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CONVERGENCE OF THE POINT VORTEX METHODS FOR EULER EQUATION ON HALF PLANE*

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Abstract

In this paper, we study the point vortex method for 2-D Euler equation of incompressible flow on the half plane, and the explicit Euler's scheme is considered with the reflection method handling the boundary condition. Optimal error bounds for this fully discrete scheme are obtained.

1. Introduction

The vortex methods are efficient numerical method of simulating incompressible flow at high Reynold's number. The convergence of the vortex methods for the initial value problems of Euler equation was first obtained by Hald[4], then the results were improved by several authors[1,2,3,5]. But in fact, many practical problems are considered in a bounded domain or an exterior domain, and the numerical boundary condition has an important effect on numerical result. The particle trajectories of exact solution will not go out from the domain, but it is not the case in pratical computation. There are three kinds of method handling the boundary condition:

(a) reflection method, in which we regard the boundary as a wall, the particles will bounce back when they hit against.

(b) absorb method, in which the particles will be thrown away while they cross the boundary of the domain.

(c)extrapolation method, in which we extend the domain Ω to $\Omega' \supset \Omega$; when the particles go out from the domain Ω', they will be thrown away. But the velocity of the particles which belong to the domain $\Omega' \backslash \Omega$ will expressed through extrapolation method.

Ying Lung-an[6] proved the convergence problem of the vortex methods with extrapolation boundary treatment (c) for two dimensional bounded domains. Ying Lung-an and the author of this paper[7] got the error estimates for fully discretized two-dimensional vortex methods for initial boundary value problems of Euler equations.

To the author's knowlege, there is no convergence analysis about the other two methods. we think that the method (b) may not converge, but the method (a) may do.

For a long time, it has been widely thought the point vortex method would not converge in any finite Sobolev space. Recently, however Goodman, Hou and Lowengrub[9] have been able to prove the stability and convergence of the point vortex method for 2-D incompressible Euler equations with smooth solutions, Hou and Lowengrub[10] proved that it was also right for 3-D Euler equations.

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The convergence of the point vortex methods for initial boundary value problems of Euler equations is not available to the author's knowlege, In this paper, we will prove the convergence of the point vortex method for Euler equations on half plane handling the boundary condition with reflection method, and in consequence we can get the convergence of the vortex blob methods with the blob parameter ε which is equivalent to the grid parameter h.

Remark. The convergence of point vortex method for Euler equations on half plane cannot directly get from Goodman, Hou and Lowengrub's paper[9]. Although the problem on half plane may be considered as an initial value problem, the vorticity is not continuous, in other word, there is a vortex sheet in the solution of initial value problem, and it is important to suppose the vorticity is smooth in the convergence of point vortex method[9].

The rest of the paper is organized as follows. In section 2, we describe the point vortex method and state our major consistency, stability and convergence results for the semi-discrete point vortex method on half plane. Moreover, we prove the convergence theorem under the assumption that the consistency and stability Lemmas are valid. The consistency and stability Lemmas are proved in section 3. Finally, we consider a time discrete point vortex method in section 4.

2. Convergence of the Semi-Discrete

The 2-D incompressible inviscid Euler equations on half plane are given by

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla\pi = 0, \quad \text{in} \quad R^2_+, \tag{2.1}$$

$$\nabla\cdot u = 0, \tag{2.2}$$

$$u_2 \mid_{x_2=0} = 0, \tag{2.3}$$

$$u \mid_{t=0} = u_0(x), \tag{2.4}$$

$$\lim_{x\to\infty} u(x,t) = u_\infty,$$

where $u = (u_1, u_2)$ stands for velocity, π stands for pressure, the density ρ is a positive constant, $R^2_+ = \{x \in R^2, \mathrm{x}_2 > 0\}$, we suppose $u_\infty = 0$. $x = (x_1, x_2)$ are points in R^2, the initial data u_0 satisfies

$$\nabla\cdot u_0 = 0, u_0\cdot n \mid_{\partial\Omega} = 0,$$

and u_0 are sufficiently smooth, then the solutions u and π are also sufficiently smooth on the domain $R^2_+ \times [0, T]$, where T is an arbitrary positive constant.

