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A LINEARIZED DIFFERENCE SCHEME FOR THE KURAMOTO-TSUZUKI EQUATION*1)

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Abstract

In this paper, a linearized three-level difference scheme is derived for the mixed boundary value problem of Kuramoto-Tsuzuki equation, which can be solved by double-sweep method. It is proved that the scheme is uniquely solvable and second order convergent in energy norm.

1. Introduction

Tsertsadze[1] studied the finite difference method for the mixed boundary value problem of Kuramoto-Tsuzuki equation

$$\frac{\partial w}{\partial t} = (1+ic_1)\frac{\partial^2 w}{\partial x^2} + w - (1+ic_2)\left|w\right|^2 w, \quad 0 < x < 1,\, 0 < t \le T \tag{1.1}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \qquad \frac{\partial w}{\partial x}(1,t) = 0, \quad 0 < t \le T \tag{1.2}$$

$$w(x,0) = w_0(x), \quad 0 \le x \le 1 \tag{1.3}$$

where c_1 and c_2 are real constants, $w(x,t)$ and $w_0(x)$ complex valued functions. Divide $[0,1]$ into$M$ subintervals and $[0,T]$ into K subintervals with meshsizes h and τ respectively. Tsertsadze[1] constructed for (1.1)-(1.3) the following difference scheme

$$\delta_t w_0^{k+\frac{1}{2}} = (1+ic_1)\frac{2}{h^2}(w_1^{k+\frac{1}{2}} - w_0^{k+\frac{1}{2}}) + w_0^{k+\frac{1}{2}} - (1+ic_2)\left|w_0^{k+\frac{1}{2}}\right|^2 w_0^{k+\frac{1}{2}},$$
$$0 \le k \le K-1 \tag{2.1}$$

$$\delta_t w_j^{k+\frac{1}{2}} = (1+ic_1)\delta_x{}^2 w_j^{k+\frac{1}{2}} + w_j^{k+\frac{1}{2}} - (1+ic_2)\left|w_j^{k+\frac{1}{2}}\right|^2 w_j^{k+\frac{1}{2}},$$
$$1 \le j \le M-1,\, 0 \le k \le K-1 \tag{2.2}$$

$$\delta_t w_M^{k+\frac{1}{2}} = (1+ic_1)\frac{2}{h^2}(w_{M-1}^{k+\frac{1}{2}} - w_M^{k+\frac{1}{2}}) + w_M^{k+\frac{1}{2}} - (1+ic_2)\left|w_M^{k+\frac{1}{2}}\right|^2 w_M^{k+\frac{1}{2}},$$
$$0 \le k \le K-1 \tag{2.3}$$

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$$w_j^0 = w_0(x_j), \qquad 0 \le j \le M \tag{2.4}$$

where $x_j = jh, t_k = k\tau, w_j^k$ the approximation of $w(x_j, t_k)$, $w_j^{k+\frac{1}{2}} = (w_j^{k+1} + w_j^k)/2$, $\delta_t w_j^{k+\frac{1}{2}} = (w_j^{k+1} - w_j^k)/\tau$, $\delta_x^2 w_j^k = (w_{j+1}^k - 2w_j^k + w_{j-1}^k)/h^2$ and proved that the difference scheme is convergent in energy norm with the convergence rate of order $O(h^{3/2})$ when $\tau = O(h^{2+\epsilon})$ $(\epsilon > 0)$. (2) is nonlinear.

In this paper, for generality, we consider inhomogeneous equation. In other words, instead of (1.1), we consider

$$\frac{\partial w}{\partial t} = (1 + ic_1)\frac{\partial^2 w}{\partial x^2} + w - (1 + ic_2)|w|^2 w + f(x,t), \quad 0 < x < 1, 0 < t \le T \tag{1.1$'$}$$

where $f(x,t)$ is a known complex valued smooth function. We develop for (1.1') and (1.2)-(1.3) the difference scheme

$$\Delta_t w_0^k = (1 + ic_1)\frac{2}{h^2}(w_1^{\hat{k}} - w_0^{\hat{k}}) + w_0^{\hat{k}} - (1 + ic_2)\left|w_0^k\right|^2 w_0^{\hat{k}} + f(\frac{h}{3}, t_k),$$
$$1 \le k \le K - 1 \tag{3.1}$$

