Journal of Computational Mathematics, Vol.15, No.4, 1997, 365–384.

MODELLING AND NUMERICAL SOLUTIONS OF A GAUGE PERIODIC TIME DEPENDENT GINZBURG-LANDAU MODEL FOR TYPE-II SUPERCONDUCTORS*1)

Zhi-ming Chen
(*Institute of Mathematics, Chinese Academy of Sciences, Beijing, China*)

Abstract

In this paper we seek the solutions of the time dependent Ginzburg-Landau model for type-II superconductors such that the associated physical observables are spatially periodic with respect to some lattice whose basic lattice cell is not necessarily rectangular. After appropriately fixing the gauge, the model can be formulated as a system of nonlinear parabolic partial differential equations with quasi-periodic boundary conditions. We first give some results concerning the existence, uniqueness and regularity of solutions and then we propose a semi-implicit finite element scheme solving the system of nonlinear partial differential equations and show the optimal error estimates both in the L^2 and energy norm. We also report on some numerical results at the end of the paper.

1. Introduction

Central to the theory of type-II superconductors is Abrikosov's characterization of the mixed state as a lattice-like arrangement of quantized flux lines, or vortices of superconducting electron pairs. The Abrikosov's vortex lattice, which has also been observed in experiments, is the solutions of the Ginzburg-Landau (GL) equations with a type of spatial periodicity. Recently there have been several authors studied the gauge periodic solutions of the *GL* superconductivity model from different point of views[1,10,11,17]. Roughly speaking, gauge periodic solutions are those solutions whose observables are spatially periodic with respect to some lattice (cf. §2). One of the key procedures in those studies is fixing the gauge. It is shown that after a preliminary gauge transformation, any gauge periodic solution can be assumed to have the form that the complex order parameter ψ satisfies some quasi-periodic boundary condition and the magnetic vector potential $\mathbf{A}$ is the sum of some periodic, divergence free function and $-\alpha G$ for some real constant α and $G(x) = (x_2, -x_1)^{\mathrm{T}}$.

The time-dependent Ginzburg-Landau (TDGL) model derived by Gor'kov and Èliashberg [15] from averaging the microscopic Bardeen-Cooper-Schrieffer theory offers a useful starting point in studying the dynamics of superconductivity. After appropriate nondimensionalization, the TDGL model can be formulated as in the following system of nonlinear partial differential equations (cf. e.g. [7], [3], [4]):

$$\eta\frac{\partial\psi}{\partial t} + i\eta\kappa\phi\psi + \Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2\psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

* Received June 29, 1995.
1)The author gratefully acknowledges the support of K.C. Wong Education Foundation, Hongkong. This work was also supported by the National Natural Science Foundation of China.

$$\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi + \mathbf{curl}\,\mathrm{curl}\mathbf{A} + \Re\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\bar{\psi}\Big] = 0, \tag{1.2}$$

where $\Re[\cdot]$ denotes the real part of the quantity in the brackets $[\cdot]$, and curl, **curl** denote the curl operators on $\mathbb{R}^2$ defined by

$$\mathrm{curl}\ \mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \quad \mathbf{curl}\, v = \Big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\Big)^{\mathrm{T}}.$$

Here ψ is a complex valued function and is usually referred to as the order parameter so that $|\psi|^2$ gives the relative density of the superconducting electron pairs; $\bar{\psi}$ is the complex conjugate of ψ; $\mathbf{A}$ is a real vector potential for the total magnetic field; ϕ is a real scalar function called electric potential; $\kappa > 0$ is the Ginzburg-Landau parameter which satisfies $\kappa > 1/\sqrt{2}$ for type-II superconductors; and $\eta > 0$ is a dimensionless constant.

The TDGL model (1.1)–(1.2) with Neumann boundary conditions has been studied in [5], [3], [7], [8], [12] with different gauge choices. The studies in [5] and [3] indicate that in contrast to the stationary case, where the Coloumb gauge div $\mathbf{A}$=0 is usually used, the Lorentz gauge div $\mathbf{A}+\phi = 0$ is more appropriate both in proving the regularity of solutions of the TDGL model and in designing numerically convergent algorithms solving the TDGL model. For more information about the subject of superconductivity, the reader may consult the two recent survey articles [2] and [9] and the references therein. We also refer to [10] for more discussions on the motivations of studying the periodic model for type-II superconductivity.

Our goal in this paper is to look for the solutions to (1.1)–(1.2) such that the associated physical observables (e.g. superconducting electron pairs, current, magnetic field, etc.) are spatially periodic, particularly for those solutions whose periodicity is supported on the hexagonal lattice. This problem was first considered in [13] on the rectangular lattice by using the Coloumb gauge. In that paper, the existence of weak solutions was obtained by using the method of lines and the problem of the asymptotic behavior for the time $t \to \infty$ was considered. In this paper, we first introduce the gauge periodic TDGL model, fix the gauge (Lorentz gauge) and then present some results concerning the existence, uniqueness and regularity of the solutions in §2. In §3 we propose a semi-implicit finite element scheme solving the gauge periodic TDGL model and in §4 we prove the optimal error estimates for the scheme both in the L^2 and energy norm. In §5 we report on a numerical example and in §6 we give some concluding remarks.

