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HIGH ACCURACY FOR MIXES FINITE ELEMENT METHODS IN RAVIART-THOMAS ELEMENT*

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Abstract

This paper deals with Raviart-Thomas element ($Q_{2,1} \times Q_{1,2} - Q_1$ element). Apart from its global superconvergence property of fourth order, we prove that a postprocessed extrapolation can globally increased the accuracy by fifth order.

1. Introduction

We consider the mixed methods of the Neumann boundary value problem

$$\begin{aligned} \mathbf{p} + \nabla u &= 0 \quad \text{in } \Omega, \\ \text{div}\mathbf{p} &= f \quad \text{in } \Omega, \\ \mathbf{p} \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain with boundaries parallel to axes, $\mathbf{n}$ is the outer unit normal to $\partial\Omega$. Denote

$$\mathbf{H}_0(\text{div}) = \{\mathbf{q} \in \mathbf{H}(\text{div}),\ \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

then we can write the weak formulation of (1) as follows: Find $(u, \mathbf{p}) \in L^2(\Omega) \times \mathbf{H}_0(\text{div})$ such that

$$(\mathbf{p}, \mathbf{q}) - (u, \text{div}\mathbf{q}) + (v, \text{div}\mathbf{p}) = (f, v), \quad \forall (v, \mathbf{q}) \in L^2(\Omega) \times \mathbf{H}_0(\text{div}). \tag{2}$$

Let $V_h \times \mathbf{P}_h \subset L^2(\Omega) \times \mathbf{H}_0(\text{div})$ be a pair of finite element spaces with respect to T_h, a uniform rectangular mesh with the size $2h$. Then the mixed finite element approximation for (2) seeks $(u_h, \mathbf{p}_h) \in V_h \times \mathbf{P}_h$ such that

$$(\mathbf{p}_h, \mathbf{q}) - (u_h, \text{div}\mathbf{q}) + (v, \text{div}\mathbf{p}_h) = (f, v), \quad \forall (v, \mathbf{q}) \in V_h \times \mathbf{P}_h. \tag{3}$$

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Here we choose $V_h \times P_h$ as one of RT elements, i.e. $Q_{2,1} \times Q_{1,2} - Q_1$ element[3], which satisfies the BB-condition and is described as

$$\begin{cases} \mathbf{P}_h = \{\mathbf{q} \in \mathbf{H}_0(\text{div}),\ \mathbf{q}|_e \in Q_{2,1}(e) \times Q_{1,2}(e), \quad \forall e \in T_h\}, \\ V_h = \{v \in L^2(\Omega),\ v|_e \in Q_1(e), \quad \forall e \in T_h\}, \end{cases} \tag{4}$$

where

$$Q_{m,n} = \text{span}\{x^i y^j,\ 0 \le i \le m,\ 0 \le j \le n\}; \qquad Q_{m,m} = Q_m.$$

Some superconvergence results for RT element have been derived by Nakata, Weiser, Wheeler, Douglas, Milner, Wang, Ewing and Lazarov([?]-[?], [?]). The asymptotic expansion was also obtained for the lowest order RT element or $Q_{1,0} \times Q_{0,1} - Q_0$ element by Wang([?]). The aim of this paper is to obtain the global superconvergence of $O(h^4)$ and the postprocessed extrapolation result of $O(h^5)$ for $Q_{2,1} \times Q_{1,2} - Q_1$ element by using integral identity, which was created by Lin *et al*([?],[?]).

2. Global Superconvergence

For $e \in T_h$, we assume that (x_e, y_e) is the center of gravity, s_1 and s_3 of the width $2k$ are the edges along y-direction, s_2 and s_4 of the width $2h$ are the edges along x-direction. Then we can define interpolation operators j_h and i_h by

$$\begin{cases} j_h\mathbf{p}|_e \in Q_{2,1}(e) \times Q_{1,2}(e), \\ \int_{s_i} (\mathbf{p} - j_h\mathbf{p})\varphi \mathbf{n} ds = 0 \quad \forall \varphi \in P_1(s_i) \quad i = 1,2,3,4, \\ \int_e (\mathbf{p} - j_h\mathbf{p})\mathbf{q} = 0 \quad \forall \mathbf{q} \in P_1(y) \times P_1(x), \end{cases} \tag{5}$$