$$\Delta_t w_j^k = (1 + ic_1){\delta_x}^2 w_j^{\hat{k}} + w_j^{\hat{k}} - (1 + ic_2)\left|w_j^k\right|^2 w_j^{\hat{k}} + f(x_j, t_k),$$
$$1 \le j \le M - 1, 1 \le k \le K - 1 \tag{3.2}$$

$$\Delta_t w_M^k = (1 + ic_1)\frac{2}{h^2}(w_{M-1}^{\hat{k}} - w_M^{\hat{k}}) + w_M^{\hat{k}} - (1 + ic_2)\left|w_M^k\right|^2 w_M^{\hat{k}} + f(1 - \frac{h}{3}, t_k),$$
$$1 \le k \le K - 1 \tag{3.3}$$

$$w_j^0 = w_0(x_j), \quad w_j^1 = w_0(x_j) + \tau w_1(x_j), \quad 0 \le j \le M \tag{3.4}$$

where

$$w_1(x) = (1 + ic_1)\frac{d^2 w_0(x)}{dx^2} + w_0(x) - (1 + ic_2)|w_0(x)|^2 w_0(x) + f(x,0)$$
$$w_j^{\hat{k}} = (w_j^{k+1} + w_j^{k-1})/2, \qquad \Delta_t w_j^k = (w_j^{k+1} - w_j^{k-1})/(2\tau).$$

The scheme (3) is a tridiagonal system of linear algebraic equations, which can be solved by double-sweep method. We suppose $\tau = \alpha h^{\frac{1}{4}+\epsilon}$, where α and ϵ are any two positive constants. In next two sections, we will prove that (3) is uniquely solvable and convergent in energy norm with convergence rate of order $O(\tau^2 + h^2)$. Farthermore, we will see that the optimal choice is $\epsilon = 3/4$ or $\tau = O(h)$.

Let $u \equiv \{u_j\}_{j=0}^M$ be a net function on $I \equiv \{x_j\}_{j=0}^M$, define the L_2 norm

$$\|u\| = \sqrt{h\left(\frac{1}{2}{u_0}^2 + \sum_{j=1}^{M-1}{u_j}^2 + \frac{1}{2}{u_M}^2\right)}.$$

2. Solvability

Theorem 1. *The difference scheme* (3) *is uniquely solvable.*

Proof. It is obvious that w^0 and w^1 are uniquely determined by (3). Now suppose w^0,w^1, $\cdots$, w^k $(1 \le k \le K-1)$ be solved uniquely. Consider the system of homogeneous equations of (3) for w^{k+1} :

$$\frac{1}{2\tau}w_0^{k+1} = (1+ic_1)\frac{1}{h^2}(w_1^{k+1} - w_0^{k+1}) + \frac{1}{2}w_0^{k+1} - \frac{1}{2}(1+ic_2)\left|w_0^k\right|^2 w_0^{k+1} \tag{4.1}$$

$$\frac{1}{2\tau}w_j^{k+1} = \frac{1}{2}(1+ic_1){\delta_x}^2 w_j^{k+1} + \frac{1}{2}w_j^{k+1} - \frac{1}{2}(1+ic_2)\left|w_j^k\right|^2 w_j^{k+1}, \quad 1 \le j \le M-1 \tag{4.2}$$

$$\frac{1}{2\tau}w_M^{k+1} = (1+ic_1)\frac{1}{h^2}(w_{M-1}^{k+1} - w_M^{k+1}) + \frac{1}{2}w_M^{k+1} - \frac{1}{2}(1+ic_2)\left|w_M^k\right|^2 w_M^{k+1}. \tag{4.3}$$