In the remainder of this section we introduce some of the notations to be used in the paper. Let $\mathcal{L}$ denote a planar lattice which consists of basis vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. After rotation, we may always assume that the lattice $\mathcal{L}$ has a basis vector that is real. Thus we assume in the following that $\mathcal{L}$ is generated by $\mathbf{t}_1 = (r_1, 0)$ and $\mathbf{t}_2 = (r_2\cos\theta, r_2\sin\theta)$ with $r_1, r_2 > 0$ and $0 < \theta < \pi$. We denote Ω the open parallelogram generated by $\mathbf{t}_1$ and $\mathbf{t}_2$. In this paper we say that a function f is periodic if $f(x+\mathbf{t}_k) = f(x)$ for $k = 1, 2$ and a.e. $x \in \mathbb{R}^2$.

For any bounded open set $\mathcal{D} \subset \mathbb{R}^2$ and each integer $m \geq 0$ and real p with $1 \leq p \leq \infty$, we denote by $W^{m,p}(\mathcal{D})$ the standard Sobolev space of real functions having all their derivatives of order up to m in the Lebesgue space $L^p(\mathcal{D})$. When $p = 2$, $W^{m,2}(\mathcal{D})$ is denoted by $H^m(\mathcal{D})$. We also use the space

$$H^m_{\text{loc}}(\mathbb{R}^2) = \{ \ u : u \in H^m(\mathcal{D}) \ \ \forall \text{ bounded open } \mathcal{D} \subset \mathbb{R}^2 \ \}.$$

If B denotes some Banach space of real scalar functions, the corresponding space of complex scalar functions will be denoted by its calligraphic form $\mathcal{B}$ and the corresponding space of real vector-valued functions, each of its components belonging to B, will be denoted by its boldfaced form $\mathbf{B}$. However, we use $\|\cdot\|_B$ to denote the norms of the Banach spaces B, $\mathcal{B}$ or $\mathbf{B}$.

For any Banach space X and any integer $m \geq 0$, real p with $1 \leq p < \infty$, denote

$$W^{m,p}(0,T;X) = \left\{u(t) \in X \text{ for a.e. } t \in (0,T), \ \int_0^T (\|u\|_X^p + \cdots + \|u^{(m)}\|_X^p) \, \mathrm{d}t \ < \infty\right\}$$

with the norm

$$\|u\|_{W^{m,p}(0,T;X)} = \left[\int_0^T \left(\|u\|_X^p + \cdots + \|u^{(m)}\|_X^p\right) \mathrm{d}t\right]^{1/p}.$$

As usual, we write $L^p(0,T;X) = W^{0,p}(0,T;X)$ and $H^1(0,T;X) = W^{1,2}(0,T;X)$. We will also make use of the following spaces

$$\begin{aligned} L^2(0,T;H^m_{\text{loc}}(\mathbb{R}^2)) &= \{ \ u : u \in L^2(0,T;H^m(\mathcal{D})) \ \ \forall \text{ bounded open } \mathcal{D} \subset \mathbb{R}^2\}, \\ H^1(0,T;H^m_{\text{loc}}(\mathbb{R}^2)) &= \{ \ u : u \in H^1(0,T;H^m(\mathcal{D})) \ \ \forall \text{ bounded open } \mathcal{D} \subset \mathbb{R}^2\}, \end{aligned}$$

where $m \geq 0$ is an integer.

Now we give a precise definition of gauge invariance. Let

$$H^{2,1}_{\text{loc}} = L^2(0,T;H^2_{\text{loc}}(\mathbb{R}^2)) \cap H^1(0,T;L^2_{\text{loc}}(\mathbb{R}^2)) \quad \text{and} \quad H^{1,0}_{\text{loc}} = L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^2)).$$

We note first that if $(\psi, \mathbf{A}, \phi) \in \mathcal{H}^{2,1}_{\text{loc}} \times \mathbf{H}^{2,1}_{\text{loc}} \times H^{1,0}_{\text{loc}}$ is a strong solution of (1.1)–(1.2), then for any $\chi \in L^2(0,T;H^3_{\text{loc}}(\mathbb{R}^2)) \cap L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^2))$, the triple $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi}) = G_\chi(\psi, \mathbf{A}, \phi)$ given by

$$\widetilde{\psi} = \psi \exp(i\kappa\chi), \quad \tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi, \quad \widetilde{\phi} = \phi - \frac{\partial \chi}{\partial t} \tag{1.3}$$

is also a strong solution of (1.1)–(1.2). In the following, two triples $(\psi, \mathbf{A}, \phi)$ and $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi})$ are said to be gauge equivalent if and only if there exists a χ such that $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi}) = G_\chi(\psi, \mathbf{A}, \phi)$.

2. The Gauge Periodic Model

2.1 Fixing the gauge

As indicated in §1, the purpose of this paper is to look for the solutions of (1.1)–(1.2) such that the observables induced from these quantities are periodic with respect to the basic parallelogram Ω. The following definition of time dependent gauge periodicity in [13] according to [1] characterizes this requirement.

