

**DISCRETE APPROXIMATIONS FOR SINGULARLY
PERTURBED BOUNDARY VALUE PROBLEMS
WITH PARABOLIC LAYERS, II^{*1)}**

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Abstract

In his series of three papers we study singularly perturbed (SP) boundary value problems for equations of elliptic and parabolic type. For small values of the perturbation parameter parabolic boundary and interior layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux do not converge uniformly with respect to this parameter. Using the method of special, adapted grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε -uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. We compare the results with those obtained by the adapted approach. Results of numerical experiments are discussed.

In the three papers we first give an introduction on the general problem, and then we consider respectively (i) Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required; (ii) Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type; (iii) Problems for SP parabolic equation with discontinuous boundary conditions.

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Part II

BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATION WITH MIXED BOUNDARY CONDITION

1. Introduction

In this part we sketch a variety of special methods which are used for constructing ε -uniformly convergent schemes. We shall demonstrate a method which achieves improved accuracy for solving singularly perturbed boundary value problem for elliptic equations with parabolic boundary layers.

In Section 4 we shall introduce a natural class, B , of finite difference schemes, in which (by the above mentioned approaches (a) and (b)) we can construct (formally) the special finite difference schemes with approximate solutions which converge parameter-uniformly to the solution of our initial boundary value problem.

In this chapter we consider a class of singularly perturbed boundary value problems which arise when diffusion processes in a moving medium are modeled. For such boundary value problems which describe transfer with diffusion, we construct a special scheme that converges parameter-uniformly. We shall show that for the construction of such schemes from class B , the use of a special condensing grid (or an adaptive mesh) is necessary. It means that the choice (to construct special parameter-uniformly convergent schemes for our class of convection diffusion problems) is quite restricted. By condensing (or adaptive) grids we can construct finite difference schemes which converge parameter-uniformly. We shall present and discuss the results of numerical computations using both the classical and the new special finite difference schemes.

2. The Class of Boundary Value Problems

2.1. The physical problem

The diffusion of a substance in a convective flow of an incompressible fluid in a two-dimension domain gives rise to an equation of the form

$$-\varepsilon\Delta u(x) + \vec{v}(x) \cdot \vec{\nabla} u(x) = F(x), \quad x \in \Omega, \quad (2.1a)$$

where $\vec{v}(x)$ and $F(x)$ are the velocity and source, respectively; $1/\varepsilon$ is the Peclet number (Reynolds number), if the substance is heat (diffusive matter or momentum)^[?]. When the substance is heat (diffusive matter or momentum) then $u(x)$ is the temperature (density or velocity) at the point x . On the boundary of domain considered (that is the wall of the container holding the fluid) we have a boundary condition that describes the exchange of the substance with the surrounding environment

$$-\alpha(u(x) - U(x)) - \frac{\partial}{\partial n} u(x) = 0, \quad x \in \partial^0\Omega. \quad (2.1b)$$

Here $\partial^0\Omega$ is the boundary of the domain (the wall of the container), $\frac{\partial}{\partial n}$ is the outward normal derivative at the boundary, α characterises intensity of exchange of the substance between the medium and the wall, where the value is given by $U(x)$. When α tends to infinity the condition (??) becomes the Dirichlet condition

$$u(x) = U(x), \quad x \in \partial^0\Omega.$$

The inflow boundary, that is the part of the boundary $\partial\Omega \setminus \partial^0\Omega$ where the stream enters the domain, we denote by $\partial^+\Omega$, and the outflow boundary by $\partial^-\Omega$

$$\vec{n}(x) \cdot \vec{v}(x) < 0, \quad x \in \partial^+\Omega; \quad \vec{n}(x) \cdot \vec{v}(x) > 0, \quad x \in \partial^-\Omega.$$

Here $\vec{n}(x)$ is a unit vector in the direction of the external normal. On the boundary $\partial^0\Omega$ we have condition

$$\vec{n}(x) \cdot \vec{v}(x) = 0, \quad x \in \partial^0\Omega.$$

On $\partial^+\Omega$ the value of $u(x)$ is given, and on outflow boundary $\partial^-\Omega$ we assume the flux to be known

$$u(x) = U^+(x), \quad x \in \partial^+\Omega, \tag{2.1c}$$

$$\frac{\partial}{\partial n}u(x) = \Psi(x), \quad x \in \partial^-\Omega. \tag{2.1d}$$

Problem (??) describes a general diffusion process into moving medium. For sufficiently large Peclet number (Reynolds number), ε can be very small. As ε tends to zero a boundary layer appears in the neighbourhood of the boundary $\partial^0\Omega$.

