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ANTIPERIODIC WAVELETS*[1)]

H.L. Chen
(*Institute of Mathematics, Academia Sinica, Beijing*)

Abstract

In this paper, we construct the orthogonal wavelet basis in the space of antiperiodic functions by appealing the spline methods. Differing from other results in papers[1,2,3,6,8], here we derive the 3-scale equation, by using this equation we construct some basic functions, those functions can be used to construct different orthonormal basis in some spline function spaces.

1. Preliminary

As we know, many authors have made great efforts in constructing the orthonormal or biorthonormal basis on the whole real line $\mathbb{R}^1$ or on the whole n-dimensional space $\mathbb{R}^n$[4,7], but in many practical problems one needs to construct orthonormal basis on some finite interval with some boundary conditions.

Here we present a method of constructing the antiperiodic orthonormal wavelets basis on the interval $I = [0, 2\pi]$.

Main difficulty in the above problem is the construction of the orthonormal basis of W_{m-1}—the orthogonal complement of V_{m-1} in V_m—the key step is that we have to construct o.n. periodic wavelets $\{A_{\nu,3}^{n,m}\}$ which satisfy 2-scale equations, therefore, we shall adopt some new strategy to construct the o.n. basis of W_{m-1} which differs from [1].

Let n, K be integers, $N \geq 1$, n odd, $n = 2n_0 + 1, 2\pi = Kh, K \geq 2n + 2$, h a real number. The point set $\{y_i\}$ are defined as follows

$$y_0 = -\frac{(n+1)}{2}h, \quad y_j = y_0 + jh, \quad j = 1, 2, \cdots \tag{1.1}$$

The B-spline function is defined by

$$B_i^n(x) = (-1)^{n+1}(y_{n+1+i} - y_i)[y_i, \cdots, y_{i+n+1}]_y(x-y)_+^n \tag{1.2}$$

Definition 1.1. $S^{n,m} := \{S | S$ *is a polynomial of degree* n *on each interval* $[jh_m, (j+1)h_m), j \in \mathbb{Z}, S \in C^{n-1}(\mathbb{R}^1)\}$, *where* $h_m = h/3^m, m \geq 0$, m *is an integer.*

Set $g_i^{n,m}(x) = B_i^n(3^m x)$, *then* $\{g_i^{n,m}\}_{i\in\mathbb{Z}}$ *is a basis of* $S^{n,m}$.

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Definition 1.2. $\overset{\circ}{\mathcal{S}}_{n,K(m)}:=$ *{S|S is a polynomial of degree n on each interval* $[jh_m,(j+1)h_m), j=0,\cdots,K(m)-1; S\in C^{n-1}(I), S^{(j)}(0)=S^{(j)}(2\pi), j=0,1,\cdots,n-1\}$. *where* $K(m)=3^mK$.

$\overset{\circ}{\mathcal{S}}_{n,K(m)}$ is the family of periodic spline functions of degree n and with $K(m)$ knots $\{jh_m\}_{j=0}^{K(m)-1}$ in $[0,2\pi)$. Set $\widetilde{B}_i^n(x) = B_i^n(x) + B_{i+K}^n(x)$, the system of functions $\{\widetilde{B}_i^n\}_{i=-n_0}^{K-n_0-1}$ constitutes a basis in $\overset{\circ}{\mathcal{S}}_{n,K}$, where $B_i^n(x)$ and $B_{i+K}^n(x)$ are defined as in (1.2), and $K(0)=K$. If we define

$$\widetilde{B}_i^{n,m}(x) := B_i^{n,m}(x) + B_{i+K(m)}^{n,m}(x) = B_i^n(3^m x) + B_{i+K(m)}^n(3^m x) \tag{1.3}$$

then, the system $\{\widetilde{B}_i^{n,m}(x)\}_{i=-n_0}^{K(m)-n_0-1}$ forms a basis of $\overset{\circ}{\mathcal{S}}_{n,K(m)}$.