Definition 2.1. *Let $\mathcal{L} = \{m\mathbf{t}_1 + n\mathbf{t}_2 : m, n$ are integers $\}$ be a planar lattice with respect to the basic parallelogram Ω. A time dependent state $(\psi, \mathbf{A}, \phi)$ is called gauge periodic if for each $s \in \mathcal{L}$, the translated state $(\psi, \mathbf{A}, \phi)(x+s,t)$ is gauge equivalent to $(\psi, \mathbf{A}, \phi)(x,t)$.*

It is obvious that this definition implies that there exists a family of functions $g^s \in L^2(0,T; H^3_{\mathrm{loc}}(\mathbb{R}^2)) \cap H^1(0,T; H^1_{\mathrm{loc}}(\mathbb{R}^2))$ such that for each $s \in \mathcal{L}$, we have

$$\begin{aligned} \psi(x+s,t) &= \psi(x,t)e^{i\kappa g^s(x,t)}, \\ \mathbf{A}(x+s,t) &= \mathbf{A}(x,t) + \nabla g^s(x,t), \\ \phi(x+s,t) &= \phi(x,t) - \frac{\partial g^s(x,t)}{\partial t}. \end{aligned} \tag{2.1}$$

Given two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, let $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$ denote their vector product. The following theorem is the main result of this section.

Theorem 2.1. *Suppose that $(\psi, \mathbf{A}, \phi) \in \mathcal{H}^{2,1}_{\mathrm{loc}} \times \mathbf{H}^{2,1}_{\mathrm{loc}} \times H^{1,0}_{\mathrm{loc}}$ is gauge periodic. Then $(\psi, \mathbf{A}, \phi)$ is gauge equivalent to $(\tilde\psi, \tilde{\mathbf{A}}, \tilde\phi)$ satisfying*

(a) $\tilde{\mathbf{A}} = \mathbf{P} + G$, *where* $\mathbf{P}$ *is periodic and* $G(x) = -\alpha(x_2, -x_1)^{\mathrm{T}}$ *for some real constant* α;

(b) $\tilde\psi(x + \mathbf{t}_k, t) = \tilde\psi(x,t)e^{i\kappa g_k(x)}$, *where* $g_k(x) = -\alpha(x \times \mathbf{t}_k)$, $k = 1, 2$;

(c) $\tilde\phi + \operatorname{div} \tilde{\mathbf{A}} = 0$.

Proof. First we note that by the well-known embedding theorem we have $(\psi, \mathbf{A}) \in C(0,T; H^1_{\mathrm{loc}}(\mathbb{R}^2))$. Let $(\psi_0, \mathbf{A}_0) = (\psi(x,0), \mathbf{A}(x,0)) \in \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^2)) \times \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$ be the initial state of the system. From the restriction of (2.1) to time $t = 0$ we know that $(\psi_0, \mathbf{A}_0)$ is steady state gauge periodic in the sense that there exists a family of functions $h^s \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that for each $s \in \mathcal{L}$,

$$\psi_0(x+s) = \psi_0(x)e^{i\kappa h^s(x)}, \quad \mathbf{A}_0(x+s) = \mathbf{A}_0(x) + \nabla h^s(x).$$

Thus from the results proved in [1], [10], [17] we know that there exists a function $\chi_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\tilde\psi_0 = \psi_0 e^{i\kappa\chi_0}, \quad \tilde{\mathbf{A}}_0 = \mathbf{A}_0 + \nabla\chi_0 \tag{2.2}$$

satisfy

(i) $\tilde{\mathbf{A}}_0 = \mathbf{P}_0 + G$, where $\mathbf{P}_0$ is some divergence free periodic function and $G(x) = -\alpha(x_2, -x_1)^{\mathrm{T}}$ for some real constant α;

(ii) $\tilde\psi_0(x + \mathbf{t}_k) = \tilde\psi_0(x)e^{i\kappa g_k(x)}$, where $g_k(x) = -\alpha(x \times \mathbf{t}_k)$, $k = 1, 2$.

We also note that the gauge transformation function $\chi_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ can be made unique by requiring that χ_0 has zero mean over the basic parallelogram Ω.

Now choose $\mathbf{Q}$ to satisfy

$$\frac{\partial \mathbf{Q}}{\partial t} - \Delta\mathbf{Q} = \frac{\partial \mathbf{A}}{\partial t} + \nabla\phi + \mathbf{curl}\,\mathrm{curl}\,\mathbf{A} \quad \forall (x,t) \in \mathbb{R}^2 \times (0,T), \tag{2.3}$$

$$\mathbf{Q} \text{ is periodic}, \quad \mathbf{Q}(x,0) = \mathbf{P}_0(x) \quad \forall x \in \mathbb{R}^2. \tag{2.4}$$

Note that, in view of (2.1), $\partial\mathbf{A}/\partial t + \nabla\phi + \mathbf{curl}\,\mathrm{curl}\,\mathbf{A} \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ is periodic. Thus there exists a unique $\mathbf{Q} \in \mathbf{H}^{2,1}_{\mathrm{loc}}$ which satisfies (2.3)–(2.4).

