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# MODIFIED DISCREPANCY PRINCIPLES WITH PERTURBED OPERATORS AND NOISY DATA*1)

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**Abstract**

We investigate a class of a posterior parameter choice for iterated Tikhonov regularization with perturbed operators and noisy data by using modified Arcangeli's method. The rate of convergence of regularization approximation is achieved.

## 1. Introduction

Let $X, Y$ be real Hilbert spaces, $T : X \to Y$ a bounded linear operator with nonclosed range $R(T), y \in D(T^+) = R(T)\dot{+}R(T)^\perp$, where $T^+$ is the Moore-Penrose inverse of $T$[1]. For each $\delta > 0$, let $y_\delta \in Y$ be such that

$$\|y - y_\delta\| \le \delta. \tag{1}$$

As we know, the problem of solving the operator equation of the first kind

$$Tx = y \tag{2}$$

is, in generality, ill-posed[2]. Also we can not ensure that $T^+y_\delta$ is a reasonable approximation of $T^+y$ since $T^+$ is a unbounded operator. In practice, one tries to construct a stable approximate solution to the equation (2) by regularization methods. A well-known regularization method for approximating $T^+y$ is the Tikhonov regularization method[3]. For each $\delta > 0$ and $\alpha > 0$, we denote by $x_{\alpha,\delta}$ the Tikhonov regularization approximation for $T^+y$. A crucial problem is the choice of regularization parameter $\alpha$ in dependence of the noisy level $\delta$ leading to optimal convergence rates.

The earliest methods of this type are Morozov's discrepancy principles, where $\alpha$ is chosen such that

$$\|Tx_{\alpha,\delta} - y_\delta\|^2 = \delta^2,$$

and Arcangeli's method, where $\alpha$ is chosen as the root of[3]

$$\|Tx_{\alpha,\delta} - y_\delta\|^2 = \frac{\delta^2}{\alpha} .$$

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However, it is shown that neither Morozov nor Arcangeli's method yields the optimal convergence rates[4,5]. In order to improve the convergence rates, J.T. King probes the iterated Tikhonov regularization[6]. In [7], H.W. Engl proposes the Modified Arcangeli's method of choosing the regularization parameter that leads to higher rates of convergence.

All the methods and results above-mentioned are only applicable if $T$ is exactly known. However, in practice, not only the right-hand member of equations but operators is approximately given. Z.Y. Hou and H.N. Li have shown two ways of choosing regularization parameter for Tikhonov regularization with perturbed operators and noisy data, and obtained the results concerning the rates of convergence[8,9]. For more information in this area, we refer the reader to [10,11].

The aim of this paper is to provide a posteriori parameter choice and higher asymptotic convergence rate for iterated Tikhonov regularization with not only noisy data but perturbed operators by using modified Arcangeli's method. The result described in the paper has much more practical value.

## 2. Modified Arcangeli's Principles

For real number $h > 0$, let $T_h : X \to Y$ be a bounded linear operator such that

$$\|T - T_h\| \leq h. \tag{3}$$

To exclude trivialitive, from now on, we assume that

$$y \in R(T), \qquad T^*y \neq 0, \qquad T^*y_\delta \neq 0, \tag{4}$$

where $T^*$ is the adjoint of $T$. For all $j \in N$, let $x^{(j)}_{\alpha,\delta,h}$ be the result of iterated Tikhonov regularization of order $j$ , i.e,

$$x^{(0)}_{\alpha,\delta,h} = 0 \ , \qquad\qquad x^{(j)}_{\alpha,\delta,h} = (T_h^*T_h + \alpha I)^{-1}(T_h^*y_\delta + \alpha x^{(j-1)}_{\alpha,\delta,h}) \ , \tag{5}$$

where $N$ denotes the set of natural number, $\alpha$ is a positive and real number and $I$ is the identity operator. It follows by induction that for any $j \in N$,

$$x^{(j)}_{\alpha,\delta,h} = \sum_{i=1}^{j} \alpha^{i-1}(T_h^*T_h + \alpha I)^{-i}T_h^*y_\delta. \tag{6}$$

For simplicity of notation we replace $T_h^*T_h, T_hT_h^*, T^*T$ and $TT^*$ by $\tilde{T}_h, \hat{T}_h, \tilde{T}$ and $\hat{T}$, respectively, below. It is easy to see that the equality

$$T^*g(\hat{T}) = g(\tilde{T})T^*$$

holds if the function $g(t)$ is continuous on $[0, +\infty)$.

