1 shows five types of line-algebraic curve branch intersections one might encounter. (a), (e) shows the intersection at regular curve point, (d) at ordinary double point - crunode, and (b), (c) at singular double point (cusp and acnode). The intersections (e), (d) are not exceptional.

The basic lemmas and the algorithm will be given in the next section, and numerical examples in the section 3.

2. The Basic Lemmas and the Algorithm

Let us first outline some simple facts. Let $f := (f_i)$ be given sequence of real numbers, B_i^n as defined in (1.3) , and let

$$B_{n,i} := B_{n,i}(f) := \sum_{j=i}^{n+i} f_j B_{j-i}^n,$$

denote the Bernstein polynomial of degree $\leq n$, based upon the values $f_i, f_{i+1}, \cdots, f_{n+i}$.

Lemma 1. *The Bernstein polynomial $B_{n,i}$ vanishes at least twice at t_0 iff t_0 is a common zero of $B_{n-1,i}(f),\ B_{n-1,i+1}(f)$.*

Proof. Recursive formula for $B_{n,i}(f)(t),\ \ B'_{n,i}(f)(t)$ produces

$$\begin{pmatrix} B_{n,i}(f)(t) \\ B'_{n,i}(f)(t) \end{pmatrix} = \begin{pmatrix} 1-t & t \\ -n & n \end{pmatrix} \begin{pmatrix} B_{n-1,i}(f)(t) \\ B_{n-1,i+1}(f)(t) \end{pmatrix}. \tag{2.5}$$

Since determinant of the matrix in (2.5) is equal to n independently of t, conclusion follows.

Lemma 2. *Let*

$$B_{n,0}(f)(1) \neq 0, \tag{2.6}$$

and $B_{n-1,0}(f)$, $B_{n-1,1}(f)$ be linearly independent. Let

$$R_n := R_n(f) := (r_{ij})_{i,j=1}^{n-1,n-1}$$

be a symmetric matrix with coefficients for $i \le j$ given as

$$r_{ij} := \sum_{k=n-i}^{\min(n-1,2n-i-j-1)} \binom{n-1}{k} \binom{n-1}{2n-i-j-1-k} \left\| \begin{matrix} f_k & f_{2n-i-j-k-1} \\ f_{k+1} & f_{2n-i-j-k} \end{matrix} \right\|. \quad (2.7)$$

$B_{n,0}(f)$ vanishes at least twice at some point t_0 iff

$$\det R_{n-r}(f) = 0, \quad r = 0, 1, \cdots, k, \quad k \ge 0.$$

Proof. By lemma 1 $B_{n,0}(f)$ has a zero of order ≥ 2 iff the vector polynomial

$$\begin{pmatrix} B_{n-1,0}(f) \\ B_{n-1,1}(f) \end{pmatrix} \quad (2.8)$$

has a zero. Recall the Bezout resultant for Bernstein polynomials in [5 , p. 336, prop. 4.1] that gives the necessary and sufficient condition for existence of a zero of a vector Bernstein polynomial in terms of its coefficients. By rewriting it for a particular polynomial (2.8) one obtains the resultant matrix (2.7) . Its determinant vanishes if (2.8) has zero, or $f_{n-1} = f_n = 0$. But the latter is not possible by (2.6) .

Moreover, [5] provides also a way of computing t_0: let the suppositions of the lemma 2 hold, and let $\det R_{n-k-1}(f) \ne 0$. Perform Gauss elimination on R to reduce it to upper triangular matrix. Let

$$\{0, 0, \cdots, h_k, h_{k-1}, \cdots, h_0\}$$

be the $(n-k-1)$-th row of the upper triangular matrix, and

$$h(u) := \sum_{r=0}^{k} h_r u^r. \quad (2.9)$$

The zeros of h, and zeros t of (2.8) are connected by the relation

$$t = \frac{u}{u+1}$$

Thus

$$t_0 \in [0, 1) \iff u \in [0, \infty).$$

Since [4] solves the problem for $n = 2, 3$, the determinants of R_2, R_3 can be found already there. In order to help the reader we compute here the lower triangles of R_4, R_5 too:

$$R_4 = \begin{pmatrix} 3f_3^2 - 3f_2f_4 & & \\ 3f_2f_3 - 3f_1f_4 & 9f_2^2 - 8f_1f_3 - f_0f_4 & \\ f_1f_3 - f_0f_4 & 3f_1f_2 - 3f_0f_3 & 3f_1^2 - 3f_0f_2 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} 4f_4^2 - 4f_3f_5 & & & \\ 6f_3f_4 - 6f_2f_5 & 24f_3^2 - 20f_2f_4 - 4f_1f_5 & & \\ 4f_2f_4 - 4f_1f_5 & 16f_2f_3 - 15f_1f_4 - f_0f_5 & 24f_2^2 - 20f_1f_3 - 4f_0f_4 & \\ f_1f_4 - f_0f_5 & 4f_1f_3 - 4f_0f_4 & 6f_1f_2 - 6f_0f_3 & 4f_1^2 - 4f_0f_2 \end{pmatrix}$$

Lemma 3. *Let*

$$B_{n-1,0}(f), \;\; B_{n-1,1}(f)$$

be linearly dependent. Then $R_n(f) = 0$*. Also,* $B_n(f) = const$ *or it has n-fold zero.*

Proof. By supposition, one can find α such that

$$B_{n-1,0}(f) = \alpha B_{n-1,1}(f).$$

This implies

$$f_i = \alpha f_{i+1} = \cdots = \alpha^{n-i} f_n, \;\; i = 0, 1, \cdots, n-1. \tag{2.10}$$

The determinants in (2.7) then vanish all,

$$\left\| \begin{matrix} f_k & f_{2n-i-j-k-1} \\ f_{k+1} & f_{2n-i-j-k} \end{matrix} \right\| = f_n^2 \left\| \begin{matrix} \alpha^{n-k} & \alpha^{i+j+k+1-n} \\ \alpha^{n-1-k} & \alpha^{i+j+k-n} \end{matrix} \right\| = f_n^2(\alpha^{i+j} - \alpha^{i+j}) = 0,$$

and first claim is confirmed. (2.10) also implies

$$B_{n,0}(f)(t) = f_n((1-t)\alpha + t)^n.$$

If $\alpha = 1$, $B_{n,0}$ is constant, otherwise vanishes n-fold at $\frac{\alpha}{\alpha-1}$.

We are now ready to outline the algorithm. Let us assume that $B_n(f) \neq const.$ Let $\mathcal{A}$ denote the algebraic curve $B_n(f) = 0$.

1. Compute $E_1 := \{s \mid 0 < s < 1, \; Q_s(1) = a_n(s) = 0\}$.

2. Compute the polynomial det $R_n(f)$, with $f := (a_i(s))$, and $a_i(s)$ given in (1.2) .

3. Determine the zeros $E_2 := \{s \mid 0 \leq s < 1, \; \det R_n(f)(s) = 0\}$. The set

$$E := \{0 = s_0 < s_1 < \cdots < s_m = 1\} := E_1 \cup E_2 \cup \{0, 1\}$$

by lemma 2 and lemma 3 contains all the exceptional values.

4. For each $s \in E$ determine the zeros $t \in [0, 1]$ of $Q_s(t)$, and add them to $\mathcal{A}$. If there are none, one can exclude that particular s from E.

5. For each pair $(s_i, s_{i+1}), \; s_i \in E, \; s_{i+1} \in E$ find all the zeros

$$t_{ij} \in (0, 1) : \; Q_s(t_{ij}) = 0, \;\; j = 1, 2, \cdots, k$$

with

$$s := \frac{s_i + s_{i+1}}{2}.$$

6. Follow the branch j of the algebraic curve from the point

$$(t_{ij}, \frac{s_i + s_{i+1}}{2})$$

to the boundaries s_i, s_{i+1}. If Δs is the chosen step in s parameter, and (s,t) already computed point on the curve, take

$$t + \Delta t, \quad \Delta t := -\frac{\partial Q_s(t)}{\partial s} / \frac{\partial Q_s(t)}{\partial t} \Delta s$$

as the initial guess for the value of the other parameter, and improve this value by Newton iteration.

Some remarks have to be added. Steps 1 and 2 require determination of the zeros of a polynomial that lie in a given interval. Thus the methods that take into account this fact such as the Sturm method would be most efficient to apply. However, general polynomial solvers are widely available. Note also that $\det R_n(f)(s)$ is (for the triangle case) in general polynomial of degree $\leq n(n-1)$, so the step 3 will be most time consuming. This follows from the fact that

$$\text{degree } r_{ij}(s) \leq (n-i)(n-j),$$

and the highest power of s in $\det R_n(f)$ is obtained by the product of diagonal elements. The step 4 is required to determine isolated points of the algebraic curve. (2.9) can be applied.