Let $\omega = -\nabla\wedge u, \omega_0 = -\nabla\wedge u_0$ and ψ be the stream function corresponding to u, then (2.1)-(2.4) is equivalent to

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = 0, \quad \text{in} \quad R^2_+, \tag{2.5}$$

$$-\Delta\psi = \omega, u = \nabla\wedge\psi, \tag{2.6}$$

$$\psi \mid_{x_2=0} = 0, \tag{2.7}$$

$$\omega \mid_{t=0} = \omega_0, \tag{2.8}$$

where $\nabla\wedge = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, then we have

$$u(x,t) = \int_{R_+^2} (K(x-x') - K(x-\underline{x'}))\omega(x',t)dx',$$

where $\underline{x}$ denotes the symmetric point of x about the line $x_2 = 0$. and

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2}.$$

If we follow the partical trajectories which are at α when $t = 0$, then the position at time $t > 0$ is given by the solution to

$$\begin{aligned} \frac{dX(\alpha,t)}{dt} &= u(X(\alpha,t),t), \\ X(\alpha,0) &= \alpha, \end{aligned} \tag{2.9}$$

with α fixed. we can then write (2.1) as

$$\frac{d}{dt}\omega(X(\alpha,t),t) = 0$$

then we have

$$\omega(X(\alpha,t),t) = \omega(\alpha,0). \tag{2.10}$$

We assume that ω_0 is smooth and has compact support in $\overline{R_+^2}$, a smooth solution is known to exist[11].

Define $\wedge_h$ to be the portion of the grid containing nonzero vorticity

$$\wedge_h = \{\alpha_i = h\cdot(i+\frac{1}{2}),\ \omega_0(\alpha_i) \neq 0,\ i = (i_1, i_2) \in Z^2\}.$$

Then $\omega(X(\alpha,t),t) = 0$ for $\alpha \notin \wedge_h$ from (2.10), we consider the following version of the 2-D point vortex method

$$\begin{aligned} \frac{d\widetilde{x}_i(t)}{dt} &= \Big(\sum_{j\neq i, jh\in\wedge_h} K(\widetilde{x}_i(t) - \widetilde{x}_j(t)) - \sum_{jh\in\wedge_h} K(\widetilde{x}_i(t) - \underline{\widetilde{x}_j(t)})\Big)\omega_j h^2 \\ &= \widetilde{u}^h(\widetilde{x}_i(t),t), \\ \widetilde{x}_i(0) &= \alpha_i. \end{aligned} \tag{2.11}$$

where $\omega_j = \omega_0(\alpha_j) = \omega(x_j(t),t)$.

Define a discrete L^p norm l^p to be

$$\|f\|_{l^p} = \Big(\sum_{jh\in\wedge_h} |f_j|^p h^2\Big)^{\frac{1}{p}}.$$

It is easy to see that

$$\|f\|_{l^\infty} = \max_i |f_i| \leq h^{-\frac{2}{p}}\|f\|_{l^p}. \tag{2.12}$$

The main result of this section is the following theorem.

Theorem 2.1. *Suppose that the initial vorticity $\omega_0(x)$ is smooth and has compact support, $1 < p < \infty$, then we have*

$$\|\widetilde{x}(t) - x(t)\|_{l^p} \le C(T)h^2|\log h|, \tag{2.13}$$
$$\|\widetilde{u}^h(\widetilde{x}(t), t) - u(x(t), t)\|_{l^p} \le C(T)h^2|\log h|. \tag{2.14}$$

We need the following Lemmas to prove Theroem 2.1.

Consistency Lemma. *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned}&\Big|\int (K(x_i(t) - X(\alpha, t)) - K(x_i(t) - \underline{X(\alpha, t)}))\omega(X(\alpha, t), t)d\alpha \\ &- (\sum_{j\neq i, jh\in\wedge_h} K(x_i(t) - x_j(t)) - \sum_{jh\in\wedge_h} K(x_i(t) - \underline{x_j(t)}))\omega_j h^2\Big| \\ &\le C(T)h^2|\log h|.\end{aligned} \tag{2.15}$$