**Lemma 1.** *Let*

$$\rho_j(\alpha) = \left\|\tilde{T}_h x^{(j)}_{\alpha,\delta,h} - T_h^*y_\delta\right\|^2 \tag{7}$$

(1) *For each* $j \in N$, $\lim_{\alpha\to 0}\alpha^q\rho_j(\alpha) = 0$ *and* $\lim_{\alpha\to+\infty}\alpha^q\rho_j(\alpha) = +\infty$ *hold. Consequently,* $\alpha^q\rho_j(\alpha) :\ [0,\ +\infty) \to R$ *is continuous and strictly increasing with respected to* $\alpha$.

(2) *For all* $j \in N, j \geq 2$ *and* $\alpha > 0, \rho_j(\alpha) \leq \rho_{j-1}(\alpha)$ *holds.*

*Proof.* Let $\{\tilde{E}_\lambda\}$ be the spectral family generated by $\tilde{T}_h$. It follows from (6) that for any $j \in N$

$$\tilde{T}_h x^{(j)}_{\alpha,\delta,h} - T^*_h y_\delta = -\alpha^j(\tilde{T}_h + \alpha I)^{-j} T^*_h y_\delta\ , \tag{8}$$

thus,

$$\rho_j(\alpha) = \int_0^\infty \left(\frac{\alpha}{\alpha+\lambda}\right)^{2j} d\left\|\tilde{E}_\lambda T^*_h y_\delta\right\|^2$$

holds.

(1) It is trivial that $\lim_{\alpha\to 0}\alpha^q\rho_j(\alpha) = 0$. By the Dominated Theorem and (8), it follows that for all $j \in N$, $\lim_{\alpha\to+\infty}\rho_j(\alpha) = \|T^*_h y_\delta\|^2$ . Thus $\lim_{\alpha\to+\infty}\alpha^q\rho_j(\alpha) = +\infty$ holds. Since the integrand in the representation of $\rho_j(\alpha)$ is continuous and strictly increasing in $\alpha$, so does $\alpha^q\rho_j(\alpha)$.

(2) It also follows immediately from (6) that for $j \geq 2$,

$$\tilde{T}_h x^{(j)}_{\alpha,\delta,h} - T^*_h y_\delta = \alpha(\tilde{T}_h + \alpha I)^{-1}(\tilde{T}_h x^{(j-1)}_{\alpha,\delta,h} - T^*_h y_\delta).$$

Together with the following inequality

$$\left\|\alpha(\tilde{T}_h + \alpha I)^{-1}\right\| \leq 1,$$

this implies that $\rho_j(\alpha) \leq \rho_{j-1}(\alpha)$ holds.

According to Lemma 1, for each $p, q, s > 0$, there exists a unique positive root of the equation

$$\alpha^q\rho_j(\alpha) = \delta^p + h^s \tag{9}$$

which we will denote by $\alpha_j(\delta, h)$ later. Let

$$\gamma = \sqrt{\delta^2 + h^2}, \qquad m = \min\{p, s\}.$$

We have that $\delta^p + h^s = O(\gamma^m)$ holds as $\gamma \to 0$, and that $\gamma$ tends to zero if and only if $\delta, h$ tend to zero independently on each other.

**Lemma 2.** *For each* $j \in N, \lim_{\gamma\to 0}\alpha_j(\delta, h) = 0$ *while for each* $j \geq 2$ *and* $\delta > 0$, $\alpha_j(\delta, h) \geq \alpha_{j-1}(\delta, h)$ *holds.*

*Proof.* Assume that there is a sequence $\gamma_n = \sqrt{\delta_n^2 + h_n^2} \to 0$ such that $\alpha_j(\delta_n, h_n) \to +\infty$ as $n \to +\infty$. Let $\alpha_{j,n} = \alpha_j(\delta_n, h_n)$. Without loss of generality, we assume that $\alpha_{j,n} > \|\tilde{T}\|$. Since the relation

$$\left\|(\tilde{T}_{h_n} + \alpha_{j,n} I)^{-1}\right\| \leq \frac{1}{\alpha_{j,n} - \left\|\tilde{T}_{h_n}\right\|}$$

holds, we have

$$\left\| x^{(j)}_{\alpha_{j,n},\delta_n,h_n} \right\| \le \frac{1}{\alpha_{j,n} - \left\| \tilde{T}_{h_n} \right\|} \left[ \| T^*_{h_n} y_{\delta_n} \| + \alpha_{j,n} \left\| x^{(j-1)}_{\alpha_{j,n},\delta_n,h_n} \right\| \right].$$