3. Numerical Examples

We shall demonstrate the algorithm for both triangular and rectangular support. Let us throughout assume that the triangle is given as

$$T = \{(x,y)|0 \leq x, y \leq x + y \leq 1\}, \;\; T_1 = \{0,0\}, \; T_2 = \{1,0\}, \; T_3 = \{0,1\},$$

and the rectangle as

$$M = \{(x,y)|0 \leq x \leq 1, \; 0 \leq y \leq 1\},$$
$$T_1 = \{0,0\}, \;\; T_2 = \{1,0\}, \;\; T_3 = \{1,1\}, \;\; T_4 = \{0,1\}.$$

The coefficients of the Bernstein polynomial over T will be given as lower triangular matrix with indexes ordered as follows

$$(i, j, n-i-j), \;\; i = n, n-1, \ldots, 0, \;\; j = n-i, n-i-1, \ldots, 0.$$

Similarly,

$$(i,j), \;\; i = 0, 1, \cdots, n, \;\; j = 0, 1, \cdots, n$$

for the rectangular case. For example, take $n = 3$. The coefficients $(f_{i,j,k})$ are given as

$$(f_{i,j,k}) = \begin{pmatrix} f_{300} & & & \\ f_{210} & f_{201} & & \\ f_{120} & f_{111} & f_{102} & \\ f_{030} & f_{021} & f_{012} & f_{003} \end{pmatrix},$$

and $(f_{i,j})$ as

$$(f_{i,j}) = \begin{pmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{pmatrix}.$$

Example 1. Let us consider first a rather complex algebraic curve that has no singular points in the support:

$$\begin{gathered} p_1(x,y) := 3 - 42x + 180x^2 - 560x^3 + 855x^4 - 546x^5 + 116x^6 - 24y + \\ 330xy - 420x^2y + 720x^3y - 1230x^4y + 588x^5y + 165y^2 - 2460xy^2 + \\ 3780x^2y^2 - 2460x^3y^2 + 795x^4y^2 - 460y^3 + 6360xy^3 - 7800x^2y^3 + 2520x^3y^3 + \\ 315y^4 - 6000xy^4 + 4410x^2y^4 + 210y^5 + 1674xy^5 - 204y^6 = 0. \end{gathered}$$

Plot of p_1 is given in figure 2.

Figure 2. The plot of p_1

A linear transformation to Bernstein basis over T gives

$$(f_{i,j,k}) = \begin{pmatrix} 3 & & & & & & \\ -4 & -1 & & & & & \\ 1 & 3 & 6 & & & & \\ -10 & 12 & -20 & 1 & & & \\ -8 & 10 & -6 & 10 & -18 & & \\ 2 & -3 & 3 & 6 & -3 & 3 & \\ 6 & -4 & -10 & 0 & 22 & -20 & 5 \end{pmatrix}$$

The coefficients $a_i(s)$ are computed as

$$\begin{aligned}(\quad & 3,\ -4+3s,\ 1+4s+s^2, \\ & -10+66s-162s^2+107s^3, \\ & -8+72s-204s^2+264s^3-142s^4, \\ & 2-25s+110s^2-140s^3+25s^4+31s^5, \\ & 6-60s+60s^2+240s^3-240s^4-336s^5+335s^6).\end{aligned}$$

The boundary zeros are determined as zeros of $a_6(s)$,

$$\begin{aligned}E_1 \ = \{ \ & 0.1205817297277968,\ 0.4617632574259461, \\ & 0.7479972987939762,\ 0.9542034415358480\}.\end{aligned}$$

The resultant matrix, even the resultant, would be too clumsy to present. The zeros of the resultant in $[0,1)$ are

$$\begin{aligned}E_2 \ = \{ \ & 0.1215264354251871,\ 0.2707817768296583, \\ & 0.5294143589879580,\ 0.6795268231893524, \\ & 0.7601818933359121,\ 0.9519920879214000\ \},\end{aligned}$$

and the set E has finally ten points. The values

$$0.529414358987958,\ 0.6795268231893524$$

are extraneous as shown in figure 3, and would be detected on step 4. Figure 4 reveals that they correspond to two exceptional points lying in $M-T$. In the Bernstein basis