Stability Lemma. *Suppose that*

$$\|\widetilde{x}(t) - x(t)\|_{l^\infty} \le h^{1+s},$$

for $0 \le t \le T^*$ for some $0 < s < 1$, then for $1 < p < \infty$, we have

$$\begin{aligned}&\Big(\sum_{ih\in\wedge_h}\Big|\Big(\sum_{j\neq i, jh\in\wedge_h} K(x_i(t) - x_j(t)) - \sum_{jh\in\wedge_h} K(x_i(t) - \underline{x_j(t)})\Big)\omega_j h^2 \\ &- \Big(\sum_{j\neq i, jh\in\wedge_h} K(\widetilde{x}_i(t) - \widetilde{x}_j(t)) - \sum_{jh\in\wedge_h} K(\widetilde{x}_i(t) - \underline{\widetilde{x}_j(t)})\Big)\omega_j h^2\Big|^p h^2\Big)^{\frac{1}{p}} \\ &\le C(T, p)\|\widetilde{x}(t) - x(t)\|_{l^p},\end{aligned} \tag{2.16}$$

where $C(T, p)$ is indepent of T^*.

Proof of Theorem 2.1: We assume the validity of the consistency and stability Lemmas, the proofs will be given in section 3. Define T^* by

$$T^* = \sup\{t : 0 \le t \le T,\ \|\widetilde{x}(t) - x(t)\|_{l^\infty} \le h^{1+s}\},$$

for some $0 < s < 1$, we write

$$\frac{d}{dt}(\widetilde{x}_i(t) - x_i(t)) = I + II,$$

where

$$\begin{aligned}I = &\int_{R_+^2} \Big(K(x_i(t) - X(\alpha, t)) - K(x_i(t) - \underline{X(\alpha, t)})\Big)\omega(X(\alpha, t), t)d\alpha \\ &- \Big(\sum_{j\neq i, jh\in\wedge_h} K(x_i(t) - x_j(t)) - \sum_{jh\in\wedge_h} K(x_i(t) - \underline{x_j(t)})\Big)\omega_j h^2,\end{aligned}$$

and

$$\begin{aligned}II = &\Big(\sum_{j\neq i, jh\in\wedge_h} K(x_i(t) - x_j(t)) - \sum_{jh\in\wedge_h} K(x_i(t) - \underline{x_j(t)})\Big)\omega_j h^2 \\ &- \Big(\sum_{j\neq i, jh\in\wedge_h} K(\widetilde{x}_i(t) - \widetilde{x}_j(t)) - \sum_{jh\in\wedge_h} K(\widetilde{x}_i(t) - \underline{\widetilde{x}_j(t)})\Big)\omega_j h^2.\end{aligned}$$

By the consistency Lemma, we have for $0 \leq t \leq T$,

$$|I| \leq C(T)h^2|\log h|.$$

And by the stability Lemma

$$\|II\|_{l^p} \leq C(T)\|\widetilde{x}(t) - x(t)\|_{l^p},$$

for $0 \leq t \leq T^*$. Therefore we obtain

$$\|\frac{d}{dt}(\widetilde{x}_i(t) - x_i(t))\|_{l^p} \leq C(T)(\|\widetilde{x}(t) - x(t)\|_{l^p} + h^2|\log h|),$$

for $0 \leq t \leq T^*$, but $(\frac{d}{dt})\|V\| \leq \|\frac{d}{dt}V\|$, by Gronwall's inequality we obtain

$$\|\widetilde{x}(t) - x(t)\|_{l^p} \leq \widetilde{C}(T)h^2|\log h|,$$

for $0 \leq t \leq T^*$, then (2.12) imply that

$$\|\widetilde{x}(t) - x(t)\|_{l^\infty} \leq \widetilde{C}(T)h^{2-\frac{2}{p}}|\log h|.$$

Since $\widetilde{C}(T)$ is independent of T^*, we obtain by taking h sufficiently small and $p > \frac{2}{1-s}$ that

$$\|\widetilde{x}(t) - x(t)\|_{l^\infty} \leq \frac{1}{2}h^{1+s},$$

for $0 \leq t \leq T^*$. This implies that $T = T^*$ and therefore we conclude that

$$\|\widetilde{x}(t) - x(t)\|_{l^p} \leq \widetilde{C}(T)h^2|\log h|, \tag{2.17}$$

for $0 < t < T$. Clearly we have as a cosequence of (2.17) that

$$\|\widetilde{u}^h(\widetilde{x}(t), t) - u(x(t,t))\|_{l^p} \leq \widetilde{C}(T)h^2|\log h|,$$

for $0 < t < T$, which proves Theorem 2.1.