This implies that $\lim_{n\to+\infty} x^{(j)}_{\alpha_{j,n},\delta_n,h_n} = 0$. Hence it follows from (3), (4) and the continuity of $T$ and $T_h$ that

$$\lim_{n\to\infty} \rho_j(\alpha_{j,n}) = \|T^* y\|^2 > 0\,,$$

which contradicts that, by the equation (9) and the assumption at the beginning of this proof, we have

$$\lim_{n\to\infty} \rho_j(\alpha_{j,n}) = \lim_{n\to\infty} (\delta_n^p + h_n^s)\alpha_{j,n}^{-q} = 0.$$

Thus,

$$\lim_{\gamma\to 0} \sup \alpha_j(\delta, h) < +\infty\,.$$

Now assume that there is a sequence $\gamma_n \to 0$ with $\alpha_{j,n} = \alpha_j(\delta_n, h_n) \to C > 0$ as $n \to \infty$. On the one hand, we have

$$\lim_{n\to\infty} \rho_j(\alpha_{j,n}) = \lim_{n\to\infty} (\delta_n^p + h_n^s)\alpha_{j,n}^{-q} = 0.$$

On the other hand, we have

$$\lim_{n\to\infty} \rho_j(\alpha_{j,n}) = \left\| C^j (\tilde{T} + CI)^{-j} T^* y \right\|^2$$

because $(\tilde{T} + \alpha_{j,n} I)^{-1}$ converges (in the operator norm ) to $(\tilde{T} + CI)^{-1}$ and (8) holds. Therefore

$$\left\| C^j (\tilde{T} + CI)^{-j} T^* y \right\|^2 = 0,$$

which contradicts (4). Hence we have that

$$\lim_{n\to\infty} \alpha_j(\delta_n, h_n) = 0$$

holds if there exists $\gamma_n \to 0$ (as $n \to \infty$) such that the limit $\lim_{n\to\infty} \alpha_j(\delta_n, h_n)$ exists.

Finally, assume that $\lim_{\gamma\to 0} \alpha_j(\delta, h) \neq 0$. By the definition of limits and the relation $\lim_{\gamma\to 0} \sup \alpha_j(\delta, h) < +\infty$ above-mentioned, we know that there exist $\varepsilon_0 > 0$ and $\gamma_n = \sqrt{\delta_n^2 + h_n^2} \to 0$ (as $n \to \infty$) such that both

$$\alpha_j(\delta_n, h_n) > \varepsilon_0$$

holds and the limit

$$\lim_{n\to\infty} \alpha_j(\delta_n, h_n)$$

exists. In view of the discussion of the second step above, we have

$$\lim_{n\to\infty} \alpha_j(\delta_n, h_n) = 0,$$

which contradicts

$$\alpha_j(\delta_n, h_n) > \varepsilon_0 > 0.$$

Therefore $\lim_{\gamma\to 0} \alpha_j(\delta, h) = 0$.

Now we turn to the proof of the remaining statement of Lemma 2. It follows from Lemma 1 and the definition of $\alpha_j(\delta, h)$, respectively, that for $j \geq 2$,

$$\alpha_{j-1}^q \rho_j(\alpha_{j-1}) \leq \alpha_{j-1}^q \rho_{j-1}(\alpha_{j-1})$$

holds and that for $j \geq 2$,

$$\alpha_j^q \rho_j(\alpha_j) = \delta^p + h^s = \alpha_{j-1}^q \rho_{j-1}(\alpha_{j-1})$$

holds. Together with the monotonicity of the function $\alpha^q \rho_j(\alpha)$ , this implies that

$$\alpha_j(\delta, h) \geq \alpha_{j-1}(\delta, h)$$

holds for $j \geq 2$.