Definition 1.3. *Given any integer l, there exists unique integer k satisfying*

$$l = k + jK(m), \quad j\in \mathbb{Z} \quad \textit{and} \quad -n_0 \le k \le -n_0 - 1 + K(m) \tag{1.4}$$

define

$$\overset{\circ}{B}{}_l^{n,m}(x) = \widetilde{B}_k^{n,m}(x), \quad x\in[0,2\pi]\ . \tag{1.5}$$

Note here we use the same symbol $\widetilde{B}_i^{n,m}, \overset{\circ}{B}{}_i^{n,m}$ as in [C1], but different in meaning since here $K(m):=3^mK$ and $h_m:=h/3^m$.

From (1.5), we conclude that $\{\overset{\circ}{B}{}_{l+\nu}^{n,m}(x)\}_{\nu=0}^{K(m)-1}$ is a basis in $\overset{\circ}{\mathcal{S}}_{n,K(m)}$. The function $\overset{\circ}{B}{}_l^{n,m}(x)$ can be extended to the whole real axis by periodicity.

We define the inner product of two function f and g on $[0,2\pi]$ by $\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$.

Definition 1.4. *Define*

$$A_{k,3}^{n,m}(x) = C_{k,3}^{n,m}\sum_{l=0}^{K(m)-1}\exp(2\pi ilk/K(m))\,\overset{\circ}{B}{}_l^{n,m}(x) \tag{1.6}$$

where

$$C_{k,3}^{n,m} = \left\{\sum_{\nu=0}^{K(m)-1}\left[\exp\left(\frac{2\pi i\nu k}{K(m)}\right)\right]\overset{\circ}{B}{}_l^{2n+1}(0)\right\}^{-\frac{1}{2}} \tag{1.7}$$

Lemma 1.1. $A_{k,3}^{n,m}(x)$ *is defined as in* (1.6), *then*

$$\langle A_{k,3}^{n,m}(\cdot), A_{j,3}^{n,m}(\cdot)\rangle = \delta_{k,j}, \quad 0\le k,j\le K(m)-1. \tag{1.8}$$

Let $V_m :=\overset{\circ}{\mathcal{S}}_{n,K(m)}, \{A_{k,3}^{n,m}\}_{k=0}^{K(m)-1}$ *is an o.n. basis in* V_m. *Where* $\delta_{k,j}$ *is the kronecker delta.*

Remark. The proof of Lemma 1.1 is analog to that in [C1]. Where the powerful tool is the Fourier expansion of the B-spline function:

$$\widetilde{B}_l^{n,m}(x) = \widetilde{B}_0^{n,m}(x-lh_m) = K^n(m)\sum_{\nu\in\mathbb{Z}}\left(\frac{\sin\dfrac{\nu\pi}{K(m)}}{\nu\pi}\right)^{n+1}\exp\left(\frac{2\pi\nu i}{T}(x-lh_m)\right) \tag{1.9}$$

$A_{k,3}^{n,m}, C_{k,3}^{n,m}$ can be presented as

$$A_{k,3}^{n,m}(x) = C_k^{n,m}K^{n+1}(m)\sum_{\nu\in\mathbb{Z}}\left(\frac{\sin\left(\dfrac{\nu K(m)+k}{K(m)}\right)\pi}{(\nu K(m)+k)\pi}\right)^{n+1}\exp\left(\frac{2\pi i(\nu K(m)+k)x}{T}\right) \tag{1.10}$$

and

$$C_{k,3}^{n,m} = K^{-n-1}(m)\left\{\sum_{\nu\in\mathbb{Z}}\left(\frac{\sin\left(\dfrac{k\pi}{K(m)}\right)}{(k+\nu K(m))\pi}\right)^{2n+2}\right\}^{-\frac{1}{2}} \tag{1.11}$$

respectively.

The following 2-scale equation differs from that in [1].