Let $w = \text{curl } (\mathbf{Q} - \mathbf{A} + G)$. Then we know from (2.3)–(2.4) that

$$\frac{\partial w}{\partial t} - \Delta w = 0 \text{ in the sense of distributions,}$$
$$w \text{ is periodic, } w|_{t=0} = \text{curl } (\mathbf{Q}_0 - \mathbf{A}_0 + G) = \text{curl } \nabla \chi_0 = 0,$$

where we have used the relation (i). Thus $w = \text{curl } (\mathbf{Q} - \mathbf{A} + G) = 0$. On the other hand, we have from the embedding theorem that $\mathbf{Q} - \mathbf{A} + G \in C(0, T; \mathbf{H}^1_{\text{loc}}(\mathbb{R}^2))$. Therefore, for each time $t \in [0, T]$, there exists a unique $\chi(\cdot, t) \in H^2_{\text{loc}}(\mathbb{R}^2)$ such that

$$\mathbf{Q}(x,t) - \mathbf{A}(x,t) + G(x) = \nabla \chi(x,t) \text{ a.e. in } \mathbb{R}^2 \text{ and } \int_\Omega \chi(x,t)\, \mathrm{dx} = 0. \tag{2.5}$$

It is clear that $\chi(x, 0) = \chi_0(x)$. By applying Poincaré inequality, we know from (2.5) that $\chi \in C(0, T; H^2_{\text{loc}}(\mathbb{R}^2))$. Again by (2.5) we have

$$\nabla \frac{\partial \chi}{\partial t} = \frac{\partial}{\partial t}(\mathbf{Q} - \mathbf{A}) \text{ in the sense of distributions,}$$

which yields $\chi \in H^1(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$. As a result of (2.3) and (2.5), we get

$$\nabla \Big(\frac{\partial \chi}{\partial t} - \Delta \chi \Big) = \nabla (\text{div } \mathbf{A} + \phi) \text{ in the sense of distributions,}$$

which implies that there exists some function $g \in L^2(0, T)$ such that

$$\frac{\partial \chi}{\partial t} - \Delta \chi = \text{div } \mathbf{A} + \phi + g(t).$$

Hence, by letting $\tilde{\chi} = \chi + \int_0^t g(\tau) \mathrm{d}\tau$, we obtain that $\tilde{\chi} \in C(0, T; H^2_{\text{loc}}(\mathbb{R}^2)) \cap H^1(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$ satisfies

$$\frac{\partial \tilde{\chi}}{\partial t} - \Delta \tilde{\chi} = \text{div } \mathbf{A} + \phi \text{ in } \mathbb{R}^2 \times (0, T), \tag{2.6}$$
$$\tilde{\chi}|_{t=0} = \chi|_{t=0} = \chi_0 \text{ in } \mathbb{R}^2. \tag{2.7}$$

Moreover, since $\text{div } \mathbf{A} + \phi \in L^2(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$ and $\chi_0 \in H^2_{\text{loc}}(\mathbb{R}^2)$, we conclude from the interior regularity theory for parabolic equations that $\tilde{\chi} \in L^2(0, T; H^3_{\text{loc}}(\mathbb{R}^2))$.

Let now $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi}) = G_{\tilde{\chi}}(\psi, \mathbf{A}, \phi)$. Then $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi})$ satisfies the parts (a) and (c) of the theorem. In order to prove the part (b) of the theorem, note first that $(\widetilde{\psi}, \tilde{\mathbf{A}}, \widetilde{\phi})$ is gauge periodic, thus we know that for $k = 1$ or 2, there exists $g^{t_k} \in L^2(0, T; H^3_{\text{loc}}(\mathbb{R}^2)) \cap H^1(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$ such that

$$\widetilde{\psi}(x + \mathbf{t}_k, t) = \widetilde{\psi}(x, t) e^{i\kappa g^{t_k}(x,t)}, \tag{2.8}$$
$$\tilde{\mathbf{A}}(x + \mathbf{t}_k, t) = \tilde{\mathbf{A}}(x, t) + \nabla g^{t_k}(x, t), \tag{2.9}$$
$$\widetilde{\phi}(x + \mathbf{t}_k, t) = \widetilde{\phi}(x, t) - \frac{\partial g^{t_k}(x, t)}{\partial t}. \tag{2.10}$$

Since $\tilde{\chi}|_{t=0} = \chi_0$, we have $\widetilde{\psi}|_{t=0} = \widetilde{\psi}_0$ the same as that in (2.2). Hence it follows from (2.8) and the relation (ii) that

$$g^{t_k}(x,0) = g_k(x) + \frac{2\pi n}{\kappa}, \quad k = 1,2 \tag{2.11}$$

for some integer n. On the other hand, it follows from the gauge relation $\widetilde{\phi} + \operatorname{div}\tilde{\mathbf{A}} = 0$ and (2.9)–(2.10) that $\frac{\partial g^{t_k}}{\partial t} - \Delta g^{t_k} = 0$ in $\mathbb{R}^2 \times (0,T)$, $k = 1,2$ which, along with (2.11), implies that $g^{t_k}(x,t) \equiv g^{t_k}(x,0) = g_k(x) + \frac{2\pi n}{\kappa}$, $k = 1,2$. Now the part (b) of the theorem follows from (2.8). □