**Lemma 3.** *Let $p, q$ and $s$ be real and positive, $\alpha_j = \alpha_j(\delta, h)$ the root of (9). For each $j \in N$, if $q > \max\{m, 2\}j - 2$, then $\lim_{\gamma\to 0} \gamma\alpha_j^{-j} = 0$.*

*Proof.* Let $\hat{E}_\lambda$ denote the spectral family generated by $\hat{T}$, $\langle\cdot,\cdot\rangle$ the inner product operation in each of the spaces $X$ and $Y$, it follows immediately that for positive $\alpha$ and any $y \in Y$,

$$\begin{aligned}
\left\|(\tilde{T}_h + \alpha I)^{-1} T_h^* y\right\|^2 &= \left\langle T_h^*(\hat{T}_h + \alpha I)^{-1} y, T_h^*(\hat{T}_h + \alpha I)^{-1} y\right\rangle \\
&= \left\langle (\hat{T}_h)^{\frac{1}{2}}(\hat{T}_h + \alpha I)^{-1} y, (\hat{T}_h)^{\frac{1}{2}}(\hat{T}_h + \alpha I)^{-1} y\right\rangle \\
&= \int_0^{+\infty} \frac{\lambda}{(\lambda+\alpha)^2} d\left\|\hat{E}_\lambda y\right\|^2 \\
&\leq \frac{1}{\alpha}\|y\|^2
\end{aligned}$$

and hence

$$\left\|(\tilde{T}_h + \alpha I)^{-1} T_h^*\right\| \leq \alpha^{-\frac{1}{2}}. \tag{10}$$

By the way, we have that

$$\left\|(\tilde{T}_h + \alpha I)^{-1} T_h\right\| \leq \alpha^{-\frac{1}{2}}, \qquad \left\|(\tilde{T}_h + \alpha I)^{-1} \tilde{T}_h\right\| \leq 1. \tag{11}$$

For each given $y \in D(T^+)$ satisfying (4), $y = TT^+y$ holds. Since

$$\sqrt{\rho_1(\alpha)} = \left\|\alpha(\tilde{T}_h + \alpha I)^{-1} T_h^* y_\delta\right\|$$

holds, it follows from (10) that

$$\sqrt{\rho_1(\alpha)} \leq \sqrt{\alpha}\delta + (\sqrt{\alpha}h + \alpha)\left\|T^+ y\right\|$$

and hence there is a positive constant $A$ independent of $\alpha, \delta$, and $h$ such that

$$\sqrt{\rho_1(\alpha)} \leq A(\gamma\sqrt{\alpha} + \alpha), \tag{12}$$

thus,

$$(\delta^p + h^s)^{\frac{1}{2}}\, \alpha_1^{-\frac{q}{2}} \leq A(\gamma\sqrt{\alpha_1} + \alpha_1) \leq A(\gamma + \alpha_1)$$

where $\alpha_1 = \alpha_1(\delta, h)$ is the solution of the equation (9). Therefore there exists a positive constant, which we denote still by $A$ without loss of the generality, such that

$$\gamma\alpha_1^{\frac{-q}{m}} \leq A(\gamma + \alpha_1)^{\frac{2}{m}},$$

provided $\gamma$ is sufficiently small. This implies that

$$\gamma\alpha_1^{-\frac{1}{m}(q-2j+2)} \leq A(\gamma\alpha_1^{j-1} + \alpha_1^{j})^{\frac{2}{m}}. \tag{13}$$

If $\frac{1}{m}(q-2j+2) \geq j$, it follows from (13) and Lemma 2 that $\lim_{\gamma\to 0} \gamma\alpha_1^{-j} = 0$. Now let $(q-2j+2)\frac{1}{m} < j$. Note that the assumptions of Lemma 3 implies that

$$q - 2j + 2 > 0 \tag{14}$$

and that for $m > 2$,

$$\frac{1}{m-2}(q-2j+2) > j. \tag{15}$$

For all $n \in N$, let

$$w_0 = 0, \qquad w_n = \left(\frac{q-2j+2}{m}\right)\sum_{i=0}^{n-1}\left(\frac{2}{m}\right)^i.$$

Together with (14), we have that with $w_1 < j$, the sequence $w_n$ is non-negtive and strictly increasing. Also, we have by induction that

$$\gamma\alpha_1^{-w_{n+1}} \leq A\left(\gamma\alpha_1^{j-1-w_n} + \alpha_1^{j-w_n}\right)^{\frac{2}{m}}$$

holds and that if $w_n < j$,

$$\lim_{\gamma\to 0} \gamma\alpha_1^{-w_{n+1}} = 0 \tag{16}$$

holds. It follows from (15) that there exists a unique $n \in N$ such that

$$w_n < j \leq w_{n+1}$$

holds. Together with (16), this implies that $\lim_{\gamma\to 0} \gamma\alpha_1^{-j} = 0$. Thus, the assertion of the Lemma 3 follows from Lemma 2.