Lemma 1.2. *Since $A_{k,3}^{n,m}(x)\in V_m\subset V_{m+1}$, we have the following 2-scale equation*

$$A_{\nu,3}^{n,m}(x) = \sum_{\lambda=0}^{2} a_{\nu,\nu+\lambda K(m)}\cdot A_{\nu+\lambda K(m),3}^{n,m+1}(x), \quad \nu = 0,1,\cdots,K(m)-1. \tag{1.12}$$

where

$$a_{\nu,\nu+\lambda K(m)} = \langle A_{\nu,3}^{n,m}, A_{\nu+\lambda K(m),3}^{n,m+1}\rangle, \quad \lambda = 0,1,2. \tag{1.13}$$

Remark. The coefficient $a_{\nu,\nu+\lambda K(m)}$ in (1.13) can be easily calculated if we appeal the formulas (1.6) and (1.7), we also notice that in the formula (1.12), the right side includes only 3 terms. In deriving (1.12) we have used the following formula

$$\widetilde{B}_l^{n,m}(x) = \sum_{k=-n-1}^{n+1} p_{n,k}\widetilde{B}_{k+3l}^{n,m+1}(x) \tag{1.14}$$

where $\widetilde{B}_l^{n,m}(x)$ is defined in (1.3) and

$$p_{n,k} = \begin{cases} 3^{-n}\displaystyle\sum_{\nu=0}^{\left[\frac{n+1-k}{2}\right]}\binom{n+1}{\nu}\binom{n+1-k-\nu}{\nu}, & k=-(n+1),\cdots,n+1 \\ 0, & \text{otherwise}\end{cases}$$

formula (1.14) can be obtained by the Fourier transform of $\widetilde{B}_l^{n,m}(x)$.

Define

$$D^{n,m}_{j,3}(x) := \begin{cases} -\overline{a}_{j,j+K(m)} a_{j,j} Q_j A^{n,m+1}_{j,3}(x) + \frac{1}{Q_j} A^{n,m+1}_{j+K(m),3}(x) - \overline{a}_{j,j+K(m)} \\ \quad \cdot a_{j,j+2K(m)} Q_j A^{n,m+1}_{j+2K(m),3}(x), \quad 0 \le j \le K(m)-1 \\ -\overline{a}_{j-K(m),j+K(m)} Q_{j-K(m)} A^{n,m+1}_{j-K(m),3}(x) + \overline{a}_{j-K(m),j-K(m)} \\ \quad \cdot Q_{j-K(m)} A^{n,m+1}_{j+K(m),3}(x), \quad K(m) \le j \le 2K(m)-1. \end{cases} \qquad (1.15)$$

where $a_{i,j} = \langle A^{n,m}_{i,3}, A^{n,m+1}_{j,3} \rangle$, $Q_k = [1/(1-|a_{k,k+K(m)}|^2)]^{\frac{1}{2}}$.

Lemma 1.3. *Denote W_m the orthogonal complementary space of V_m in V_{m+1}, $V_{m+1} = V_m \oplus W_m$, $V_m \perp W_m$. , $\{D^{n,m}_{j,3}(x)\}_{j=0}^{2K(m)-1}$ is an o.n. basis of W_m, i.e.*

$$\langle D^{n,m}_{j_1,3}, D^{n,m}_{j_2,3} \rangle = \delta_{j_1,j_2}, \qquad 0 \le j_1, j_2 \le 2K(m)-1 \qquad (1.16)$$

and

$$\langle D^{n,m}_{j,3}, A^{n,m}_{k,3} \rangle = 0, \quad 0 \le j \le 2K(m)-1, 0 \le k \le K(m)-1. \qquad (1.17)$$

Proof. By applying the orthonormal property of the systems $\{A^{n,m+1}_{j,3}\}_{j=0}^{K(m+1)-1}$, i.e. $\langle A^{n,m+1}_{j_1,3}, A^{n,m+1}_{j_2,3} \rangle = \delta_{j_1,j_2}$, and $\langle A^{n,m}_{j,3}, A^{n,m}_{j,3} \rangle = |a_{j,j}|^2 + |a_{j,j+K(m)}|^2 + |a_{j,j+2K(m)}|^2 = 1$, we can prove the following equalities

$$\langle A^{n,m}_{j,3}, D^{n,m}_{l,3} \rangle = Q_j a_{j,j+K(m)} \left[\frac{1}{Q_j^2} - |a_{j,j}|^2 - |a_{j,j+2K(m)}|^2 \right] \delta_{jl} = 0 \cdot \delta_{j,l} = 0$$

for $0 \le j, l \le K(m)-1$.