Multiplying (4.1)-(4.3) by $\bar{w}_0^{k+1}$, $2\bar{w}_j^{k+1}$ and $\bar{w}_M^{k+1}$ respectively, then adding the results, we obtain

$$\begin{aligned}
&\left\|w^{k+1}\right\|^2/\tau \\
=&(1+ic_1)\left[\frac{1}{h^2}\bar{w}_0^{k+1}(w_1^{k+1} - w_0^{k-1}) + \sum_{j=1}^{M-1}\bar{w}_j^{k+1}{\delta_x}^2 w_j^{k+1}\right. \\
&\left.+\frac{1}{h^2}\bar{w}_M^{k+1}(w_{M-1}^{k+1} - w_M^{k+1})\right]h + \left\|w^{k+1}\right\|^2 \\
&-(1+ic_2)\left[\frac{1}{2}\left|w_0^k\right|^2\cdot\left|w_0^{k+1}\right|^2 + \sum_{j=1}^{M-1}\left|w_j^k\right|^2\cdot\left|w_j^{k+1}\right|^2 + \frac{1}{2}\left|w_M^k\right|^2\cdot\left|w_M^{k+1}\right|^2\right]h \\
=&-(1+ic_1)\sum_{j=0}^{M-1}\left|\frac{1}{h}(w_{j+1}^{k+1} - w_j^{k+1})\right|^2 + \left\|w^{k+1}\right\|^2 \\
&-(1+ic_2)\left[\frac{1}{2}\left|w_0^k\right|^2\cdot\left|w_0^{k+1}\right|^2 + \sum_{j=1}^{M-1}\left|w_j^k\right|^2\cdot\left|w_j^{k+1}\right|^2 + \frac{1}{2}\left|w_M^k\right|^2\cdot\left|w_M^{k+1}\right|^2\right]h.
\end{aligned}$$

Taking the real part, we have

$$\left\|w^{k+1}\right\|^2/\tau \le \left\|w^{k+1}\right\|^2.$$

Thus $\left\|w^{k+1}\right\|^2 = 0$ when $\tau < 1$. That is, (4) has only trivial solution. Therefore, (3) determines w^{k+1} uniquely. By the inductive principle, this completes the proof.

3. Convergence

Lemma 1. *If a and b are positive and v_1,v_2,$\cdots$,v_l nonnegtive and satisfy*

$$v_k \le (1+a\tau)v_{k-1} + b\tau, \quad k = 2,3,\cdots,l$$

then

$$v_k \le (v_1 + \frac{b}{a})\exp[a(k-1)\tau], k = 1,2,\cdots,l$$

Theorem 2. *Suppose* (1.1') *and* (1.2)-(1.3) *have sufficiently smooth solution, then the difference scheme* (3) *is convergent in energy norm with the convergence rate of order* $O(\tau^2 + h^2)$.

Proof. Differentiate (1.1') with respect to x, we obtain

$$\frac{\partial}{\partial t}(\frac{\partial w}{\partial x}) = (1+ic_1)\frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial x} - (1+ic_2)(2\,|w|^2\,\frac{\partial w}{\partial x} + w^2\frac{\partial \bar{w}}{\partial x}) + \frac{\partial}{\partial x}f(x,t) \qquad (5.1)$$

Noticing

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \qquad (5.2)$$

we have

$$(1+ic_1)\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} + \left.\frac{\partial}{\partial x}f(x,t)\right|_{x=0} = 0$$

or,

$$\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} = -\frac{1}{1+ic_1}\left.\frac{\partial}{\partial x}f(x,t)\right|_{x=0} \qquad (5.3)$$

Substituting (5.2) and (5.3) into Taylor expansion

$$w|_{x=h} = w|_{x=0} + h\left.\frac{\partial w}{\partial x}\right|_{x=0} + \frac{1}{2}h^2\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} + \frac{1}{6}h^3\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} + O(h^4)$$

we obtain

$$\begin{aligned}\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} &= \frac{2}{h^2}(w|_{x=h} - w|_{x=0}) - \frac{1}{3}h\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} + O(h^2)\\ &= \frac{2}{h^2}(w|_{x=h} - w|_{x=0}) + \frac{1}{3}h\frac{1}{1+ic_1}\left.\frac{\partial}{\partial x}f(x,t)\right|_{x=0} + O(h^2).\end{aligned}$$