**Lemma 4.** *For each $i \in N$ and $\alpha, h \in (0,\ \|T\|)$, there exists a constant $C_i$, which is independent of $\alpha, h$, such that*

$$\left\|(T+\alpha I)^i - (T_h + \alpha I)^i\right\| \leq C_i h$$

*holds.*

*Proof.* This can be proved by induction.

**Remark 1.** If we replace $T$ and $T_h$ by $\hat{T}$ (or $\tilde{T}$) and $\hat{T}_h$ (or $\tilde{T}_h$), respectively, the assertion of Lemma 4 is valid.

**Lemma 5.** *For each $j \in N$ and any $\alpha, \delta, h \in (0, \|T\|)$,*

$$\left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\| \leq A\gamma\alpha^{-j}$$

*holds, where $A$, a constant, is independent of $\alpha, \delta, h$ and $x_{\alpha,\delta}^{(j)}$ is the result of iterated Tikhonov regularization of order $j$ as described by* (5)*, but with an exactly known operator $T$. Moreover, if $T^+y \in R(\tilde{T}^j)$, we have that*

$$\left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\| \leq A\gamma\alpha^{-\frac{1}{2}}$$

*holds.*

*Proof.* By the definition of $x_{\alpha,\delta,h}^{(j)}$ and $x_{\alpha,\delta}^{(j)}$, we have

$$\begin{aligned} x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} = &\sum_{i=1}^{j} \alpha^{i-1} \left[ \left( \tilde{T}_h + \alpha I \right)^{-i} T_h^* - \left( \tilde{T} + \alpha I \right)^{-i} T^* \right] (y_\delta - y) \\ &+ \sum_{i=1}^{j} \alpha^{i-1} T_h^* \left[ \left( \hat{T}_h + \alpha I \right)^{-i} - \left( \hat{T} + \alpha I \right)^{-i} \right] y \\ &+ \sum_{i=1}^{j} \alpha^{i-1} \left( T_h^* - T^* \right) \left[ \left( \hat{T} + \alpha I \right)^{-i} y \right] . \end{aligned}$$

For the sake of convenience, now let

$$G_{1,i} = \left\| \alpha^{i-1} T_h^* (\hat{T}_h + \alpha I)^{-i} \right\|, \ G_{2,i} = \left\| (\hat{T} + \alpha I)^i - (\hat{T}_h + \alpha I)^i \right\|,$$

$$G_{3,i} = \left\| (\hat{T} + \alpha I)^{-i} T \right\| \left\| T^+ y \right\| .$$

In view of (10) and Remark 1, it follows that for each $j \in N$,

$$\left\| \sum_{i=1}^{j} \alpha^{i-1} \left[ \left( \tilde{T}_h + \alpha I \right)^{-i} T_h^* - \left( \tilde{T} + \alpha I \right)^{-i} T^* \right] (y_\delta - y) \right\| \leq A\gamma\alpha^{-\frac{1}{2}} ,$$

$$\begin{aligned} \left\| \sum_{i=1}^{j} \alpha^{i-1} T_h^* \left[ \left( \hat{T}_h + \alpha I \right)^{-i} - \left( \hat{T} + \alpha I \right)^{-i} \right] y \right\| &\leq \sum_{i=1}^{j} \prod_{k=1}^{3} G_{k,i} \\ &\leq A\gamma\alpha^{-\frac{1}{2}} \sum_{i=1}^{j} \left\| (\hat{T} + \alpha I)^{-i} T \right\| \left\| T^+ y \right\| \\ &\leq A\gamma\alpha^{-\frac{1}{2}} \left\| T^+ y \right\| \sum_{i=1}^{j} \alpha^{-i+\frac{1}{2}} \leq A\gamma\alpha^{-j} \end{aligned}$$

and

$$\left\| \sum_{i=1}^{j} \alpha^{i-1} \left(T_h^* - T^*\right) \left[ \left(\hat{T} + \alpha I\right)^{-i} y \right] \right\| \leq j \, \|T^+ y\| \, h\alpha^{-\frac{1}{2}} \leq A\gamma\alpha^{-\frac{1}{2}}$$

hold where $A$ is a constant. Therefore, we have

$$\left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\| \leq A\gamma\alpha^{-j} \, .$$