Figure 3. The lines $s \in E$ and the plot of $p_1 = 0$ over T

over M, p_1 now reads

$$(f_{i,j}) = \begin{pmatrix} 3 & -1 & 6 & 1 & -18 & 3 & 5 \\ -4 & 7/6 & -10 & -15/2 & -7 & 35/3 & -18 \\ 1 & 32/3 & -98/15 & 32/5 & 206/15 & 15 & -31 \\ -10 & 11/2 & -39/5 & 72/5 & 81/5 & -4 & -23 \\ -8 & 7 & -24/5 & 74/5 & -61/15 & -115/3 & 45 \\ 2 & 31/6 & -41/3 & -19/2 & -161/3 & -199/3 & 247 \\ 6 & 0 & -18 & -17 & -51 & 55 & 815 \end{pmatrix}.$$

The roles of the lines $T_1 \to (1-s)T_2 + sT_3$ are now playing that of the lines

$$(1-x)T_1 + xT_2 \to (1-x)T_4 + xT_3.$$

As a result,

$$\begin{aligned} E_1 \;\; = \{ \;\; & 0.0378849393303741, 0.1062272779679897, \\ & 0.4022080383277741, 0.8129587231380680 \; \} \end{aligned}$$

and

$$\begin{aligned} E_2 \;\; = \{ \;\; & 0.0468897629167010, 0.1863282899219849, 0.2178113399491106, \\ & 0.2393089960526824, 0.3019553483048598, 0.3893871112700215, \\ & 0.4047902547627562, 0.5359584747568275, 0.5387740195311480, \\ & 0.5390230579738621, 0.5461798336859473, 0.7770167302864178 \; \}. \end{aligned}$$

The algebraic curve as well as the lines that correspond to E are given in figure 4.

Figure 4. The lines $x \in E$ and the plot of $p_1 = 0$ over M

Example 2. Consider now the cubic algebraic curve plotted in figure 1 (c). Its power basis expansion reads

$$p_2(x,y) \;:=\; 2891 - 27588x + 87120x^2 - 85184x^3 - 864y + 1728y^2.$$

The B-form over T is computed as

$$(f_{i,j,k}) = \frac{1}{1728} \begin{pmatrix} 2891 & & & \\ -6305 & 2603 & & \\ 13539 & -6593 & 2891 & \\ -22761 & 13251 & -6305 & 3755 \end{pmatrix}.$$

The coefficients $a_i(s)$ are

$$\begin{aligned} \tfrac{1}{1728}(\;\; & 2891, \; -6305 + 8908s, \\ & (4513 - 7404s)(3 - 4s), \\ & -22761 + 108036s - 166704s^2 + 85184s^3). \end{aligned}$$

Figure 5. The lines $s \in E$ and the plot of $p_2 = 0$ over T

There is only one zero of $a_3(s)$ in $(0,1)$,

$$E_1 = \{0.4656123746463041\}.$$

The resultant matrix R_3 is small enough, i.e.

$$\begin{aligned} 23328 r_{11} &= 621819 - 3224166s + 6401224s^2 - 5669264s^3 + 1848256s^4 \\ 23328 r_{12} &= -152823 + 485452s - 495776s^2 + 137128s^3 \\ 23328 r_{22} &= 9559 + 63646s - 97928s^2. \end{aligned}$$

Its determinant is given by

$$\begin{aligned} \tfrac{121}{5038848}(& -1332331 + 12024254s - 45309539s^2 + 91231976s^3 \\ & -103225400s^4 + 61890368s^5 - 15289264s^6). \end{aligned}$$

It has only one (double) zero in $[0,1)$, i.e.

$$E_2 = \{0.5\}.$$

This reveals the isolated point (acnode) of the algebraic curve, $(x,y) = (\frac{1}{4}, \frac{1}{4})$, that would be by more general methods hard to uncover.

As for the rectangular case,

$$(f_{i,j}) = \frac{1}{1728} \begin{pmatrix} 2891 & 2603 & 2891 & 3755 \\ -6305 & -6593 & -6305 & -5441 \\ 13539 & 13251 & 13539 & 14403 \\ -22761 & -23049 & -22761 & -21897 \end{pmatrix}.$$

$$E_1 = \{ \ 0.5380115821196769, 0.6105187724683388 \ \}$$

and

$$E_2 = \{ \ 0.25, 0.5227272727272727 \ \}.$$

The final result is plotted in figure 6.

Figure 6. The lines $x \in E$ and the plot of $p_2 = 0$ over M

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