Noticing

$$\left.\frac{\partial w}{\partial t}\right|_{x=0} = (1+ic_1)\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} + w|_{x=0} - (1+ic_2)\left.|w|^2\,w\right|_{x=0} + f(0,t)$$

we get

$$\begin{aligned}\left.\frac{\partial w}{\partial t}\right|_{x=0} &= (1+ic_1)\left[\frac{2}{h^2}(w|_{x=h} - w|_{x=0}) + \frac{1}{3}h\frac{1}{1+ic_1}\left.\frac{\partial}{\partial x}f(x,t)\right|_{x=0} + O(h^2)\right]\\ &\quad + w|_{x=0} - (1+ic_2)\left.|w|^2\,w\right|_{x=0} + f(0,t)\\ &= (1+ic_1)\frac{2}{h^2}(w|_{x=h} - w|_{x=0}) + w|_{x=0} - (1+ic_2)\left.|w|^2\,w\right|_{x=0}\\ &\quad + f(\frac{h}{3},t) + O(h^2).\end{aligned} \qquad (6.1)$$

Similarly, we have

$$\left.\frac{\partial w}{\partial t}\right|_{x=1} = (1+ic_1)\frac{2}{h^2}(w|_{x=1-h} - w|_{x=1}) + w|_{x=1} - (1+ic_2)\left.|w|^2\,w\right|_{x=1} + f(1-\frac{h}{3},t) + O(h^2). \qquad (6.2)$$

Define net functions

$$W_j^k = w(x_j,t_k), \quad \phi_j^k = W_j^k - w_j^k.$$

Averging the equations (6.1) when $t = t_{k-1}$ and $t = t_{k+1}$, we obtain

$$\Delta_t W_0^k = (1+ic_1)\frac{2}{h^2}(W_1^{\hat{k}} - W_0^{\hat{k}}) + W_0^{\hat{k}} - (1+ic_2)\left|W_0^k\right|^2 W_0^{\hat{k}} + f(\frac{h}{3}, t_k) + O(\tau^2+h^2). \quad (7.1)$$

Similarly, we have

$$\Delta_t W_M^k = (1+ic_1)\frac{2}{h^2}(W_{M-1}^{\hat{k}} - W_M^{\hat{k}}) + W_M^{\hat{k}} - (1+ic_2)\left|W_M^k\right|^2 W_M^{\hat{k}} + f(1-\frac{h}{3}, t_k) + O(\tau^2+h^2). \quad (7.2)$$

Farthermore, from Taylor expansion, we have

$$\Delta_t W_j^k = (1+ic_1)\delta_x{}^2 W_j^{\hat{k}} + W_j^{\hat{k}} - (1+ic_2)\left|W_j^k\right|^2 W_j^{\hat{k}} + f(x_j, t_k) + O(\tau^2+h^2). \quad (7.3)$$

$$W_j^0 = w_0(x_j), \qquad W_j^1 = w_0(x_j) + \tau w_1(x_j) + O(\tau^2+h^2). \quad (7.4)$$

Subtracting (3) from (7), we obtain the error equations

$$\Delta_t \phi_0^k = (1+ic_1)\frac{2}{h^2}(\phi_1^{\hat{k}} - \phi_0^{\hat{k}}) + \phi_0^{\hat{k}} - (1+ic_2)\Big[(W_0^k\bar{\phi}_0^k + \bar{w}_0^k\phi_0^k)W_0^{\hat{k}}$$
$$+ \left|w_0^k\right|^2 \phi_0^{\hat{k}}\Big] + P_0^k, \quad 1 \le k \le K-1 \quad (8.1)$$

$$\Delta_t \phi_j^k = (1+ic_1)\delta_x^2 \phi_j^{\hat{k}} + \phi_j^{\hat{k}} - (1+ic_2)\Big[(W_j^k\bar{\phi}_j^k + \bar{w}_j^k\phi_j^k)W_j^{\hat{k}}$$
$$+ \left|w_j^k\right|^2 \phi_j^{\hat{k}}\Big] + P_j^k, \quad 1 \le j \le M-1,\ 1 \le k \le K-1 \quad (8.2)$$

$$\Delta_t \phi_M^k = (1+ic_1)\frac{2}{h^2}(\phi_{M-1}^{\hat{k}} - \phi_M^{\hat{k}}) + \phi_M^{\hat{k}} - (1+ic_2)\Big[(W_M^k\bar{\phi}_M^k + \bar{w}_M^k\phi_M^k)W_M^{\hat{k}}$$
$$+ \left|w_M^k\right|^2 \phi_M^{\hat{k}}\Big] + P_M^k, \quad 1 \le k \le K-1 \quad (8.3)$$

$$\phi_j^0 = 0, \quad \phi_j^1 = Q_j, \quad 0 \le j \le M \quad (8.4)$$

where P_j^k and Q_j are the truncation errors of difference scheme (3) and there exists a constant c_0 such that

$$\left|P_j^k\right| \le c_0(\tau^2+h^2), \quad |Q_j| \le c_0(\tau^2+h^2), \quad 0 \le j \le M, \quad 1 \le k \le K-1 \quad (9)$$