If $T^+ y \in R(\tilde{T}^j)$ , let $T^+ y = \tilde{T}^j w$ . Noting

$$\left\| (\hat{T} + \alpha I)^{-i} \hat{T}^i T x \right\|^2 = \int_0^{+\infty} \left[ \frac{\lambda}{\lambda + \alpha} \right]^{2i} d \left\| \hat{E}_\lambda T x \right\|^2 \leq \|T\|^2 \, \|x\|^2 \, ,$$

we re-estimate $\left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\|$ ( also in view of (10) and Remark 1 ) as follows

$$\left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\| \leq 2A\gamma\alpha^{-\frac{1}{2}} + \alpha^{-\frac{1}{2}} h \left( \sum_{i=1}^{j} C_i \right) \|T\| \max_{1 \leq i \leq j} \left\| \tilde{T}^{j-i} w \right\| \, .$$

This implies the assertion of Lemma 5. Therefore Lemma 5 is proved.

We now establish the regularity of the modified Arcangeli' s method, which is the first crucial result in this paper.

**Theorem 1.** *For each* $\delta > 0\,, h > 0$ *and* $y_\delta \in Y$ *fulfilling* (1), *let* $x_{\alpha,\delta,h}^{(j)}$ *be the result of the iterated Tikhonov regularization of order* $j \geq 1$ *as described by* (5), *where* $\alpha_j = \alpha_j(\delta, h)$ *is the unique solution of* (9) *for some* $p, q, s > 0$. *Assume that* (3) *and* (4) *hold. If* $q > \max\{m, 2\}j - 2$ , *then*

$$\lim_{\gamma \to 0} x_{\alpha_j,\delta,h}^{(j)} = T^+ y,$$

*where* $m = \min\{p, s\}$ *and* $\gamma = \sqrt{\delta^2 + h^2}$.

*Proof.* For each $j \in N$ , let $x_\alpha^{(j)}$ be the result of the iterated Tikhonov regularization of order $j \geq 1$ as described by (5), but with the operator $T$ and the exactly known data $y$ , i.e,

$$x_\alpha^{(j)} = \sum_{i=1}^{j} \alpha^{i-1} \left( \tilde{T} + \alpha I \right)^{-i} T^* y \, .$$

Similarly to the calculations that lead to the results of Lemma 5 we can conclude that

$$\left\| x_{\alpha,\delta}^{(j)} - x_\alpha^{(j)} \right\| \leq B\delta\alpha^{-\frac{1}{2}}$$

with a suitable costant $B$. Together with (6) and Lemma 5 it follows that

$$\left\| x_{\alpha,\delta,h}^{(j)} - T^+ y \right\| \leq \left\| x_{\alpha,\delta,h}^{(j)} - x_{\alpha,\delta}^{(j)} \right\| + \left\| x_{\alpha,\delta}^{(j)} - x_\alpha^{(j)} \right\| + \left\| x_\alpha^{(j)} - T^+ y \right\|$$

$$\leq (A + B)\gamma\alpha^{-j} + \left\| \alpha^j (\tilde{T} + \alpha I)^{-j} T^+ y \right\| \, ,$$

in particular,

$$\left\|x_{\alpha_j,\delta,h}^{(j)} - T^+y\right\| \le (A+B)\gamma\alpha_j^{-j} + \left\|\alpha_j^j\left(\tilde{T}+\alpha_j I\right)^{-j} T^+y\right\|$$

$$\le (A+B)\gamma\alpha_j^{-j} + \left[\int_0^{+\infty}\left(\frac{\alpha_j}{\alpha_j+\lambda}\right)^{2j} d\left\|\tilde{E}_\lambda T^+y\right\|^2\right]$$

where $\alpha_j = \alpha_j(\delta,h)$ is the unique solution of (9) and $\tilde{E}_\lambda$ is the spectral family generated by $\tilde{T}$. This implies by the Dominated Convergence Theorem and Lemma 3 that

$$\lim_{\gamma\to 0}\left\|x_{\alpha_j,\delta,h}^{(j)} - T^+y\right\| = 0.$$

## 3. Convergence Rates

Now we turn to the estimation of convergence rate.