$$\int_e (u - i_h u)v = 0 \quad \forall v \in Q_1(e). \tag{6}$$

We immediately find from integration by parts that

$$(v, \text{div}(\mathbf{p} - j_h\mathbf{p})) = 0 \quad \forall v \in V_h.$$

In fact, the projection j_h satisfying term above is Fortin's operator (see [?]) and in this paper it is locally defined. This definition can be also seen in [?] and [?]. i_h is the local L^2-projection operator. Since $\text{div}\mathbf{q} \in V_h$, we can see that

$$(u - i_h u, \text{div}\mathbf{q}) = 0, \quad \forall \mathbf{q} \in \mathbf{P}_h.$$

Lemma 1. *If* $\mathbf{p} \in [W^{5,r}(\Omega)]^2$, *then we have*

$$\begin{aligned} & (\mathbf{p}_h - j_h\mathbf{p}, \mathbf{q}) - (u_h - i_h u, div\mathbf{q}) + (div(\mathbf{p}_h - j_h\mathbf{p}), v) \\ = \ & \frac{2}{45} h^4 \int_\Omega (p_1)_{xxxx} q_1 + \frac{2}{45} k^4 \int_\Omega (p_2)_{yyyy} q_2 + h^5 r_h(\mathbf{p}, \mathbf{q}) \quad \forall (\mathbf{q}, v) \in \mathbf{P}_h \times V_h \end{aligned}$$

with

$$|r_h(\mathbf{p}, \mathbf{q})| \le c\|\mathbf{p}\|_{5,r}\|\mathbf{q}\|_{\text{div},r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1,\ 1 \le r, r' \le \infty.$$

where $\|\cdot\|_{\mathrm{div},r'} = \|\cdot\|_{0,r'} + \|\mathrm{div}\cdot\|_{0,r'}$. *In the following, we denote* $\|\cdot\|_m = \|\cdot\|_{m,2}$ *and* $\|\cdot\|_{\mathrm{div},r'} = \|\cdot\|_{H(\mathrm{div})}$ *for* $r' = 2$.

Remark 1: From the proof of Lemma 1 in Section 5, it is easy to see that, for $\mathbf{p} \in [W^{4,r}(\Omega)]^2$,

$$|(\mathbf{p} - j_h\mathbf{p}, \mathbf{q})| \leq ch^4\|\mathbf{p}\|_{4,r}\|\mathbf{q}\|_{0,r'}. \tag{7}$$

It follows from the stability result of Brezzi([**?**]) that the following Theorem is true.

Theorem 1. *If* $\mathbf{p} \in [H^4(\Omega)]^2$, *then*

$$\|\mathbf{p}_h - j_h\mathbf{p}\|_{H(div)} + \|u_h - i_h u\|_0 \leq ch^4\|\mathbf{p}\|_4.$$

Let us turn to L^∞ superconvergence. For this purpose we first introduce two pairs of regularized Green's functions at $z \in \Omega$ by

$$\begin{aligned} \mathbf{G}_1 + \nabla\lambda_1 &= 0 \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{G}_1 &= \delta_1^h \quad \text{in } \Omega, \\ \lambda_1 &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathbf{G}_2 + \nabla\lambda_2 &= \delta_2^h \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{G}_2 &= 0 \quad \text{in } \Omega, \\ \lambda_2 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{9}$$

where δ_1^h and δ_2^h are the regularized Dirac functions at $z \in \Omega$ satisfying

$$\begin{aligned} (v, \delta_1^h) &= v(z) \quad \forall v \in V_h, \\ (\mathbf{q}, \delta_2^h) &= \mathbf{q}(z) \quad \forall \mathbf{q} \in \mathbf{P}_h. \end{aligned}$$

Choosing right point z respectively can yield (see [**?**])

$$\|v\|_\infty \leq 2|(v, \delta_1^h)| \tag{10}$$

$$\|\mathbf{q}\|_\infty \leq 2|(\mathbf{q}, \delta_2^h)| \tag{11}$$

Wang proved in [**?**] that

$$\|\mathbf{G}_1^h\|_0 \leq c|\log h|^{\frac{1}{2}}, \tag{12}$$

$$\|\mathbf{G}_2^h\|_{0,1} \leq c|\log h|. \tag{13}$$

Theorem 2. *If* $\mathbf{p} \in [W^{4,\infty}(\Omega)]^2$, *then*

$$\|\mathbf{p}_h - j_h\mathbf{p}\|_{0,\infty} + |\log h|^{\frac{1}{2}}\|u_h - i_h u\|_{0,\infty} \leq ch^4|\log h|\|\mathbf{p}\|_{4,\infty}.$$