2.2. The class of boundary value problems

Now we describe the class of two-dimensional convection-diffusion problems with mixed boundary conditions, for which we shall study the convergence behaviour. Notice that we consider here mixed boundary conditions, where usually only Dirichlet boundary conditions are studied.

On the rectangular domain $D = \{x : 0 < x_i < d_i, \quad i = 1, 2\}$ we consider the elliptic boundary value problem

$$L_{(??)}u(x) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} - b(x) \frac{\partial}{\partial x_1} - c(x) \right\} u(x) = f(x), \quad x \in D, \tag{2.2a}$$

$$u(x) = \varphi(x), \quad x \in \Gamma^+, \tag{2.2b}$$

$$l_{(??)}u(x) \equiv -\varepsilon\alpha \frac{\partial}{\partial n}u(x) - (1 - \alpha)u(x) = \psi(x), \quad x \in \Gamma^0, \tag{2.2c}$$

$$\frac{\partial}{\partial n}u(x) = \eta(x), \quad x \in \Gamma^-. \tag{2.2d}$$

Here $a_s, b, c, f, \varphi, \psi$, and η are sufficiently smooth functions, $\alpha \in [0, 1]$, $\varepsilon \in (0, 1]$, $a_0 \leq a_1(x)$, $a_2(x) \leq a^0$; $b(x) \geq b_0$; $c(x) \geq 0$; $x \in \overline{D}$; $a_0, b_0 > 0$, and

$$\begin{aligned} \Gamma^+ &= \Gamma \cap \{x \mid x_1 = 0\}, \\ \Gamma^- &= \Gamma \cap \{x \mid x_1 = d_1\}, \\ \Gamma^0 &= \Gamma \cap \{x \mid 0 < x_1 < d_1\}. \end{aligned}$$

This class of problems includes, for example, the following boundary value problem for a regular differential equation

$$L_{(??)}U(y) \equiv \left\{ \sum_{s=1,2} A_s(y) \frac{\partial^2}{\partial y_s^2} - B(y) \frac{\partial}{\partial y_1} \right\} U(y) = F(y), \quad (2.3a)$$

$y \in \tilde{D}$, with regular boundary conditions

$$\begin{aligned} U(y) &= \Phi(y), & y &\in \tilde{\Gamma}^+, \\ l_{(??)}U(y) &\equiv -\alpha \frac{\partial}{\partial n} U(y) - (1 - \alpha)U(y) = \Psi(y), & y &\in \tilde{\Gamma}^0, \\ \frac{\partial}{\partial n} U(y) &= 0, & y &\in \tilde{\Gamma}^-, \end{aligned} \quad (2.3b)$$

on the rectangular domain $\tilde{D} = \{y : 0 < y_i < \tilde{d}_i, i = 1, 2\}$, $\tilde{d}_i = \varepsilon^{-1} d_i$, if the size of domain \tilde{D} is sufficiently large.

2.3. The construction of ε -uniformly convergent schemes

When ε tends to zero in the neighbourhood of Γ^0 , boundary layers appear which are described by parabolic equations. Hence these layers are known as parabolic boundary layers.

Although classical difference approximations (see, for example, [?, ?]) converge for (??) to the solution of the boundary value problem for each fixed value of ε (see Theorem ??), the accuracy of the numerical solution depends on the value of ε and decreases, sometimes to complete loss of accuracy, when ε is less or comparable with the step-size of the uniform grid. This means that classical finite difference schemes do not converge uniformly with respect to the parameter ε , (see theorem ??). Therefore, for the boundary value problem (??) it is of interest to construct special schemes the solution of which does converge ε -uniformly.