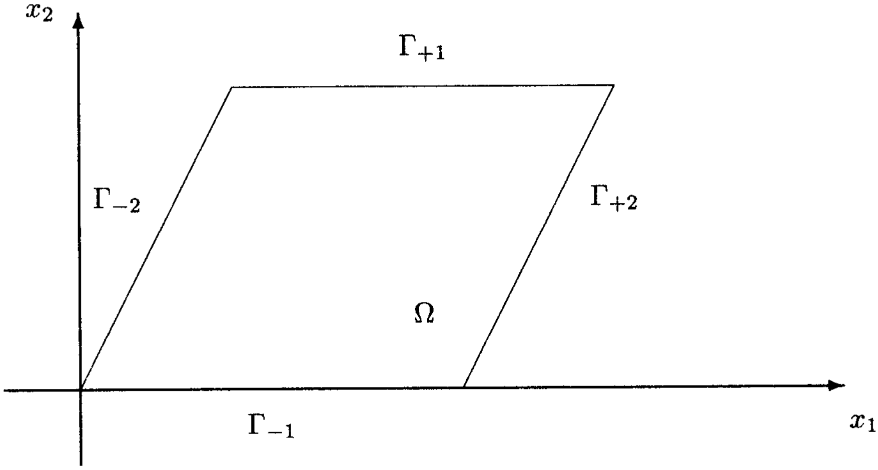

Fig. 1 The basic parallelogram Ω and its four segments.

Denote the four sides of the parallelogram Ω by $\Gamma_{+1}, \Gamma_{-1}, \Gamma_{+2}, \Gamma_{-2}$ using the convention of Fig.1. The corresponding unit outer normal vectors will be denoted by $\mathbf{n}_{+1}, \mathbf{n}_{-1}, \mathbf{n}_{+2}, \mathbf{n}_{-2}$, respectively. Note that for $k = 1$ or 2, Γ_{+k} is the locus of points $y \in \mathbb{R}^2$ such that $y = x + \mathbf{t}_k$ for $x \in \Gamma_{-k}$. It is also clear that $\mathbf{n}_{+k} = -\mathbf{n}_{-k}$ for $k = 1,2$.

As a result of Theorem 2.1, the gauge periodic TDGL model can be formulated as follows:

$$\eta\frac{\partial\psi}{\partial t} - i\eta\kappa \operatorname{div}\mathbf{P}\psi + \Big(\frac{i}{\kappa}\nabla + \mathbf{P} + G\Big)^2\psi + (|\psi|^2 - 1)\psi = 0 \quad \text{in } \Omega \times (0,T), \tag{2.12}$$

$$\frac{\partial\mathbf{P}}{\partial t} - \Delta\mathbf{P} + \Re\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{P}\psi + G\psi\Big)\bar{\psi}\Big] = 0 \quad \text{in } \Omega \times (0,T), \tag{2.13}$$

$$\psi(x + \mathbf{t}_k, t) = \psi(x,t)e^{i\kappa g_k(x)} \quad \text{on } \Gamma_{-k} \times (0,T), \quad k = 1,2 \tag{2.14}$$

$$\mathbf{P}(x + \mathbf{t}_k, t) = \mathbf{P}(x,t) \quad \text{on } \Gamma_{-k} \times (0,T), \quad k = 1,2 \tag{2.15}$$

$$\psi(x,0) = \psi_0(x), \quad \mathbf{P}(x,0) = \mathbf{P}_0(x) \quad \text{on } \Omega. \tag{2.16}$$

In order to complete the problem, we add the following natural boundary conditions:

$$\Big(\frac{i}{\kappa}\nabla\psi + G\psi\Big)\Big|_{(x+\mathbf{t}_k,t)} \cdot \mathbf{n}_{+k} = \Big[\Big(\frac{i}{\kappa}\nabla\psi + G\psi\Big)e^{i\kappa g_k}\Big]_{(x,t)} \cdot \mathbf{n}_{+k} \text{ on } \Gamma_{-k} \times (0,T), \tag{2.17}$$

$$\nabla \mathbf{P}(x+\mathbf{t}_k,t) = \nabla \mathbf{P}(x,t) \quad \text{on} \quad \Gamma_{-k}\times(0,T), \quad k=1,2. \tag{2.18}$$

Here $\nabla \mathbf{P} = (\partial P_i/\partial x_j)_{i,j=1}^2$ stands for the gradient matrix of $\mathbf{P}$. The condition (2.17) can be obtained by differentiating the relation in Theorem 2.1(b) with respect to x and using the following obvious identity

$$G(x+\mathbf{t}_k) - G(x) = -\nabla g_k(x) \quad \text{for} \quad k=1,2.$$

Remark 2.1. *For the sake of completeness, we give here some of the physical implications of the parameter α in the model. For details we refer to the discussions in* [1] *and* [10]. *First, the periodicity of $\mathbf{P}$ implies that*

$$\int_\Omega \operatorname{curl}\mathbf{A}\mathrm{dx} = \int_\Omega \operatorname{curl}\mathbf{P}\mathrm{dx} + 2\alpha|\Omega| = 2\alpha|\Omega|,$$

where $|\Omega|$ is the area of the parallelogram Ω. Thus we know that α is equal to the half of the average magnetic field $\overline{B}$ over Ω defined by

$$\overline{B} = \frac{1}{|\Omega|}\int_\Omega h\mathrm{dx} = \frac{1}{|\Omega|}\int_\Omega \operatorname{curl}\mathbf{A}\mathrm{dx}.$$