3. Consistency and Stability

First we note that the flow map and the inverse flow map are smooth functions, since the flow is incompressible. Thus there exist positive constants $C_1(T)$ and $C_2(T)$ depending on ω_0 and T only, such that

$$C_1(T)|\alpha - \beta| \leq |X(\alpha, t) - X(\beta, t)| \leq C_2(T)|\alpha - \beta|. \tag{3.1}$$

Proof of the consistency Lemma: Suppose that the support of the initial vorticity Ω_0 is contained in $B(0, \frac{R}{4}) \cap R^2_+$, where $B(\alpha_i, r) = \{\alpha, |\alpha - \alpha_i| \leq r\}$, we define $E(\alpha_i, R) = B(\alpha_i, R) \cap R^2_+$, $E^*(\alpha_i, R) = \underline{B(\alpha_i, R) \cap R^2_-}$, where $\underline{E} = \{x, \underline{x} \in E\}$. Then we have

$$\begin{aligned}
&E(\alpha_i, R) \supset B(0, \frac{R}{4}) \cap R^2_+, \qquad \forall \alpha_i \in B(0, \frac{R}{4}) \cap R^2_+,\\
&E^*(\alpha_i, R) \supset B(0, \frac{R}{4}) \cap R^2_+, \qquad \forall \alpha_i \in B(0, \frac{R}{4}) \cap R^2_+,\\
&E(\alpha_i, R) \cup \underline{E^*(\alpha_i, R)} = B(\alpha_i, R).
\end{aligned}$$

Thus we can decompose the consistency error into two parts as follows

$$\begin{aligned}
&\int_{R^2_+} \Big(K(x_i(t) - X(\alpha,t)) - K(x_i(t) - \underline{X(\alpha,t)})\Big)\omega(X(\alpha,t),t)d\alpha \\
&-\Big(\sum_{j\neq i, jh\in\wedge_h} K(x_i(t) - x_j(t)) - \sum_{jh\in\wedge_h} K(x_i(t) - \underline{x_j(t)})\Big)\omega_j h^2 \\
&= I + II,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
I = &\int_{E(\alpha_i,R)} K(x_i(t) - X(\alpha,t))(\omega(X(\alpha,t),t) - \omega(x_i(t),t))d\alpha \\
&- \int_{E^*(\alpha_i,R)} K(x_i(t) - \underline{X(\alpha,t)})(\omega(X(\alpha,t),t) - \omega(x_i(t),t))d\alpha \\
&- (\sum_{j\neq i, jh\in E(\alpha_i,R)} K(x_i(t) - x_j(t)) \\
&- \sum_{jh\in E^*(\alpha_i,R)} K(x_i(t) - \underline{x_j(t)})) \\
&(\omega(x_j(t),t) - \omega(x_i(t),t))h^2,
\end{aligned}$$

and

$$\begin{aligned}
II = &\omega(x_i(t),t)(\int_{E(\alpha_i,R)} K(x_i(t) - X(\alpha,t))d\alpha \\
&- \int_{E^*(\alpha_i,R)} K(x_i(t) - \underline{X(\alpha,t)})d\alpha \\
&- (\sum_{j\neq i, jh\in E(\alpha_i,R)} K(x_i(t) - x_j(t)) \\
&- \sum_{jh\in E^*(\alpha_i,R)} K(x_i(t) - \underline{x_j(t)}))h^2).
\end{aligned}$$

We first estimate the II term. Define a new kernel K^d by

$$\begin{aligned}
K^d(x_i(t) - X(\alpha,t)) = &(K(x_i(t) - X(\alpha,t)) - DS_1(\alpha,\alpha_i)) \\
&- (K(x_i(t) - \underline{X(\alpha,t)}) - DS_2(\alpha,\alpha_i)),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
DS_1(\alpha,\alpha_i) &= \frac{1}{2\pi}\frac{\nabla X(\alpha_i,t)\cdot(\alpha-\alpha_i)}{|\nabla X(\alpha_i,t)\cdot(\alpha-\alpha_i)|^2}, \quad \alpha\neq\alpha_i, \\
DS_2(\alpha,\alpha_i) &= \frac{1}{2\pi}\frac{\nabla X(\alpha_i,t)\cdot(\underline{\alpha}-\alpha_i)}{|\nabla X(\alpha_i,t)\cdot(\underline{\alpha}-\alpha_i)|^2},
\end{aligned}$$