Proof. From (8) and (10) we have

$$\begin{aligned} \|u_h - i_h u\|_{0,\infty} &\leq 2|(u_h - i_h u, \mathrm{div}\mathbf{G}_1^h)| \\ &= 2|(u - i_h u, \mathrm{div}\mathbf{G}_1^h) - (\mathbf{p} - j_h\mathbf{p}, \mathbf{G}_1^h) - (\mathrm{div}(j_h\mathbf{p} - \mathbf{p}_h), \lambda_1^h)| \\ &= 2|(\mathbf{p} - j_h\mathbf{p}, \mathbf{G}_1^h)| \end{aligned}$$

which, combining with (7) and (12), yields

$$\|u_h - i_h u\|_{0,\infty} \le ch^4 |\log h|^{\frac{1}{2}} \|\mathbf{p}\|_4.$$

Similarly, we have

$$\begin{aligned}\|\mathbf{p}_h - j_h\mathbf{p}\|_{0,\infty} &\le 2|(\mathbf{p}_h - j_h\mathbf{p}, \mathbf{G}_2^h) + (\lambda_2^h, \mathrm{div}(\mathbf{p}_h - j_h\mathbf{p}))| \\ &\le 2|(\mathbf{p} - j_h\mathbf{p}, \mathbf{G}_2^h) - (u - i_h u, \mathrm{div}\mathbf{G}_2^h) - (i_h u - u_h, \mathrm{div}\mathbf{G}_2^h)| \\ &\le 2|(\mathbf{p} - j_h\mathbf{p}, \mathbf{G}_2^h)|\end{aligned}$$

which implies that

$$\|\mathbf{p}_h - j_h\mathbf{p}\|_{0,\infty} \le ch^4 |\log h| \|\mathbf{p}\|_{4,\infty}.$$

Theorem 2 is proved.

Assume that T_h has been obtained from T_{2h} by subdividing each element τ of T_{2h} into four congruent rectangles e_1, e_2, e_3, and e_4. Then we can define two postprocessing operators J_{2h} and I_{2h} by

$$\begin{cases} J_{2h}\mathbf{p}|_\tau \in P_3(\tau) \times P_3(\tau), \\ \int_{l_i} (\mathbf{p} - J_{2h}\mathbf{p})\varphi \mathbf{n} ds = 0 \quad \forall \varphi \in P_1(l_i), \quad i = 1, 2, \cdots, 6, \\ \int_{e_i} (\mathbf{p} - J_{2h}\mathbf{p}) = 0 \quad i = 1, 2, 3, 4, \end{cases}$$

$$\begin{cases} I_{2h}u|_\tau \in P_3(\tau), \\ \int_{e_i} (u - I_{2h}u)\varphi = 0 \quad \forall \varphi \in P_1(e_i), \quad i = 1, 2, 3, \\ \int_{e_4} (u - I_{2h}u) = 0, \end{cases}$$

where l_i $(i = 1, \cdots, 6)$ are four edges and two central lines of τ. It is easy to see that

$$\begin{cases} J_{2h}j_h = J_{2h}, \\ \|J_{2h}\mathbf{q}\|_{0,r} \le c\|\mathbf{q}\|_{0,r}, \quad \forall \mathbf{q} \in \mathbf{P}_h, \\ \|J_{2h}\mathbf{p} - \mathbf{p}\|_{0,r} \le ch^4 \|\mathbf{p}\|_{4,r}, \end{cases} \quad \begin{cases} I_{2h}i_h = I_{2h}, \\ \|I_{2h}v\|_{0,r} \le c\|u\|_{0,r}, \quad \forall v \in V_h, \\ \|I_{2h}u - u\|_{0,r} \le ch^4 \|u\|_{4,r}. \end{cases}$$