For the case of the Dirichlet problem (??), an ε -uniformly convergent finite difference scheme is found in [?, ?].

3. Classical Difference Scheme

To solve the problem (??) we first use a classical finite difference method. On the set \overline{D} we introduce the rectangular grid

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2, \quad (3.4)$$

where $\overline{\omega}_s$ is a , in general non-uniform, grid on the interval $[0, d_s]$ and N_s is the number of nodes of the grid $\overline{\omega}_s$, $s = 1, 2$. Define $h_s^i = x_s^{i+1} - x_s^i$, $h_s = \max_i h_s^i$, $h \leq M N^{-1}$, where $h = \max_s h_s$, $N = \min_s N_s$, $s = 1, 2$; $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$. For problem (??) we use the difference scheme

$$\Lambda_{(??)}z(x) = f(x), \quad x \in D_h, \quad (3.5a)$$

$$z(x) = \varphi(x), \quad x \in \Gamma_h^+, \quad (3.5b)$$

$$\lambda_{(??)}z(x) = \psi(x), \quad x \in \Gamma_h^0, \tag{3.5c}$$

$$\delta_{\overline{x_1}}z(x) = \eta(x), \quad x \in \Gamma_h^-, \tag{3.5d}$$

where

$$\Lambda_{(??)}z(x) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x) \delta_{\widehat{x_s x_s}} z(x) - b(x) \delta_{\overline{x_1}}z(x) - c(x)z(x),$$

$$\lambda_{(??)}z(x) \equiv \begin{cases} \varepsilon \alpha \delta_{x_2} z(x) - (1 - \alpha)z(x), & x_2 = 0, \\ -\varepsilon \alpha \delta_{\overline{x_2}} z(x) - (1 - \alpha)z(x), & x_2 = d_2, \end{cases}$$

$\delta_{\widehat{x_s x_s}} z(x)$ is the second divided difference on a non-uniform grid, and $\delta_{x_s} z(x)$ and $\delta_{\overline{x_s}} z(x)$ are the first forward and backward divided differences.

The difference scheme (??), (??) is monotone (that is the maximum principle holds)^[?]. By means of the maximum principle, and using the estimates of the derivatives^[?], we find that the solution of the scheme (??)-(??) converges (for a fixed value of the parameter ε) as

$$|u(x) - z(x)| \leq M\varepsilon^{-4}N^{-1}, \quad x \in \overline{D}_h. \tag{3.6}$$

Theorem 3.1 *Let $u \in C^4(\overline{D})$. Then, for a fixed value of the parameter ε , the solution of the scheme (??)-(??) converges to the solution of the boundary value problem (??) with an error bound given by (??).*

Clearly (??) does not imply ε -uniform convergence of the difference scheme. In fact it can be shown that it is impossible to obtain ε -uniform convergence for the difference scheme (??)-(??) on a fixed ε -independent mesh. The proof is found in [?]. We summarise this result in the following theorem.

Theorem 3.2 (see [?]) *On an ε -independent grid of type (??), the solution of the classical finite difference scheme (??), (??) does not converge ε -uniformly to the solution of the boundary value problem (??).*

We want to make the following interesting observation. We consider problem (??). If we take in $l_{(??)}$ the parameter $\varepsilon = 1$ (leaving ε unchanged in (??) and $\Lambda_{(??)}$, but adapting it in $\lambda_{(??)}$), then no singular part will appear as a first term in the expansion w.r.t. ε . Hence, the classical scheme will be ε -uniform convergent in this case.