for $|\alpha - \alpha_i|$ small, we can expand $X(\alpha,t)$ around $\alpha = \alpha_i$ and get

$$X(\alpha,t) - x_i(t) = \nabla X(\alpha_i,t)\cdot(\alpha-\alpha_i) + O(|\alpha-\alpha_i|^2). \tag{3.4}$$

Thus

$$\begin{aligned}|X(\alpha,t)-x_i(t)| &= |\nabla X(\alpha_i,t)\cdot(\alpha-\alpha_i)+O(|\alpha-\alpha_i|^2)|\\ &\geq C_1(T)|\alpha-\alpha_i|\end{aligned}$$

then

$$\begin{aligned}|\nabla X(\alpha_i,t)\cdot(\alpha-\alpha_i)| &\geq C_1(T)|\alpha-\alpha_i|-C_3(T)|\alpha-\alpha_i|^2\\ &\geq \frac{1}{2}C_1(T)|\alpha-\alpha_i|,\end{aligned} \tag{3.5}$$

for $|\alpha-\alpha_i|>0$ small. Therefore DS_1 is well defined.

Similar to (3.1), we have

$$C_1'(T)|\underline{\alpha}-\beta| \leq |\underline{X(\alpha,t)}-X(\beta,t)| \leq C_2'(T)|\underline{\alpha}-\beta|.$$

And so DS_2 is well defined for $|\underline{\alpha}-\alpha_i|$ small. Clearly $DS_1(\alpha,\alpha_i)$ is an odd function in $(\alpha-\alpha_i)$. $DS_2(\alpha,\alpha_i)$ is an odd function in $(\underline{\alpha}-\alpha_i)$. In virtue of symmetry, we have

$$\begin{aligned}&\int_{E(\alpha_i,R)} DS_1(\alpha,\alpha_i)d\alpha-\int_{E^*(\alpha_i,R)} DS_2(\alpha,\alpha_i)d\alpha\\ &=\sum_{j\neq i, jh\in E(\alpha_i,R)} DS_1(\alpha_j,\alpha_i)h^2-\sum_{jh\in E^*(\alpha_i,R)} DS_2(\alpha_j,\alpha_i)h^2\\ &=0.\end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6)

$$|K^d(x_i(t)-X(\alpha,t))| \leq C(T). \tag{3.7}$$

By differentiating $K^d(x_i(t)-X(\alpha,t))$ with respect to α and arguing in the same way as in proving (3.7), we can show that for $\alpha\neq\alpha_i$, we have

$$|D_\alpha^\gamma K^d(x_i(t)-X(\alpha,t))| \leq \frac{C(T)}{|\alpha-\alpha_i|^{|\gamma|}}. \tag{3.8}$$

Define S_j by

$$S_j=\Big\{\alpha=(\alpha_1,\alpha_2):\quad (j_k-\frac{1}{2})h\leq\alpha_k\leq(j_k+\frac{1}{2})h,\quad 1\leq k\leq 2\}\cap R_+^2\Big\}. \tag{3.9}$$

Then from (3.7) we obtain

$$|\int_{S_j} K^d(x_i(t)-X(\alpha,t))| \leq \widetilde{C}(T)h^2. \tag{3.10}$$

Now it follows from the error estimate for composite midpoint rule approximation [5] that

$$\begin{aligned}|II| =&|\omega(x_i(t),t)\Big(\int K^d(x_i(t)-X(\alpha,t))d\alpha\\ &-\sum K^d(x_j(t)-x_i(t))h^2\Big)|\\ &\leq C(T)h^2\sum\sum_{|\gamma|=2}\max_{\alpha\in S_j}|D_\alpha^\gamma(K^d(x_j(t)-x_i(t)))|,\end{aligned} \tag{3.11}$$

where we have used (3.6) and (3.10), by (3.1) we have for $\alpha \in S_j$, with $j \neq i$