$$\langle A^{n,m}_{j,3}, D^{n,m}_{l+K(m),3} \rangle = (-a_{j,j} a_{j,j+2K(m)} Q_j + a_{j,j} Q_j a_{j,j+2K(m)}) \delta_{j,l} = 0 \cdot \delta_{j,l} = 0$$

for $0 \le j, l \le K(m)-1$

$$\begin{aligned} \langle D^{n,m}_{j,3}, D^{n,m}_{l+K(m),3} \rangle &= (a_{j,j} Q_j^2 \overline{a}_{j,j+K(m)} a_{j,j+2K(m)} - a_{jj} Q_j^2 \overline{a}_{j,j+K(m)} a_{j,j+2K(m)}) \delta_{j,l} \\ &= 0 \end{aligned}$$

for $0 \le j, l \le K(m)-1$.

$$\langle D^{n,m}_{j,3}, D^{n,m}_{k,3} \rangle = (|a_{j,j+K(m)}|^2 + 1 - |a_{j,j+K(m)}|^2) \delta_{j,k} = \delta_{j,k}$$

for $0 \le j, k \le K(m)-1$.

$$\langle D^{n,m}_{j+K(m),3}, D^{n,m}_{k+K(m),3} \rangle = (1 - |a_{j,j+K(m)}|^2) Q_j^2 \delta_{j,k} = \delta_{j,k}$$

for $0 \le j, k \le K(m)-1$.
thus we have all the conclusions in lemma 1.3.

Remark. Since $\{A^{n,m+1}_{0,3}, A^{n,m+1}_{1,3}, \cdots, A^{n,m+1}_{K(m+1)-1,3}\}$ and $\{A^{n,m}_{0,3}, \cdots, A^{n,m}_{K(m)-1,3}, D^{n,m}_{0,3}, \cdots, D^{n,m}_{2K(m)-1,3}\}$ are o.n. bases in V_{m+1}, there is an unitary matrix which transforms the former to the latter, this transform is presented in the formulas (1.12) and (1.15)

2. Antiperiodic Wavelets

A function $f(x)$ is called antiperiod π if $f(x+\pi) = -f(x), \quad \forall x \in I\!R^1$. If $f(x)$ is differentiable up to order S and satisfies $f^{(j)}(x+\pi) = -f^{(j)}(x)$, $x \in R^1$ $j = 0, 1, \cdots, S$, then f is called antiperiod π of order S.

In order to construct the bases of the class of anti-periodic functions, we use the spline method.

In this section we use the same symbol of the B-spline function as defined in (1.2), but here we stipulate that $T = 2\pi$, and the intervals $J = [0, \pi]; I = [0, 2\pi]$.

Let L be a positive integer such that $\pi = Lh$;

$$L \geq n+1, \quad K = 2L, \quad Kh = 2\pi. \tag{2.1}$$

Definition 2.1. $E_i^{n,m}(x) = \overset{\circ}{B}{}_i^{n,m}(x) - \overset{\circ}{B}{}_{i+L(m)}^{n,m}(x), \quad i \in Z\!\!Z.$

Theorem 2.1. *$E_i^{n,m}(x)$ is an antiperiodic function of order $n-1$, with antiperiod π.*

Proof. for $0 \leq j \leq n-1$, by 2π periodicity of the function $\overset{\circ}{B}{}_i^{n,m}(x)$, we have

$$\begin{aligned} D^j E_i^{n,m}(x+\pi) &= D^j\{\overset{\circ}{B}{}_i^{n,m}(x+\pi) - \overset{\circ}{B}{}_{i+L(m)}^{n,m}(x+\pi)\} \\ &= D^j\{\overset{\circ}{B}{}_i^{n,m}(x-\pi) - \overset{\circ}{B}{}_{i+L(m)}^{n,m}(x+\pi)\} \\ &= D^j\{\overset{\circ}{B}{}_i^{n,m}(x-L(m)h) - \overset{\circ}{B}{}_{i+L(m)}^{n,m}(x+L(m)h)\} \\ &= D^j\{\overset{\circ}{B}{}_{i+L(m)}^{n,m}(x) - \overset{\circ}{B}{}_i^{n,m}(x)\} = -D^j E_i^{n,m}(x)\ . \end{aligned}$$