Therefore, under the assumption of Theorem 1, we have the global L^2 superconvergence

$$\begin{aligned}&\|J_{2h}\mathbf{p}_h - \mathbf{p}\|_0 + \|I_{2h}u_h - u\|_0 \\ &\le \|J_{2h}(\mathbf{p}_h - j_h\mathbf{p})\|_0 + \|J_{2h}\mathbf{p} - \mathbf{p}\|_0 + \|I_{2h}(u_h - i_h u)\|_0 + \|I_{2h}u - u\|_0 \\ &\le ch^4(\|\mathbf{p}\|_4 + \|u\|_4),\end{aligned}$$

and under the assumption of Theorem 2, we have the global L^∞ superconvergence

$$\|J_{2h}\mathbf{p}_h - \mathbf{p}\|_{0,\infty} + |\log h|^{\frac{1}{2}} \|I_{2h}u_h - u\|_{0,\infty} \le ch^4 |\log h| (\|\mathbf{p}\|_{4,\infty} + \|u\|_{4,\infty}).$$

Remark 2: From the proof of Lemma 1 in Section 4, it is not difficult to see that all superconvergence results above lost one half order for the Dirichilet boundary value problem.

3. Postprocessed Extrapolation

Let us first establish the asymptotic error expansion.

Theorem 3. *Under the assumption of Lemma* 1*, we have the error expansion*

$$\begin{aligned} \mathbf{p}_h - j_h\mathbf{p} &= h^4\mathbf{w}_1 + k^4\mathbf{w}_2 + h^5\rho_{h,p}, \\ u_h - i_h u &= h^4\chi_1 + k^4\chi_2 + h^5\rho_{h,u}, \end{aligned}$$

with

$$\|\rho_{h,p}\|_0 + \|\rho_{h,u}\|_0 \le c.$$

Proof. Considering auxaliary problems

$$\begin{aligned} -\nabla\chi_1 &= \mathbf{w}_1 && \text{in } \Omega, \\ \mathrm{div}\mathbf{w}_1 &= \frac{2}{45}(p_1)_{xxxx} && \text{in } \Omega, \\ \chi_1 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\nabla\chi_2 &= \mathbf{w}_2 && \text{in } \Omega, \\ \mathrm{div}\mathbf{w}_2 &= \frac{2}{45}(p_2)_{yyyy} && \text{in } \Omega, \\ \chi_2 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

from Lemma 1, we have

$$\begin{aligned} (\mathbf{p}_h &- j_h\mathbf{p} - h^4\mathbf{w}_1^h + k^4\mathbf{w}_2^h, \mathbf{q}) + (u_h - i_h u - h^4\chi_1^h - k^4\chi_2^h, \mathrm{div}\mathbf{q}) \\ &+ (\mathrm{div}(\mathbf{p}_h - j_h\mathbf{p} - h^4\mathbf{w}_1^h - k^4\mathbf{w}_2^h), v) \\ &= h^5 r_h(\mathbf{p}, \mathbf{q}), \end{aligned}$$

which, in view of stability result in [**?**], yields

$$\|\mathbf{p}_h - j_h\mathbf{p} - h^4\mathbf{w}_1^h - k^4\mathbf{w}_2^h\|_0 + \|u_h - i_h u - h^4\chi_1^h - k^4\chi_2^h\|_0 \le ch^5\|\mathbf{p}\|_5.$$

Using regularity property and optimal error estimate we obtain

$$\|\mathbf{p}_h - j_h\mathbf{p} - h^4\mathbf{w}_1 - k^4\mathbf{w}_2\|_0 + \|u_h - i_h u - h^4\chi_1 - k^4\chi_2\|_0 \le ch^5\|\mathbf{p}\|_5.$$

Theorem 3 is proved.

By the extrapolation we obtain

$$\begin{aligned} &\Big\|\frac{1}{15}[(16\mathbf{p}_{\frac{h}{2}} - \mathbf{p}_h) - (16j_{\frac{h}{2}}\mathbf{p} - j_h\mathbf{p})]\Big\|_0 \\ &+\Big\|\frac{1}{15}[(16u_{\frac{h}{2}} - u_h) - (16i_{\frac{h}{2}}u - i_h u)]\Big\|_0 \le ch^5\|\mathbf{p}\|_5. \end{aligned}$$