$$|x_i(t) - X(\alpha, t)| \geq C_1(T)|\alpha_i - \alpha| \geq (1 - \frac{3^{\frac{1}{2}}}{2})C_1(T)|\alpha_i - \alpha_j|, \tag{3.13}$$

therefore we have

$$\begin{aligned} |II| &\leq C(T)h^2\Big(\sum_{j\neq i, jh\in E(\alpha_i,R)} \frac{1}{|\alpha_i - \alpha_j|^2} + \sum_{jh\in E^*(\alpha_i,R)} \frac{1}{|\underline{\alpha_i} - \alpha_j|^2}\Big) \\ &\leq C(T)h^2 \int_{C_1 h \leq |\alpha - \alpha_i| < R} \frac{1}{|\alpha_i - \alpha|^2} d\alpha \\ &\leq \widetilde{C}(T)h^2|\log h|. \end{aligned} \tag{3.14}$$

Similarly, we can show that

$$|I| \leq C(T)h^2|\log h|. \tag{3.15}$$

This completes the proof of consistency Lemma.

Similar to [10], the stability Lemma can be proved easily by using the known stability result for the corresponding vortex blob method, by noting that

$$|\underline{\widetilde{x}} - \underline{x}| = |\widetilde{x} - x|.$$

4. Time Discrete Method

We now consider the time discrete version of the point vortex method, the forward Euler scheme is applied to the ordinary differential equations (2.11), and we use the reflection method handling the boundary condition. Let $\widetilde{x}_i^n$ be an approximation to $X(\alpha_i, t_n)$ with $t_n = n\Delta t$, we define the time discrete point vortex method by

$$\widetilde{x}_i^{n+1} = \begin{cases} \widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n), & \text{if } \widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n) \in R_+^2, \\ \underline{\widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n)}, & \text{if } \widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n) \notin R_+^2, \end{cases}$$

$$\widetilde{x}_i^0 = \alpha_i, \tag{4.1}$$

where $\widetilde{u}^h(\widetilde{x}_i^n, t_n)$ is defined as in (2.11).

Theorem 4.1. *Under the assumption of Theorem* 2.1*, the time discrete point vortex method* (4.1) *is converge if* $\Delta t = O(h^2)$*, more precisely, there exist a positive constant* $h_0(T)$ *such that for* $2 < p < \infty$ *and all* $0 < h < h_0(T)$

$$\|x^n - \widetilde{x}^n\|_{l^p} \leq C(T)(h^2|\log h| + \Delta t), \tag{4.2}$$

$$\|u(x^n, t_n) - \widetilde{u}(\widetilde{x}^n, t_n)\|_{l^p} \leq C(T)(h^2|\log h| + \Delta t) \tag{4.3}$$

where $x^n = \{x_i^n\}, x_i^n = X(\alpha_i, t_n)$.

Proof. Define

$$e_i^n = x_i^n - \widetilde{x}_i{}^n,$$

$$\begin{aligned} T^* =&\sup\{t_n, 0 \le t \le T,\ \|e^n\|_{l^\infty} \le h^{1+s}, \\ &\max_i |\widetilde{u}^h(\widetilde{x}_i^n, t_n)| \le C, \quad \text{for } 0 < s < 1\}. \end{aligned}$$

If $\widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n) \notin R_+^2$, then

$$(\widetilde{x}_i^n)_2 + \Delta t \widetilde{u}_2^h(\widetilde{x}_i^n, t_n) < 0.$$

Thus we get

$$|(\widetilde{x}_i^n)_2| \le \Delta t |\widetilde{u}_2^h(\widetilde{x}_i^n, t_n)| \le C\Delta t, \qquad \text{for} \quad 0 < t_n < T^*,$$

by (4.1)

$$\begin{aligned} (\widetilde{x}_i^{n+1})_2 \le& (\widetilde{x}_i^n)_2 + \Delta t \widetilde{u}_2^h(\widetilde{x}_i^n, t_n) \\ &+ 2|(\widetilde{x}_i^n)_2| + 2\Delta t |\widetilde{u}_2^h(\widetilde{x}_i^n, t_n)| \\ &\le (\widetilde{x}_i^n)_2 + \Delta t \widetilde{u}_2^h(\widetilde{x}_i^n, t_n) + 4\Delta t |\widetilde{u}_2^h(\widetilde{x}_i^n, t_n)|, \end{aligned}$$