4. The Fitted Difference Scheme

For parabolic problems with parabolic layers, it was shown in [?] that there does not exist a difference scheme only based on fitting of the coefficients, for which the solution converges ε -uniformly to the solution. Here we show a similar result for the elliptic boundary value problem (??). Let us consider the problem

$$L_{(??)}u(x) \equiv \varepsilon^2 \Delta u(x) - \frac{\partial}{\partial x_1} u(x) = 0, \quad x \in D, \tag{4.7a}$$

$$u(x) = 0, \quad x \in \Gamma^+, \quad (4.7b)$$

$$l_{(??)}u(x) = \psi(x), \quad x \in \Gamma^0, \quad (4.7c)$$

$$\frac{\partial}{\partial n}u(x) = 0, \quad x \in \Gamma^-, \quad (4.7d)$$

where

$$\psi(x) = \begin{cases} \psi_0(x_1), & x \in \Gamma^0, \quad x_2 = 0, \\ 0, & x \in \Gamma^0, \quad x_2 \neq 0, \end{cases}$$

and the function $\psi_0(x_1)$, $x_1 \in [0, d_1]$ is sufficiently smooth. The solution of problem (??) is the singular solution.

Let us introduce a class, called class A, of finite difference schemes for problem (??), for the construction of which we use uniform meshes:

$$\overline{D}_h^u = \{\overline{D}_h(??), \text{ where } \overline{\omega}_s = \overline{\omega}_s^u \text{ are uniform grids, } s = 1, 2\}. \quad (4.8)$$

and also (for the approximation of equation (??)) a standard five-point, fitted finite difference operator

$$\Lambda_{(??)}z(x) \equiv \left\{ \sum_{s=1,2} (A_s \delta_{x_s \overline{x}_s} + B_s \delta_{x_s} - C) \right\} z(x) = E, \quad x \in D_h. \quad (4.9)$$

Here the coefficients A_s , B_s , C , E are functionals of the coefficients of equation (??) and also depend on x , h_1 , h_2 , and ε . We suppose that for $h_2 \varepsilon^{-1} \rightarrow 0$ and $h_1 \rightarrow 0$ these coefficients A_s , B_s , C , E approximate the data of equation (??), in the uniform norm, in the neighbourhood of at least one point the boundary layer region.

Theorem 4.1 *In the class A of finite difference schemes there does not exist a difference scheme of which the solution converges ε -uniformly to the solution of the boundary value problem (??).*

The proof of this theorem is rather complex. An outline of the principle steps is found in [?, ?].

Remark 1. A statement similar to Theorem ?? is also true in the case when the difference schemes are constructed on a more general stencil with a finite number of nodes.

The results of Theorem ?? and Remark 1 can be explained as follows. All solutions of problem (??) (defined by different functions $\psi(x)$) are singular solutions. Those solutions can not be represented as linear combinations of a finite number of fixed functions of boundary layer type (boundary layer functions).

Let us introduce class B of finite difference schemes for problem (??), for the construction of which we use rectangular grids $\overline{D}_h(??)$, which are generally non-uniform and a five-point finite difference operator (in general a fitted operator) of the standard form. The coefficients of the difference operator are, as before, functionals of the coefficients of the equation (??) and also depend on x , ε and on the distance between

the nodes of the stencil used. Again, we suppose that for $h \rightarrow 0$ the coefficients of the difference operator approximate (in the uniform norm) the coefficients of equation (??) on the set \overline{D}_h .

We remark that class B is a natural class for constructing finite difference schemes for problem (??) as it includes both fitted methods and methods with special condensing grids. This is in contrast to class A, which contains only schemes on uniform meshes. Consequences of Theorem ?? include results such as:

Corollary 4.2

In the case of boundary value problems of type (??), class B of finite difference schemes does not contain any difference scheme which, on grids with arbitrary distribution of nodes, can achieve ε -uniform convergence of the solution to solution of boundary value problem (??) by the use of a fitted method.

Corollary 4.3

In the case of boundary value problems of type (??), the use of special condensing grids (or adaptive meshes) is necessary for the construction of ε -uniformly class B finite difference schemes.

5. Difference Scheme of Method of Special Condensing Mesh

We now construct an ε -uniformly convergent scheme for the boundary value problem (??). We use a special condensing mesh (in the neighbourhood of the boundary layers), where the distribution of the nodes is defined by a-priori estimates of the solution and its derivatives. This approach is similar to that in [?, ?, ?], where the Dirichlet problem was studied.