Therefore, α is gauge invariant. It is also known that the strength of the external magnetic field H_e, which does not explicitly appear in the model, determines the $\overline{B}$ and thus also α. Another remarkable feature of the gauge periodicity is that the size of the lattice cell Ω determines α. To see that, we use Theorem 2.1(*b*) *to get*

$$\psi(x+\mathbf{t}_1+\mathbf{t}_2) = \psi(x+\mathbf{t}_1)e^{i\kappa g_2(x+\mathbf{t}_1)} = \psi(x)e^{i\kappa[g_1(x)+g_2(x+\mathbf{t}_1)]},$$

and

$$\psi(x+\mathbf{t}_1+\mathbf{t}_2) = \psi(x+\mathbf{t}_2)e^{i\kappa g_1(x+\mathbf{t}_2)} = \psi(x)e^{i\kappa[g_2(x)+g_1(x+\mathbf{t}_2)]},$$

Since, by some easy calculations,

$$g_1(x)+g_2(x+\mathbf{t}_1)-g_2(x)-g_1(x+\mathbf{t}_2) = -2\alpha r_1 r_2\sin(\theta),$$

hence, there exists some integer n such that

$$\alpha = \frac{n\pi}{\kappa r_1 r_2 \sin(\theta)} = \frac{n\pi}{\kappa|\Omega|}. \quad \square \tag{2.19}$$

Remark 2.2. *We now give some remarks about the Coulomb gauge used in* [13]. *Let $\Omega=(0,1)\times(0,1)$. It is proved in* [13, *Theorem* 2.1] *that any time dependent gauge periodic triple $(\psi,\mathbf{A},\phi)$ is gauge equivalent to some $(\Psi,\mathbf{P}+\mathbf{C},\Phi)$ such that*

(*a*) $\mathbf{P}$ *is periodic with mean value zero;*

(*b*) $\operatorname{div}\mathbf{P}=0$;

(*c*) $\mathbf{C}(z,t)=k(t)(x_2,-x_1)^{\mathrm{T}}$ *for some time dependent function $k(t)$;*

(*d*) $\Psi(z+s,t)=e^{\kappa k(t)(\bar{s}z-s\bar{z})/2}\Psi(z,t)$ *for $s=1$ or $s=i$;*

(*e*) $\Phi(z+s,t)=\Phi(z,t)-k'(t)(\bar{s}z-s\bar{z})/2i$ *for $s=1$ or $s=i$,*

where $z=x_1+ix_2$. However, by using the arguments leading to (2.19), *we can see from* (*d*) *that*

$$k(t) = -\frac{n(t)\pi}{\kappa} \tag{2.20}$$

with $n(t)$ being some integer valued function. Thus the continuity assumption on $k(t)$ implies that $k(t)$ must be a time independent constant satisfying (2.20) *for some integer n. At this time we know from* (*e*) *that Φ is also periodic.* $\square$

2.2 Solvability and regularity

We begin this section by introducing some function spaces. For each integer $m \geq 1$, we define the space of periodic functions

$$\mathbf{H}^m_{\rm per}(\mathbb{R}^2) = \{\mathbf{A} \in \mathbf{H}^m_{\rm loc}(\mathbb{R}^2) : \mathbf{A}(x+\mathbf{t}_k) = \mathbf{A}(x) \ \text{ for } k=1,2 \text{ and } \forall x \in \mathbb{R}^2 \ \}$$

and the space of quasi-periodic functions

$$\mathcal{H}^m_{\rm qp}(\mathbb{R}^2) = \{\psi \in \mathcal{H}^m_{\rm loc}(\mathbb{R}^2) : \psi(x+\mathbf{t}_k) = \psi(x)e^{i\kappa g_k(x)} \ \text{ for } k=1,2 \text{ and } \forall x \in \mathbb{R}^2 \ \}.$$

From these we define the corresponding spaces of functions restricted to Ω

$$\mathbf{H}^m_{\rm per}(\Omega) = \{ \ \mathbf{A}|_\Omega : \ \mathbf{A} \in \mathbf{H}^m_{\rm per}(\mathbb{R}^2)\} \ \text{ and } \ \mathcal{H}^m_{\rm qp}(\Omega) = \{\psi|_\Omega : \psi \in \mathcal{H}^m_{\rm qp}(\mathbb{R}^2)\}.$$

Let $\mathcal{W}_{\rm qp}(0,T) = L^2(0,T;\mathcal{H}^1_{\rm qp}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\mathbf{W}_{\rm per}(0,T) = L^2(0,T;\mathbf{H}^1_{\rm per}(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))$. We now give a precise definition of the weak formulation of the problem (2.12)–(2.18).

Problem (P) Find $(\psi, \mathbf{P}) \in \mathcal{W}_{\rm qp}(0,T) \times \mathbf{W}_{\rm per}(0,T)$ such that