and

$$\begin{aligned} &\widetilde{u}_2^h(\widetilde{x}_i^n, t_n) = \widetilde{u}_2^h(\widetilde{x}_i^n, t_n) - u_2(x_i^n, t_n) + u_2(x_i^n, t_n) \\ &u_2(x_i^n, t_n) = u_2(x_i^n, t_n) - u_2(\widetilde{x}_i^n, t_n) + u_2(\widetilde{x}_i^n, t_n) \\ &\le C|x_i^n - \widetilde{x}_i^n| + u_2(\widetilde{x}_i^n, t_n). \end{aligned}$$

By the boundary condition and smoothness of u

$$u_2(\widetilde{x}_i^n, t_n) \le C|(\widetilde{x}_i^n)_2| \le C\Delta t.$$

Then

$$\begin{aligned} \widetilde{x}_i^{n+1} \le& \widetilde{x}_i^n + \Delta t \widetilde{u}^h(\widetilde{x}_i^n, t_n) \\ &+ 4\Delta t (C|\widetilde{x}_i^n - x_i^n| + C\Delta t + |\widetilde{u}^h(\widetilde{x}_i^n, t_n) - u(x_i^n, t_n)|). \end{aligned} \tag{4.4}$$

It is well known that the Euler method is first-order accuracy. Therefore we have

$$\begin{aligned} e_i^{n+1} \le& e_i^n + \Delta t (\widetilde{u}^h(\widetilde{x}_i^n, t_n) - u(x_i^n, t_n)) \\ &+ 2\Delta t (|\widetilde{x}_i^n - x_i^n| + \Delta t + |\widetilde{u}^h(\widetilde{x}_i^n, t_n) - u(x_i^n, t_n)|) \\ &+ O(\Delta t^2), \end{aligned} \tag{4.5}$$

we write

$$\begin{aligned} \widetilde{u}^h(\widetilde{x}_i^n, t_n) - u(x_i^n, t_n) =& (\widetilde{u}^h(\widetilde{x}_i^n, t_n) - u^h(x_i^n, t_n)) \\ &+ (u^h(x_i^n, t_n) - u(x_i^n, t_n)) \\ =& \text{Stability} + \text{Consistency}, \end{aligned} \tag{4.6}$$

where $u^h(x_i^n, t_n)$ is the point vortex method discretization for the velocity integral corresponding for the exact particle position $\{x_i^n\}$. By the Consistency Lemma and Stability Lemma, we have

$$\|e^{n+1}\|_{l^p} \le \|e^n\|_{l^p} + C(T)\Delta t (\|e^n\|_{l^p} + Ch^2 |\log h| + \Delta t).$$

Which implies that

$$\|e^n\|_{l^p} \le C_1(T)(h^2|\log h| + \Delta t), \tag{4.7}$$

$$\begin{aligned}\|\widetilde{u}^h(\widetilde{x}_i^n, t_n) - u(x_i^n, t_n)\|_{l^p} &\le C(T)(\|\widetilde{x}^n - x^n\|_{l^p} + h^2|\log h|) \\ &\le C(T)(h^2|\log h| + \Delta t),\end{aligned} \tag{4.8}$$

where C_1 depends on ω_0 and T only, in particular, we obtain

$$\|e^n\|_{l^\infty} \le C_1(T)h^{2-\frac{2}{p}}|\log h| < \frac{1}{2}h^{1+s}, \tag{4.9}$$

$$|\widetilde{u}^h(\widetilde{x}_i^n, t_n)| \le |u(x_i^n, t_n)| + C_1(T)h^{-\frac{2}{p}}(h^2|\log h| + \Delta t) \le C, \tag{4.10}$$

by $\Delta t = O(h^2)$. for h small enough and $p > \frac{2}{1-s}$. Therefore (4.9), (4.10) would allow us to take one more step $T^* + \Delta t$ if $T^* < T$. Hence $T^* = T$, and (4.7), (4.8) holds on the entire interval $0 \le t_n \le T$. This completes the proof of Theorem 4.1.

Remark: Using the proof of convergence in this paper, we can get a convergence result for the vortex blob method while the blob parameter ε is equalent to the grid parameter h.

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