Theorem 2.2. *The system $\{E_i^{n,m}(X) \mid \ i = -n_0, \cdots, L(m) - n_0 - 1\}$ is linearly independent on J, where n is odd, $n = 2n_0 + 1$.*

Proof. Assume there are constants $\{C_i\}_{-n_0}^{-n_0-1+L(m)}$, s.t.

$$C_{-n_0} E_{-n_0}^{n,m}(x) + \cdot + C_{-n_0-1+L(m)} E_{-n_0-1+L(m)}^{n,m}(x) = 0, \quad x \in J \tag{2.2}$$

Set

$$a_i = \begin{cases} C_i, & i = -n_0, \cdots, -n_0 - 1 + L(m) \\ C_{i-L(m)}, & i = -n_0 + L(m), \cdots, -n_0 - 1 + K(m) \end{cases} \tag{2.3}$$

then from (2.2) and the antiperiodicity of $E_j^{n,m}(x)$ for all $j \in \{-n_0, \cdots, -n_0 - 1 + L(m)\}$, (2.2) is valid on $[0, 2\pi]$, from Definition 2.1 (2.2) and (2.3) we have the following equality

$$\sum_{i=-n_0}^{-n_0-1+K(m)} a_i \overset{\circ}{B}{}_i^{n,m}(x) \equiv 0, \quad x \in [0, 2\pi],$$

therefore $a_i = 0$ for all i since $\{\overset{\circ}{B}{}_i^{n,m}\}_{i=-n_0}^{-n_0-1+K(m)}$ is a linearly independent system in $\overset{\circ}{S}_{n,K(m)}([0, 2\pi])$.

In Definition 2.1, we have defined a class of functions of antiperiodicity on J, these are splines which constitute a class of functions, we give a name in the following

Definition 2.2. *$V_m^a = \mathcal{S}_{n,m}^a = lin\{E_i^{n,m} | i = -n_0, \cdots, -n_0 - 1 + L(m)\}$ The dimension of V_m^a is $L(m)(= 3^m L)$, each element in V_m^a is an antiperiodic spline function of degree n with knots in the set $\{jh_m | j \in \mathbb{Z}, h_m = h/3^m\}$.*

By using the method provided by [5], we can easily construct the antiperiodic spline interpolation function.

Theorem 2.3. *Let $L(x)$ be the fundamental periodic spline interpolation function*

$$L(x) = \sum_{\nu=0}^{K(m)-1} C_{\nu,3} \sum_{\mu=0}^{K(m)-1} \exp(2\pi i\nu\mu/K(m)) \overset{\circ}{B}{}_{\mu}^{n,m}(x) , \tag{2.4}$$

where $C_{\nu,3} = (C_{\nu,3}^{n_0,m})^2/K(m)$, $n = 2n_0 + 1$, $K(m) = 3^m K$, $K(m)h_m = 2\pi; C_{\nu,3}^{n_0,m}$ and $\overset{\circ}{B}{}_{\mu}^{n,m}(x)$ are defined in Part 1, then $L(x)$ is in $\overset{\circ}{\mathcal{S}}_{n,K(m)}([0, 2\pi])$, and satisfies

$$L(kh_m) = \delta_{k,0}, \quad k = 0, 1, \cdots, K(m) - 1 \tag{2.5}$$

Define

$$L_{n,m}^a(x) = L(x) - L(x - \pi) \tag{2.6}$$

then $L_{n,m}^a(x) \in \mathcal{S}_{n,m}^a$ satisfies

$$L_{n,m}^a(kh_m) = \delta_{k,0}, \quad 0 \le k \le L(m) - 1. \tag{2.7}$$

Proof. The function $L(x)$ in $\overset{\circ}{\mathcal{S}}_{n,K(m)}([0, 2\pi])$ is obvious and (2.5) can be easily checked.