Denote

$$s = \max_{0\le x\le 1,\, 0\le t\le T} |w(x,t)|\,. \quad (10)$$

We prove by inductive method that

$$\left\|\phi^k\right\| \le c(\tau^2+h^2), \quad 0 \le k \le K \quad (11)$$

where

$$c = c_0\sqrt{1 + \frac{1}{2+(1+|c_2|)(2s+1)s}}\exp\{3[2+(1+|c_2|)(2s+1)s]T\}.$$

From (8.4) and (9), we have

$$\left\|\phi^0\right\| = 0, \quad \left\|\phi^1\right\| \le c_0(\tau^2+h^2). \quad (12)$$

Therefore (11) is valid for $k=0$ and $k=1$. Now suppose that (11) is true for k from 0 to $l(1\le l\le K-1)$. It follows from the inductive assumption that

$$\left|\phi_j^k\right| \le c(\tau^2+h^2)h^{-\frac{1}{2}} \le \tilde{c}(h^{2\epsilon}+h^{\frac{3}{2}}) \le 1, \quad 0\le j\le M, 1\le k\le l \tag{13.1}$$

for small ϵ and therefore

$$\left|w_j^k\right| = \left|W_j^k-\phi_j^k\right| \le \left|W_j^k\right| + \left|\phi_j^k\right| \le s+1, \quad 0\le j\le M, 1\le k\le l. \tag{13.2}$$

For $1\le k\le l$, multiplying (8.1-3) by $\frac{1}{2}\bar{\phi}_0^{\hat{k}}$, $\bar{\phi}_j^{\hat{k}}$ and $\frac{1}{2}\bar{\phi}_M^{\hat{k}}$ respectively, then adding the results, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}\bar{\phi}_0^{\hat{k}}\Delta_t\phi_0^k + \sum_{j=1}^{M-1}\bar{\phi}_j^{\hat{k}}\Delta_t\phi_j^k + \frac{1}{2}\bar{\phi}_M^{\hat{k}}\Delta_t\phi_M^k\right)h \\
&= -(1+ic_1)\sum_{j=0}^{M-1}\left|\frac{1}{h}(\phi_{j+1}^{\hat{k}}-\phi_j^{\hat{k}})\right|^2 h + \left\|\phi^{\hat{k}}\right\|^2 \\
&- (1+ic_2)\left[\frac{1}{2}(W_0^k\bar{\phi}_0^k+\bar{w}_0^k\phi_0^k)W_0^{\hat{k}}\bar{\phi}_0^{\hat{k}}\right. \\
&+ \left.\sum_{j=1}^{M-1}(W_j^k\bar{\phi}_j^k+\bar{w}_j^k\phi_j^k)W_j^{\hat{k}}\bar{\phi}_j^{\hat{k}} + \frac{1}{2}(W_M^k\bar{\phi}_M^k+\bar{w}_M^k\phi_M^k)W_M^{\hat{k}}\bar{\phi}_M^{\hat{k}}\right]h \\
&- (1+ic_2)\left(\frac{1}{2}\left|w_0^k\right|^2\cdot\left|\phi_0^{\hat{k}}\right|^2 + \sum_{j=1}^{M-1}\left|w_j^k\right|^2\cdot\left|\phi_j^{\hat{k}}\right|^2 + \frac{1}{2}\left|w_M^k\right|^2\cdot\left|\phi_M^{\hat{k}}\right|^2\right)h \\
&+ \left(\frac{1}{2}P_0^k\cdot\bar{\phi}_0^{\hat{k}} + \sum_{j=1}^{M-1}P_j^k\cdot\bar{\phi}_j^{\hat{k}} + \frac{1}{2}P_M^k\cdot\bar{\phi}_M^{\hat{k}}\right)h, \quad 1\le k\le l.
\end{aligned}$$