$$\psi(x,0) = \psi_0(x), \quad \mathbf{P}(x,0) = \mathbf{P}_0(x) \tag{2.21}$$

and

$$\begin{aligned}
\eta \int_0^T \int_\Omega \frac{\partial \psi}{\partial t}\bar{\omega}\,\mathrm{dxdt} &- i\eta\kappa \int_0^T \int_\Omega \operatorname{div}\mathbf{P}\psi\bar{\omega}\,\mathrm{dxdt} \\
&+ \int_0^T \int_\Omega \Big(\frac{i}{\kappa}\nabla\psi + \mathbf{P}\psi + G\psi\Big)\Big(-\frac{i}{\kappa}\nabla\bar{\omega} + \mathbf{P}\bar{\omega} + G\bar{\omega}\Big)\mathrm{dxdt} \\
&+ \int_0^T \int_\Omega (|\psi|^2 - 1)\psi\bar{\omega}\,\mathrm{dxdt} = 0 \quad \forall \omega \in L^2(0,T;\mathcal{H}^1_{\rm qp}(\Omega)),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\int_0^T \int_\Omega \frac{\partial \mathbf{P}}{\partial t}\mathbf{Q}\,\mathrm{dxdt} &+ \int_0^T \int_\Omega \Big(\operatorname{div}\mathbf{P}\operatorname{div}\mathbf{Q} + \operatorname{curl}\mathbf{P}\operatorname{curl}\mathbf{Q}\Big)\mathrm{dxdt} + \int_0^T \int_\Omega \\
&\Re\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{P}\psi + G\psi\Big)\bar{\psi}\Big]\mathbf{Q}\,\mathrm{dxdt} = 0 \quad \forall \mathbf{Q} \in L^2(0,T;\mathbf{H}^1_{\rm per}(\Omega)).
\end{aligned} \tag{2.23}$$

Let $Q_T = \Omega \times (0,T)$. Denote by $\mathcal{H}^{2,1}_{\rm qp}(Q_T) = L^2(0,T;\mathcal{H}^2_{\rm qp}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\mathbf{H}^{2,1}_{\rm per}(Q_T) = L^2(0,T;\mathbf{H}^2_{\rm per}(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))$. We have the following theorem concerning the solvability of the Problem (P).

Theorem 2.2. *Let $(\psi_0, \mathbf{P}_0) \in \mathcal{H}^1_{\rm qp}(\Omega) \times \mathbf{H}^1_{\rm per}(\Omega)$ such that $|\psi_0| \leq 1$ a.e. on Ω. Then the Problem (P) has a unique strong solution $(\psi, \mathbf{A}) \in \mathcal{H}^{2,1}_{\rm qp}(Q_T) \times \mathbf{H}^{2,1}_{\rm per}(Q_T)$ satisfying $|\psi| \leq 1$ a.e. in Q_T.*

The uniqueness of strong solutions of Problem (P) can be proved by the standard argument as in [5]. The existence of strong solutions can be proved either by using the method of lines similar to that in [13] or by using the Leray-Schauder's fixed point theorem as in [5]. The necessary regularity results for the associated linear elliptic problems can be found in [10, §4.1]. Here we omit the details of the proof.

Moreover, we can prove the following regularity results for the solutions of the Problem (P) by the method in [3, §2].

Theorem 2.3. *Let $(\psi_0, \mathbf{P}_0) \in \mathcal{H}^2_{\rm qp}(\Omega) \times \mathbf{H}^2_{\rm per}(\Omega)$ such that $|\psi_0| \leq 1$ a.e. on Ω. Then the solution $(\psi, \mathbf{A})$ of the Problem (P) satisfies that*

$$\psi \in C(0,T;\mathcal{H}^2_{\rm qp}(\Omega)) \cap H^1(0,T;\mathcal{H}^1_{\rm qp}(\Omega)) \ \text{ and } \ \psi_t \in \mathcal{L}^4(Q_T);$$
$$\mathbf{P} \in C(0,T;\mathbf{H}^2_{\rm per}(\Omega)) \cap H^1(0,T;\mathbf{H}^1_{\rm per}(\Omega)) \ \text{ and } \ \mathbf{P}_t \in \mathbf{L}^4(Q_T).$$

3. The Approximations

In this section we turn to the numerical solutions of the gauge periodic TDGL model (2.12)–(2.18). The numerical scheme presented below is similar to that used in [3] for the TDGL model with Neumann boundary conditions. The novel feature of the method here is the treatment of the "periodicity" conditions imposed on functions belonging to $\mathcal{H}^1_{\text{qp}}(\Omega)$ and $\mathbf{H}^1_{\text{per}}(\Omega)$. We will also prove a new L^2 error estimate which is optimal with respect to the rate of convergence.

We make use of backward Euler scheme to discretize the Problem (P) in time. Let M be a positive integer and $\Delta t = T/M$ be the time step. For any $n = 0, 1, \cdots, M$, we define $t^n = n\Delta t$ and $I^n = (t^{n-1}, t^n]$. Furthermore, we denote $\partial\eta^n = (\eta^n - \eta^{n-1})/\text{dt}$ for any given sequence $\{\eta^n\}_{n=0}^M$ and $\eta^n = \eta(\cdot, t^n)$ for any given function $\eta \in C(0, T; X)$ with some Banach space X.

In space we utilize linear finite element approximations. Let $\{\Delta_h\}_{h>0}$ be a family of regular and quasi-uniform triangulations of Ω such that $\Omega = \cup_{K\in\Delta_h} K$. Denote by h the largest diameter of any of the triangles in Δ_h and by $\mathcal{N}_h$ the set of nodes of the triangulation Δ_h.