For getting the global high accuracy we define other pair of operators by

$$\begin{cases} J'_{2h}\mathbf{p}|_\tau \in P_4(\tau)\times P_4(\tau), \\ \int_{l_i}(\mathbf{p}-J'_{2h}\mathbf{p})\varphi\mathbf{n}ds = 0 \quad \forall\varphi\in P_1(l'_i), \quad i=1,2,\cdots,12, \\ \int_{e_i}(\mathbf{p}-J'_{2h}\mathbf{p}) = 0 \quad i=1,2,3, \end{cases}$$

$$\begin{cases} I'_{2h}u|_\tau \in P_4(\tau) \\ \int_{e_i}(u-I'_{2h}u)\varphi = 0 \quad \forall\varphi\in Q_1(e_i), \quad i=1,2,3, \\ \int_{e_4}(u-I'_{2h}u)\varphi = 0 \quad \forall\varphi\in P_1(e_4), \end{cases}$$

where l'_i $(i=1,\cdots,12)$ are all edges of $e_1,\ e_2,\ e_3$ and e_4. We can check that

$$\begin{cases} J'_{2h}j_h = J'_{2h}, \quad J'_{2h}j_{\frac{h}{2}} = J'_{2h}, \\ \|J'_{2h}\mathbf{q}\|_{0,r}\le c\|\mathbf{q}\|_{0,r}, \quad \forall\mathbf{q}\in\mathbf{P}_h, \\ \|J'_{2h}\mathbf{p}-\mathbf{p}\|_{0,r}\le ch^5\|\mathbf{p}\|_{5,r}, \end{cases} \quad \begin{cases} I'_{2h}i_h = I'_{2h}, \quad I'_{2h}i_{\frac{h}{2}} = I'_{2h}, \\ \|I'_{2h}v\|_{0,r}\le c\|v\|_{0,r}, \quad \forall v\in V_h, \\ \|I'_{2h}u-u\|_{0,r}\le ch^5\|u\|_{5,r}. \end{cases}$$

Hence we obtain the postprocessed extrapolation results

$$\left\|\frac{1}{15}J'_{2h}(16\mathbf{p}_{\frac{h}{2}}-\mathbf{p}_h)-\mathbf{p}\right\|_0 + \left\|\frac{1}{15}I'_{2h}(16u_{\frac{h}{2}}-u_h)-u\right\|_0 \le ch^5\|\mathbf{p}\|_5.$$

Using pointwise extrapolation technique (cf [?]), we have

$$\left\|\frac{1}{15}J'_{2h}(16\mathbf{p}_{\frac{h}{2}}-\mathbf{p}_h)-\mathbf{p}\right\|_{0,\infty} \le c(c(r)+|\log h|)h^{5-\frac{2}{r}}\|\mathbf{p}\|_{5,r}, \quad 2\le r<\infty,$$

and

$$\left\|\frac{1}{15}I'_{2h}(16u_{\frac{h}{2}}-u_h)-u\right\|_{0,\infty} \le ch^5(\|\mathbf{p}\|_{5,\infty}+\|u\|_{5,\infty}).$$

4. Proof of Lemma 1

Since

$$\begin{aligned} & (\mathbf{p}_h-j_h\mathbf{p},\mathbf{q})-(u_h-i_hu,\mathrm{div}\mathbf{q})+(\mathrm{div}(\mathbf{p}_h-j_h\mathbf{p}),v) \\ = \ & (\mathbf{p}-j_h\mathbf{p},\mathbf{q})-(u-i_hu,\mathrm{div}\mathbf{q})+(\mathrm{div}(\mathbf{p}-j_h\mathbf{p}),v) \\ = \ & (\mathbf{p}-j_h\mathbf{p},\mathbf{q}), \end{aligned}$$

we only need to prove

$$(p_1-j_hp_1,q_1) = \frac{2}{45}h^4\int_\Omega (p_1)_{xxxx}q_1 + h^5r_h(\mathbf{p},\mathbf{q}).$$

In the following, we assume

$$|r_{h,e}(\mathbf{p},\mathbf{q})| \le c\|\mathbf{p}\|_{5,r,e}\|\mathbf{q}\|_{\mathrm{div},r',e)}.$$

Define the error function $E = \frac{1}{2}[(x-x_e)^2 - h^2]$, then

$$x - x_e = \frac{1}{6}(E^2)_{xxx}, \quad (x-x_e)^2 = \frac{1}{45}(E^3)_{xxxx} + \frac{1}{5}h^2,$$

$$E^2 = \frac{1}{420}(E^4)_{xxxx} - \frac{2}{21}h^2(E^2)_{xx} + \frac{2}{15}h^4.$$