Taking the real part and using (10), (13) and (9), we get

$$\begin{aligned}
&(\left\|\phi^{k+1}\right\|^2 - \left\|\phi^{k-1}\right\|^2)/(4\tau) \\
\le& \left\|\phi^{\hat{k}}\right\|^2 + (1+|c_2|)(2s+1)s\left(\frac{1}{2}\left|\phi_0^k\right|\cdot\left|\phi_0^{\hat{k}}\right| + \sum_{j=1}^{M-1}\left|\phi_j^k\right|\cdot\left|\phi_j^{\hat{k}}\right| + \frac{1}{2}\left|\phi_M^k\right|\cdot\left|\phi_M^{\hat{k}}\right|\right)h \\
&+ \left\|P^k\right\|^2 + \left\|\phi^{\hat{k}}\right\|^2 \\
\le& \left\|\phi^{\hat{k}}\right\|^2 + \frac{1}{2}(1+|c_2|)(2s+1)s(\left\|\phi^k\right\|^2 + \left\|\phi^{\hat{k}}\right\|^2) + \left\|P^k\right\|^2 + \left\|\phi^{\hat{k}}\right\|^2 \\
\le& \frac{1}{2}[2+\frac{1}{2}(1+|c_2|)(2s+1)s](\left\|\phi^{k+1}\right\|^2 + \left\|\phi^{k-1}\right\|^2) \\
&+ \frac{1}{2}(1+|c_2|)(2s+1)s\left\|\phi^k\right\|^2 + [c_0(\tau^2+h^2)]^2
\end{aligned}$$

that is,

$$\begin{aligned}
&\{1-2[2+\frac{1}{2}(1+|c_2|)(2s+1)s]\tau\}\left\|\phi^{k+1}\right\|^2 \\
\le&\{1+2[2+\frac{1}{2}(1+|c_2|)(2s+1)s]\tau\}\left\|\phi^{k-1}\right\|^2
\end{aligned}$$

$$+ 2(1 + |c_2|)(2s + 1)s\tau \left\|\phi^k\right\|^2 + 4\tau[c_0(\tau^2 + h^2)]^2, \quad 1 \le k \le l.$$

Therefore, when $6[2 + \frac{1}{2}(1 + |c_2|)(2s + 1)s]\tau \le 1$,

$$\begin{aligned}\left\|\phi^{k+1}\right\|^2 \le & \{1 + 6[2 + \frac{1}{2}(1 + |c_2|)(2s + 1)s]\tau\} \left\|\phi^{k-1}\right\|^2 \\ & + 3(1 + |c_2|)(2s + 1)s\tau \left\|\phi^k\right\|^2 + 6\tau[c_0(\tau^2 + h^2)]^2, \quad 1 \le k \le l.\end{aligned}$$

It follows easily from this inequality that

$$\begin{aligned}&\max(\left\|\phi^{k+1}\right\|^2, \left\|\phi^k\right\|^2) \\ \le & \{1 + 6[2 + (1 + |c_2|)(2s + 1)s]\tau\} \max(\left\|\phi^k\right\|^2, \left\|\phi^{k-1}\right\|^2) \\ & + 6\tau[c_0(\tau^2 + h^2)]^2, \quad 1 \le k \le l.\end{aligned}$$

Applying Lemma 1 and noticing (12), we know

$$\begin{aligned}&\max(\left\|\phi^{l+1}\right\|^2, \left\|\phi^l\right\|^2) \\ \le & \{\max(\left\|\phi^1\right\|^2, \left\|\phi^0\right\|^2) + \frac{[c_0(\tau^2 + h^2)]^2}{2 + (1 + |c_2|)(2s + 1)s}\} \exp\{6[2 + (1 + |c_2|)(2s + 1)s]l\tau\} \\ \le & [1 + \frac{1}{2 + (1 + |c_2|)(2s + 1)s}][c_0(\tau^2 + h^2)]^2 \exp\{6[2 + (1 + |c_2|)(2s + 1)s]T\}\end{aligned}$$

or,

$$\left\|\phi^{l+1}\right\| \le c(\tau^2 + h^2).$$

That means, (11) is valid for $k = l + 1$. This completes the proof.

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