Consider the special grid

$$\overline{D}_h^* = \overline{\omega}_1 \times \overline{\omega}_2^* , \quad (5.10)$$

where $\overline{\omega}_2^* = \overline{\omega}_2^*(\sigma)$ is a special piecewise uniform mesh, $\overline{\omega}_1$ is a uniform mesh, σ is a parameter which depends on ε and N_2 . The mesh $\overline{\omega}_2^*(\sigma)$ is constructed as follows. The interval $[0, d_2]$ is divided into three parts $[0, \sigma]$, $[\sigma, d_2 - \sigma]$, $[d_2 - \sigma, d_2]$, $0 < \sigma \leq d_2/4$. Each subinterval $[0, \sigma]$ and $[d_2 - \sigma, d_2]$ is divided into $N_2/4$ equal cells and the subinterval $[\sigma, d_2 - \sigma]$ into $N_2/2$ equal cells. Suppose $\sigma = \sigma(\varepsilon, N_2) = \min[d_2/4, m\varepsilon \ln N_2]$ where m is arbitrary number.

The difference scheme (??), (??) belongs to class B. The scheme is constructed using an a priori adapted mesh. Distribution of the nodes on grid \overline{D}_h^* ensures ε -uniform approximation of the boundary value problem. This is formalised in the following theorem (see also [?]).

Theorem 5.1 *The solution of difference scheme (??), (??) converges ε -uniformly to the solution of boundary value problem (??). The following bound holds for the error*

$$|u(x) - z(x)| \leq M N^{-1/3}, \quad x \in \overline{D}_h^*. \quad (5.11)$$

The proof of this theorem will appear in a future paper.

6. Numerical Results

Theoretically (see Theorem ??) it has been shown that the classical difference scheme (??) on the uniform grid (??) does not converge ε -uniformly in the l^∞ -norm to the solution of the boundary value problem (??). But it could be the case that the error $\max_{\overline{D}_h} |z(x) - u(x)|$ is relatively small for the classical scheme, which would reduce the need for a special scheme.

On the other hand, Theorem ?? shows that the special scheme (??),(??) converges ε -uniformly, but no indication is given about the value of the order constant M in (??) and the order of convergence is rather small. It might be that the error is relatively large for any reasonable values of N_1, N_2 . This would reduce the practical value of the special scheme. The following numerical experiments address these issues.

6.1. The model problem

To see the effect of the special scheme in practice, for the approximation of the model problem we study the singularly perturbed elliptic equation with a mixed boundary condition

$$L_{(??)} u(x) \equiv \varepsilon^2 \Delta u(x) - \frac{\partial}{\partial x_1} u(x) = -1, \quad x \in D, \quad (6.12)$$

$$l_{(??)} u(x) = \psi(x), \quad x \in \Gamma^0,$$

$$u(x) = 0, \quad x \in \Gamma^+, \quad \frac{\partial}{\partial x_2} u(x) = 0, \quad x \in \Gamma^-,$$

where

$$l_{(??)} u(x) \equiv \begin{cases} \alpha \varepsilon (\partial / \partial x_2) u(x) - (1 - \alpha) u(x), & x_2 = 0, \\ -\alpha \varepsilon (\partial / \partial x_2) u(x) - (1 - \alpha) u(x), & x_2 = 1. \end{cases}$$

We compare the numerical results for the classical scheme (??), (??) and the special scheme (??), (??). Here $D = \{x : 0 < x_1, x_2 < 1\}$,

$$\psi(x) = \begin{cases} x_1, & x \in \Gamma^0, x_2 = 0, \\ 0, & x \in \Gamma^0, x_2 = 1. \end{cases}$$

For the solution of problem (??), we have the representation

$$u(x) = U(x) + W(x), \quad x \in \overline{D},$$

where $U(x) = x_1$, $x \in \overline{D}$, is the outer solution, and $W(x)$ represents the parabolic boundary layer in the neighbourhood of the edges at $x_2 = 0$ and $x_2 = 1$. We have the following bounds on the solution

$$-1 \leq u(x) \leq 1, \quad x \in \overline{D}.$$

Due to Theorem ?? the solution of the discrete problem with the adapted mesh converges ε -uniformly to the solution of our model problem (?). The function $u_h(x)$, which is the solution of the special scheme (?),(?) is shown in Figure 1.