By 2π-periodicity of function $L(x)$, we have

$$L_{n,m}^a(x + \pi) = L(x + \pi) - L(x) = -[L(x) - L(x - \pi)] = -L_{n,m}^a(x)$$

thus $L_{n,m}^a(x) \in \mathcal{S}_{n,m}^a$,

We apply the 2π-periodicity of $L(x)$ again and property (2.5)

$$L_{n,m}^a(kh_m) = L(kh_m) - L(kh_m + L(m)h_m) = \delta_{k,0} - \delta_{k+L(m),0} = \delta_{k,0}$$

since $0 \le k + L(m) \le K(m) - 1$ for all $k, 0 \le k \le L(m) - 1$, we obtain (2.7)

Definition 2.3. *The inner product on $J = [0, \pi]$ is defined by $(f, g) = \frac{1}{\pi} \int_0^{\pi} f(x) \overline{g(x)} dx$.*

Lemma 2.4. *If f and g are antiperiod π, then*

$$\frac{1}{\pi} \int_0^{\pi} f(x)\overline{g(x)}dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}dx \tag{2.8}$$

namely

$$(f, g) = \langle f, g \rangle$$

Proof.

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_0^{\pi} f(x)\overline{g(x)}dx$$

$$+\frac{1}{2\pi}\int_\pi^{2\pi} f(x)\overline{g(x)}dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\overline{g(x)}dx = (f,g)\ .$$

Definition 2.4. *Define*

$$A_{n,m}^{a,j}(x) = \frac{1}{2}[A_{2j-1,3}^{n,m}(x) - A_{2j-1,3}^{n,m}(x-\pi)] \tag{2.9}$$

Theorem 2.5. $\{A_{n,m}^{a,j}\}_{j=0}^{L(m)-1}$ *is an orthonormal basis in* $S_{n,m}^a (= V_m^a)$.
Before giving the proof of theorem 2.5, we have to establish the following lemma.
Lemma 2.1.

$$A_{\nu,3}^{n,m}(x-kh_m) = \exp\left(\frac{-2\pi i\nu\cdot k}{K(m)}\right) A_{\nu,3}^{n,m}(x). \tag{2.10}$$

Proof. by applying the Fourier expansion of $A_{\nu,3}^{n,m}(\cdot)$ (see (1.10)), then (2.10) is immediate.

Proof of theorem 2.5. By periodicity of the function $A_{2j-1,3}^{n,m}(\cdot)$, and lemma 2.1 we have $\langle A_{2j_1-1,3}^{n,m}(\cdot), A_{2j_2-1,3}^{n,m}(\cdot+\pi)\rangle = \langle A_{2j_1-1,3}^{n,m}(\cdot-\pi), A_{2j_2-1,3}^{n,m}(\cdot)\rangle = (-1)^{2j_1-1}\delta_{j_1,j_2}$, since $A_{n,m}^{a,j}$ is antiperiodic with antiperiod π,

$$(A_{n,m}^{a,j_1}, A_{n,m}^{a,j_2}) = \langle A_{n,m}^{a,j_1}, A_{n,m}^{a,j_2}\rangle = \frac{1}{4}\{2\delta_{j_1,j_2} - 2(-1)^{2j_1-1}\delta_{j_1,j_2}\} = \delta_{j_1,j_2},$$

$$0 \le j_1, j_2 \le L(m)-1,$$

since $A_{n,m}^{a,j} \in S_{n,m}^a$, thus $\{A_{n,m}^{a,j}\}_{j=0}^{L(m)-1}$ is an o.n. bais in $S_{n,m}^a$.

Remark. Suppose that $L(m) = 2^m L$, $h_m = h/2^m$, $\pi = L(m)h_m$; and suppose $A_j^{n,m}(x) = A_j^{n,0}(2^m x)$ as in [C$_1$], similar to (2.9), we define

$$A_{n,m}^{a,j}(x) = \frac{1}{2}[A_{2j-1}^{n,m}(x) - A_{2j-1}^{n,m}(x-\pi)] \tag{2.11}$$