Hence the Taylor expansion shows that

$$\begin{aligned}\int_e (p_1 - j_h p_1)q_1 &= \int_e (p - j_h p)\Big\{q_1(x_e, y) + \frac{1}{3}(E^2)_{xxx}(q_1)_x(x_e, y) \\ &\quad + \Big[\frac{1}{90}(E^3)_{xxxx} + \frac{1}{10}h^2\Big](q_1)_{xx}\Big\} \\ &\equiv \text{I+II+III}. \end{aligned} \tag{14}$$

Using definition (5), integration by parts and the property of $E = 0$ on s_1 and s_3, we have

$$\begin{aligned} \text{I} &= 0, \\ \text{III} &= \frac{1}{90}\int_e (p_1)_{xxxx}E^3(q_1)_{xx} \\ &= \frac{1}{90}\int_e (p_1)_{xxxx}E^3[(\text{div}\mathbf{q})_x - (q_2)_{xy}] \\ &= \frac{1}{90}\int_e (p_1)_{xxxx}E^3(\text{div}\mathbf{q})_x - \frac{1}{90}(\int_{s_4} - \int_{s_2})(p_1)_{xxxx}E^3(q_2)_x dx \\ &\quad + \frac{1}{90}\int_e (p_1)_{xxxxy}E^3(q_2)_x \\ &= h^5 r_{h,e}(\mathbf{p}, \mathbf{q}) - \frac{1}{90}(\int_{s_4} - \int_{s_2})(p_1)_{xxxx}E^3(q_2)_x dx, \\ \text{II} &= -\frac{1}{3}\int_e (p_1)_{xxx}E^2(q_1)_x(x_e, y) \\ &= -\frac{1}{3}\int_e (p_1)_{xxx}[\frac{1}{420}(E^4)_{xxxx} - \frac{2}{21}h^2(E^2)_{xx} + \frac{2}{15}h^4][(q_1)_x - E_x(q_1)_{xx}] \\ &= -\frac{1}{3}\int_e (p_1)_{xxxxx}[\frac{1}{420}(E^4)_{xx} - \frac{2}{21}h^2E^2][(q_1)_x - E_x(q_1)_{xx}] \\ &\quad - \frac{2}{45}h^4(\int_{s_3} - \int_{s_1})(p_1)_{xxx}q_1 dy + \frac{2}{45}h^4\int_e (p_1)_{xxxx}q_1 \\ &\quad - \frac{2}{45}h^4\int_e (p_1)_{xxxx}E(q_1)_{xx} \\ &\equiv h^5 r_{h,e}(\mathbf{p}, \mathbf{q}) - \frac{2}{45}h^4(\int_{s_3} - \int_{s_1})(p_1)_{xxx}q_1 dy + \frac{2}{45}h^4\int_e (p_1)_{xxxx}q_1 \\ &\quad + \int_e g(x)(p_1)_{xxxx}(q_1)_{xx}, \end{aligned} \tag{15}$$

where

$$g(x) = -\frac{2}{45}h^4 E,$$

$$
\begin{aligned}
&\int_e g(x)(p_1)_{xxxx}(q_1)_{xx}\\
=\ &\int_e g(x)(p_1)_{xxxx}[(\mathrm{div}\mathbf{q})_x-(q_2)_{xy}]\\
=\ &\int_e g(x)(p_1)_{xxxx}(\mathrm{div}\mathbf{q})_x+(\int_{s_4}-\int_{s_2})g(x)(p_1)_{xxxx}(q_2)_x dx+\int_e g(x)(p_1)_{xxxxy}(q_2)_x.
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\mathrm{II}\ =\ &\frac{2}{45}h^4\int_e(p_1)_{xxxx}q_1+(\int_{s_4}-\int_{s_2})g(x)(p_1)_{xxxx}(q_2)_x dx+h^5 r_{h,e}(\mathbf{p},\mathbf{q})\\
&-\frac{2}{45}h^4(\int_{s_3}-\int_{s_1})(p_1)_{xxx}q_1 dy.
\end{aligned}
$$

The fact the line integrals above terms will disappear during the summation of each element $e\in T_h$ can completes the proof of Lemma 1.

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