**Proposition 1.** *For each $\delta, h, \alpha > 0$, $y_\delta \in Y$ fulfilling* (1)*, let $x_{\alpha,\delta,h}^{(j)}$ be the result of iterated Tikhonov regularization of order $j \ge 1$ as described by* (5)*. Assume that* (3) *and* (4) *hold. If $T^+y \in R(\tilde{T})$, then there exists $A$, a constant independent of $\alpha, \delta, h$, such that*

$$\left\|x_{\alpha,\delta,h}^{(j)} - T^+y\right\| \le A\left(\frac{\gamma}{\sqrt{\alpha}} + \alpha^j\right), \qquad j \in N.$$

*Proof.* It follows from the assumption $T^+y \in R(\tilde{T}^j)$ and Lemma 5 that

$$\left\|x_{\alpha,\delta,h}^{(j)} - T^+y\right\| \le A\gamma\alpha^{-\frac{1}{2}} + \left\|\alpha^j(\tilde{T}+\alpha I)^{-j}T^+y\right\|$$

$$\le A\gamma\alpha^{-\frac{1}{2}} + \alpha^j\left\|w\right\|$$

holds with a suitable constant $A$ and $w \in X$ such that $\tilde{T}^j w = T^+y$.

**Lemma 6.** *Let $\delta, h, \alpha, y_\delta$ and $T^+y$ be as in Proposition* 1*. Assume that* (3) *and* (4) *hold. For each positive $p, q, s$, if $2q > (2j-1)m - 4j$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \le \gamma^m \alpha_j^{-q-2j} \le C_2$$

*as $\gamma \to 0$, where $\alpha_j = \alpha_j(\delta,h)$ is the solution of* (9) *for some $p, q, s > 0$.*

*Proof.* Whithout loss of generality, let $\alpha < 1$ and $T^+y = \tilde{T}^j w$. Similarly to the calculations that lead to (12), we have that for $\alpha, \delta, h > 0$,

$$\sqrt{\rho_j(\alpha)} \le \left\|\alpha^j(\tilde{T}_h+\alpha I)^{-j}T_h^*\right\|\delta + \left\|\alpha^j(\tilde{T}+\alpha I)^{-j}\tilde{T}^j\tilde{T}w\right\|$$

$$+\left\|\alpha^j T_h^*(\hat{T}_h+\alpha I)^{-j}\left[(\hat{T}_h+\alpha I)^j - (\hat{T}+\alpha I)^j\right](\hat{T}+\alpha I)^{-j}\hat{T}^j Tw\right\|$$

$$+\left\|\alpha^j(T_h^* - T^*)(\hat{T}+\alpha I)^{-j}\,\hat{T}^j Tw\right\|$$

$$\leq \delta\alpha^{\frac{1}{2}} + \alpha^j \left\|\tilde{T}w\right\| + A\alpha^{\frac{1}{2}}h\,\|w\| + \alpha^j h\,\|w\|\,\|T\|$$

holds, where $A$, a constant, is independent of $\alpha, \delta, h$. Hence there is a constant, which we denote also by $A$, such that

$$\gamma\alpha^{-\frac{q+1}{m}} \leq A(\gamma + \alpha^{j-\frac{1}{2}})^{\frac{2}{m}}.$$

In particular, we have

$$\gamma\alpha_j^{-\frac{q+1}{m}} \leq A(\gamma + \alpha_j^{j-\frac{1}{2}})^{\frac{2}{m}} \tag{17}$$

where $\alpha_j = \alpha_j(\delta, h)$ is the solution of (9). By (18) we can claim that

$$\lim_{\gamma\to 0} \gamma\alpha_j^{-j+\frac{1}{2}} = 0 \tag{18}$$

holds under the assumptions of Lemma 6. Indeed if

$$\frac{q+1}{m} \geq j - \frac{1}{2}\,,$$

it follows from (18) and Lemma 2 that $\lim_{\gamma\to 0} \gamma\alpha_j^{-j+\frac{1}{2}} = 0$ holds. Now let $\frac{q+1}{m} < j - \frac{1}{2}$ . Note that the assumption of Lemma 6 implies that for $m > 2$ ,

$$\frac{q+1}{m-2} > j - \frac{1}{2}$$

holds. For all $n \in N$ , let

$$w_0 = 0, \quad w_n = \frac{q+1}{m} \sum_{i=0}^{n-1} \left(\frac{2}{m}\right)^i .$$