Figure 1. Solution computed with the adapted mesh, $\varepsilon = 0.01$, $\alpha = 0.5$, $N = 32$ and $m = 1.0$.

6.2. The behaviour of the numerical solution of the classical scheme

To see the difference between the use of the uniform and the adapted grid, for the approximation of (?) we first use the classical scheme (?), (?). We solve the problem for different values of the mesh width $h_1 = h_2 = N^{-1}$ and for different values of the parameters ε and α . The results for a set of numerical experiments is given in Table 1.

From Table 1 we can see that the solution of scheme (?)-(?) does not converge ε -uniformly. The errors, for a fixed value of N , depend on the parameters ε and α . For $\varepsilon \geq 0.1$ and $\alpha = 0.0, 0.1, 0.5, 1.0$ the error behaviour is regular: when N increases, the error decreases. For $\varepsilon = 10^{-3}$ and $\alpha = 0.0, 0.1, 0.5$ and for $\varepsilon = 10^{-2}$ and $\alpha = 0.0$, for some values of N the error increases with increasing N . For $\alpha = 0.5, 1.0$ and a fixed N the error increases with decreasing ε . In particular, for $\varepsilon = 10^{-3}$ and $\alpha = 0.5, 1.0$ the errors for $N \leq 128$ are of the same order or larger than (in ℓ^∞ -norm) the solution of the BVP. Thus, the numerical results illustrate that the lack of ε -uniform convergence leads to large errors indeed.

6.3. The behaviour of the numerical solution of the special scheme

In Table 2 we show the behaviour of (?), (?), with $m = m_{(?) } = 1$, applied to the model problem (?) From Table 2 we can see that the solution of the scheme (?)-(?) does converge ε -uniformly indeed. The errors for a fixed value of

Table 1. Table of errors $E(N, \varepsilon, \alpha)$ for the classical scheme

N		4	8	16	32	64	128
ε	α						
1	0.00	0.105	0.633(-1)	0.329(-1)	0.158(-1)	0.696(-2)	0.241(-2)
10^{-1}		0.305	0.144	0.655(-1)	0.291(-1)	0.122(-1)	0.403(-2)
10^{-2}		0.247	0.127	0.822(-1)	0.107	0.882(-1)	0.246(-1)
10^{-3}		0.246	0.121	0.588(-1)	0.283(-1)	0.157(-1)	0.200(-1)
1	0.10	0.691(-1)	0.381(-1)	0.170(-1)	0.744(-2)	0.314(-2)	0.104(-2)
10^{-1}		0.312	0.149	0.701(-1)	0.310(-1)	0.129(-1)	0.423(-2)
10^{-2}		0.247	0.194	0.185	0.156	0.109	0.583(-1)
10^{-3}		0.246	0.209	0.216	0.216	0.211	0.197
1	0.5	0.852(-1)	0.401(-1)	0.183(-1)	0.842(-2)	0.359(-2)	0.119(-2)
10^{-1}		0.706	0.473	0.254	0.121	0.526(-1)	0.176(-1)
10^{-2}		1.20	1.26	1.15	0.893	0.557	0.279
10^{-3}		1.28	1.43	1.49	1.48	1.41	1.27
1	1.0	0.123	0.610(-1)	0.288(-1)	0.132(-1)	0.562(-2)	0.187(-2)
10^{-1}		1.52	0.752	0.344	0.154	0.643(-1)	0.211(-1)
10^{-2}		18.2	10.3	5.19	2.40	1.02	0.408
10^{-3}		187.	109.	57.9	29.5	14.6	7.01

In this table the error $E(N, \varepsilon, \alpha)$ is defined by

$$E(N, \varepsilon, \alpha) = \max_{x \in \overline{D}_h} |e(x; N, \varepsilon, \alpha)|, \quad (6.13a)$$

$$e(x; N, \varepsilon, \alpha) = z(x) - u^*(x), \quad (6.13b)$$

where $u^*(x)$ is the piecewise interpolation of $z_m^{*256}(x)$, $m = m_{??} = 1$ (see Table 2), and $z(x) \equiv z^N(x)$ is the solution of (??),(??) with $h_1 = h_2 = N^{-1}$. Notice that $u^*(x)$ is an accurate approximation of $u(x)$.