[illegible] $^{n,m}(x)$ is defined in [1], i.e. $K(m)(= 3^m K)$ is replaced by $2^m K$, and $L(m)(=$ [illegible] replaced by $2^m L$, then

$$\begin{aligned} A_{n,m}^{a,j}(x) &= \frac{1}{2}[A_{2j-1}^{n}(2^m x) - A_{2j-1}^{n}(2^m x - 2^m\pi)] \\ &= \frac{1}{2}[A_{2j-1}^{n}(2^m x) - A_{2j-1}^{n}(2^m x)] \equiv 0 \quad \text{(for all} \quad m \ge 1). \end{aligned}$$

Since $A_l^n(y)$ is a $2\pi(= T)$ period function, and the function $E_i^{n,m}(x)$ (Def. 2.1) is also identically zero. Therefore, we adopt 3-scale instead of 2-scale, and we have to establish the important formulas for the construction of the o.n. basis $\{D_{j,3}^{n,m}\}_{j=0}^{2K(m)-1}$ (see (1.15)), where $K(m) = 3^m K$.

Since Dim $V^a_{m+1} = L(m+1)(= 3^{m+1}L)$, Dim $V^a_m = L(m)(= 3^m L)$, the complementary subspace of V^a_m in V^a_{m+1} is denoted by W^a_m, Dim $W^a_m = 2L(m)$. We define a function $D^{a,j}_{n,m}$

$$D^{a,j}_{n,m}(x) = \frac{1}{2}(D^{n,m}_{2j-1,3}(x) - D^{n,m}_{2j-1,3}(x-\pi)),$$

where $D^{n,m}_{l,3}(x)$ is defined in Part 1, formula (1.15).

Theorem 2.6. *The system of functions* $\{D^{a,j}_{n,m}(x)\}_{j=1}^{2L(m)}$ *constitute an o.n. basis of* W^a_m, *i.e.*

$$(D^{a,j_1}_{n,m}, D^{a,j_2}_{n,m}) = \delta_{j_1,j_2}, \quad 1 \le j_1, j_2 \le 2L(m). \tag{2.12}$$

and

$$(D^{a,j}_{n,m}, A^{a,l}_{n,m}) = 0, \quad 1 \le j \le 2L(m), \quad 0 \le l \le L(m)-1,$$

thus,

$$V^a_{m+1} = W^a_m \oplus V^a_m, \quad W^a_m \perp V^a_m \,. \tag{2.13}$$

Proof. By the 2π periodicity of the function $D^{n,m}_{2j-1,3}(x)$, we can derive $D^{a,j}_{n,m}(x+\pi) = -D^{a,j}_{n,m}(x)$, from (1.15), since $D^{n,m}_{j,3}(x) \in V_{m+1}$, therefore , $D^{a,j}_{n,m}(x) \in V^a_{m+1}$, for $j = 1, \cdots, 2L(m)$.

By using (1.15) we can establish the similar formular as (2.10) for $D^{a,j}_{n,m}(x)$, combine this with (1.16) we easily obtain formula (2.12), in fact,

$$\begin{aligned}
&(D^{a,j_1}_{n,m}(\cdot), D^{a,j_2}_{n,m}(\cdot)) = \tfrac{1}{4}\{\langle D^{n,m}_{2j_1-1,3}(\cdot), D^{n,m}_{2j_2-1,3}(\cdot)\rangle \\
&+\langle D^{n,m}_{2j_1-1,3}(\cdot-\pi), D^{n,m}_{2j_2-1,3}(\cdot-\pi)\rangle - \langle D^{n,m}_{2j_1-1,3}(\cdot), D^{n,m}_{2j_2-1,3}(\cdot-\pi)\rangle \\
&-\langle D^{n,m}_{2j_1-1,3}(\cdot-\pi), D^{n,m}_{2j_2-1,3}(\cdot)\rangle = \tfrac{1}{4}\{2\delta_{j_1,j_2} - 2(-1)^{2j_1-1}\delta_{j_1,j_2}\} \\
&= \delta_{j_1,j_2},
\end{aligned}$$

(2.13) can be easily verified by using formula (1.17) and Definition 2.3.

By using the functions $A^{a,j}_{n,m}$ and $D^{aj}_{n,m}$, the reconstruction and decomposition formulas can be established, but this will be presented in another paper.

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