By the same method as used in the proof of Lemma 3 we can prove that if $w_n < j - \frac{1}{2}$ , then

$$\lim_{\gamma\to 0} \gamma\alpha_1^{-w_{n+1}} = 0$$

and that there exists a unique $n \in N$ such that

$$w_n < j - \frac{1}{2} \leq w_{n+1}\,.$$

Together with Lemma 2, this implies that

$$\lim_{\gamma\to 0} \gamma\alpha_j^{-j+\frac{1}{2}} = 0.$$

By (18) and (19) we now have that

$$\limsup_{\gamma\to 0} \gamma^m\, \alpha_j^{-q-2j} < +\infty\ .$$

In order to prove Lemma 6, it suffices to show that

$$\liminf_{\gamma\to 0} \gamma^m\, \alpha_j^{-q-2j} > 0\,. \tag{19}$$

Without loss of the generality let $T^+y = \tilde{T}^j w \neq 0$, and let

$$Z_1 = \alpha^j T_h^*(\hat{T}_h + \alpha I)^{-j}\left[(\hat{T}+\alpha I)^j - (\hat{T}_h+\alpha I)^j\right](\hat{T}+\alpha I)^{-j} y$$

$$Z_2 = \alpha^j (T_h^* - T^*)(\hat{T}+\alpha I)^{-j} y$$

$$Z_3 = \alpha^j T^*(\hat{T}+\alpha I)^{-j} y\,.$$

On the one hand, we have that

$$\sqrt{\rho_j(\alpha)} \geq \left|\left\|\alpha^j(\tilde{T}_h+\alpha I)^{-j}T_h^* y\right\| - \left\|\alpha^j(\tilde{T}_h+\alpha I)^{-j}T_h^*(y_\delta - y)\right\|\right|$$

and

$$\left\|\alpha^j(\tilde{T}_h+\alpha I)^{-j}T_h^* y\right\| \geq \left|\|Z_1+Z_2\| - \|Z_3\|\right|$$

hold, and that

$$\|Z_1+Z_2\| \leq A\gamma\alpha^{\frac{1}{2}}$$

and

$$\left\|\alpha^j(\tilde{T}_h+\alpha I)^{-j}T_h^*(y_\delta - y)\right\| \leq A\gamma\alpha^{\frac{1}{2}}$$

hold with a suitable constant $A$. On the other hand, we have by the Dominated Convergence Theorem that

$$\lim_{\gamma\to 0}\left\|T^*(\hat{T}+\alpha I)^{-j}y\right\| = \lim_{\gamma\to 0}\left\|(\tilde{T}+\alpha I)^{-j}T^* y\right\|$$

$$= \lim_{\gamma\to 0}\left\|(\tilde{T}+\alpha I)^{-j}\,\tilde{T}^j\,\tilde{T}w\right\|$$

$$= \left\|\tilde{T}w\right\| > 0.$$

Together with (19) we have finally that

$$\lim_{\gamma\to 0}\inf \alpha_j^{-j}\sqrt{\rho_j(\alpha_j)} \geq \left\|\tilde{T}w\right\| > 0$$

holds. This implies by the definition of $\alpha_j = \alpha_j(\delta, h)$ that (20) holds. We now have proved Lemma 6.

The next theorem is the second crucial result in this paper. It shows that the convergence rates can be improved if we choose suitable $p, q, s$ in (9) and $j \in N$.

**Theorem 2.** *For each $\delta, h > 0$ and $y_\delta \in Y$ fulfilling* (1), *let $x^{(j)}_{\alpha_j,\delta,h}$ be the result of the iterated Tikhonov regularization of order $j \geq 1$ as described by* (5), *where $\alpha_j = \alpha_j(\delta, h)$ is the unique solution of* (9) *for some $p, q, s > 0$. Assume that* (3) *and* (4) *hold, and*

$$2q = (2j+1)m - 4j\,.$$

*If $T^+y \in R(\tilde{T}^j)$, then*

$$\left\|x^{(j)}_{\alpha_j,\delta,h} - T^+y\right\| = O\left(\gamma^{\frac{2j}{2j+1}}\right)$$

*holds as $\gamma \to 0$.*

*Proof.* It follows from Proposition 1 that for $j \in N$,

$$\left\| x_{\alpha_j,\delta,h}^{(j)} - T^+ y \right\| = O\left(\gamma \alpha_j^{-\frac{1}{2}} + \alpha_j^j\right)$$

holds as $\gamma \to 0$. An easy calculation shows that the assumption of Theorem 2 implies that the assumption of Lemma 6 are fulfilled and hence we have that

$$\alpha_j = \alpha_j(\delta, h) = O\left(\gamma^{\frac{m}{q+2j}}\right)$$

holds as $\gamma \to 0$. This implies that

$$\left\| x_{\alpha_j,\delta,h}^{(j)} - T^+ y \right\| = O\left(\gamma^{\frac{2j}{2j+1}}\right)$$

holds as $\gamma \to 0$.

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