$\varepsilon = 1.0, 10^{-1}, 10^{-2}, 10^{-3}$ and $\alpha = 0.0, 0.1, 0.5, 1.0$ have all a regular behaviour and decrease for increasing N . For a fixed value of α and N the error stabilises for decreasing ε : the errors for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ are practically the same. For $\varepsilon \leq 10^{-2}$ and a fixed value of N we find the largest error for $\alpha = 1.0$. In particular, for $\varepsilon \leq 10^{-2}$, $\alpha = 1.0$ and $N = 128$ the error is less than 6%. Also here, the numerical results illustrate the practical value of ε -convergent methods.

7. Conclusion

For the elliptic boundary value problem (??), where a small parameter multiplies the highest derivative, we have analysed different approaches for the construction of discrete methods. We present methods for which the accuracy of the discrete solution does not depend on the value of the small parameter, but only on the number of points

Table 2. Table of errors $E(N, \varepsilon, \alpha)$ for the special scheme

N		4	8	16	32	64	128
ε	α						
1	0.0	0.105	0.633(-1)	0.329(-1)	0.158(-1)	0.696(-2)	0.241(-2)
10^{-1}		0.262	0.144	0.655(-1)	0.291(-1)	0.122(-1)	0.403(-2)
10^{-2}		0.246	0.147	0.807(-1)	0.361(-1)	0.148(-1)	0.497(-2)
10^{-3}		0.246	0.147	0.807(-1)	0.361(-1)	0.148(-1)	0.497(-2)
1	0.1	0.691(-1)	0.381(-1)	0.170(-1)	0.744(-2)	0.314(-2)	0.104(-2)
10^{-1}		0.276	0.149	0.701(-1)	0.310(-1)	0.129(-1)	0.423(-2)
10^{-2}		0.246	0.170	0.887(-1)	0.461(-1)	0.240(-1)	0.924(-2)
10^{-3}		0.246	0.169	0.887(-1)	0.461(-1)	0.241(-1)	0.925(-2)
1	0.5	0.852(-1)	0.401(-1)	0.183(-1)	0.842(-2)	0.359(-2)	0.119(-2)
10^{-1}		0.611	0.473	0.254	0.121	0.526(-1)	0.176(-1)
10^{-2}		0.539	0.511	0.361	0.217	0.111	0.420(-1)
10^{-3}		0.535	0.511	0.361	0.217	0.111	0.420(-1)
1	1.0	0.123	0.610(-1)	0.288(-1)	0.132(-1)	0.562(-2)	0.187(-2)
10^{-1}		1.14	0.752	0.344	0.154	0.643(-1)	0.211(-1)
10^{-2}		0.977	0.889	0.554	0.301	0.144	0.521(-1)
10^{-3}		0.963	0.888	0.554	0.301	0.144	0.521(-1)

In this table the function $E(N, \varepsilon, \alpha)$ is defined by (??), but now $z(x) = z_m^{*N}(x)$ in (??) is the solution of (??),(??) with $m = m_{(??)} = 1$ and $N_1 = N_2 = N$

in the discretisation.

We show that in a natural class of finite difference schemes, for the problem considered, no ε -uniform methods exist on a uniform grid (Theorem ??). As a consequence, for the construction of ε -uniform methods the use of an adapted non-uniform mesh is necessary. With a special, adapted, non-uniform mesh and a simple classical difference scheme, we are able to construct an ε -uniform approximation

To illustrate the practical importance of our study, for a model problem we show by a numerical example that, on a uniform grid, the classical difference scheme is not ε -uniformly convergent. In our example, the error (with a Neumann boundary condition) is not less than 700% of the solution, for $N=128$ and $\varepsilon = 10^{-3}$. The same example shows that we might obtain an ε -uniformly convergent solution if we use the adapted mesh. Now the error is not larger than 6% of the solution, for any value of the parameter ε . Thus, the numerical example illustrates that the theoretical considerations have practical implications indeed.

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(To be continued)