We define the finite element space $V_h = \{v \in C(\bar{\Omega}) \colon v|_K \text{ is linear for all } K \in \Delta_h\}$ and let I_h denote the usual interpolation operator from $C(\bar{\Omega})$ into V_h, i.e., for any $v \in C(\bar{\Omega})$, $I_h v$ is the unique element of V_h such that $I_h v(x) = v(x)$ for all $x \in \mathcal{N}_h$. It is well-known that there exists a constant C independent of h such that the following interpolation estimate holds:

$$\|v - I_h v\|_{H^j(\Omega)} \le Ch^{2-j}\|v\|_{H^2(\Omega)} \quad \forall v \in H^2(\Omega) \quad j = 0, 1.$$

In order to approximate the problem, we introduce the spaces

$$\begin{aligned}\mathcal{V}_h = \{\psi \in & \mathcal{C}(\bar{\Omega}) : \psi|_K \text{ is linear for all } K \in \Delta_h \text{ and} \\ & \psi(x + \mathbf{t}_k) = \psi(x)e^{i\kappa g_k(x)} \quad \forall x \in \mathcal{N}_h \cap \Gamma_{-k}, \ k = 1, 2 \ \}\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}_h = \{\mathbf{Q} \in & \mathbf{C}(\bar{\Omega}) : \mathbf{Q}|_K \text{ is linear for all } K \in \Delta_h \text{ and} \\ & \mathbf{Q}(x + \mathbf{t}_k) = \mathbf{Q}(x) \quad \forall x \in \mathcal{N}_h \cap \Gamma_{-k}, \quad k = 1, 2 \ \}\end{aligned}$$

We observe that $\mathbf{V}_h \subset \mathbf{H}^1_{\text{per}}(\Omega)$ but $\mathcal{V}_h \not\subset \mathcal{H}^1_{\text{qp}}(\Omega)$ because the functions in $\mathcal{V}_h$ satisfy the quasi-periodic constraint only at the nodes on the boundary of Ω. However, the finite element space $\mathcal{V}_h$ do provide a good approximation of the space $\mathcal{H}^1_{\text{qp}}(\Omega)$ (see [11, Lemma 3.1]).

Now we are in the position to introduce the following discrete problem.

Problem (DP) For $n = 1, 2, \cdots, M$, find $(\psi_h^n, \mathbf{P}_h^n) \in \mathcal{V}_h \times \mathbf{V}_h$ such that

$$\psi_h^0 = I_h\psi_0, \quad \mathbf{P}_h^0 = I_h\mathbf{P}_0 \tag{3.1}$$

and

$$\begin{aligned}\eta\int_\Omega \partial\psi_h^n\bar{\omega}_h \text{dx} &- i\eta\kappa\int_\Omega \text{div}\,\mathbf{P}_h^n\psi_h^n\bar{\omega}_h\text{dx} + \int_\Omega\Big(\frac{i}{\kappa}\nabla\psi_h^n + \mathbf{P}_h^n\psi_h^n + G\psi_h^n\Big) \\ &\cdot\Big(-\frac{i}{\kappa}\nabla\bar{\omega}_h + \mathbf{P}_h^n\bar{\omega}_h + G\bar{\omega}_h\Big)\text{dx} + \int_\Omega(|\psi_h^n|^2 - 1)\psi_h^n\bar{\omega}_h\text{dx} = 0 \ \forall\omega_h \in \mathcal{V}_h,\end{aligned} \tag{3.2}$$

$$\int_\Omega \partial \mathbf{P}_h^n \mathbf{Q}_h \mathrm{dx} + \int_\Omega \Big(\mathrm{div}\, \mathbf{P}_h^n \mathrm{div}\, \mathbf{Q}_h + \mathrm{curl}\ \mathbf{P}_h^n \mathrm{curl}\ \mathbf{Q}_h\Big)\mathrm{dx} + \int_\Omega \Big[\Big(\frac{i}{\kappa}\nabla\psi_h^{n-1} + \mathbf{P}_h^{n-1}\psi_h^{n-1} + G\psi_h^{n-1}\Big)\bar{\psi}_h^{n-1}\Big]\mathbf{Q}_h \mathrm{dx} = 0 \ \ \forall \mathbf{Q}_h \in \mathbf{V}_h. \tag{3.3}$$

We note that at each time step n, (3.3) is a linear system of equations with positive definite coefficient matrix, which can be solved by standard methods. As soon as we know $\mathbf{P}_h^n$ from (3.3), we substitute it into (3.2) and solve the nonlinear system of equations to get ψ_h^n. For a discussion on the existence and uniqueness of the solution ψ_h^n to (3.2) we refer to [3]. In practical computations, the nonlinear system of equations (3.2) may be solved by Newton's iterative method.

We have the following theorem concerning the approximation properties of Problem (DP) to Problem (P).

Theorem 3.1. *Let Δt be sufficiently small. Assume that the solution $(\psi, \mathbf{P})$ of Problem (P) satisfies that $\psi \in H^1(0,T;\ \mathcal{H}^2_{\mathrm{qp}}(\Omega))$ and $\mathbf{P} \in H^1(0,T;\mathbf{H}^2_{\mathrm{per}}(\Omega))$. Then, we have*

$$\max_{1\le n\le M}\Big[\|\psi^n - \psi_h^n\|^2_{L^2(\Omega)} + \|\mathbf{P}^n - \mathbf{P}_h^n\|^2_{L^2(\Omega)}\Big] \le C(h^4 + \Delta t^2), \tag{3.4}$$

$$\sum_{n=1}^{M} \Delta t\Big[\|\psi^n - \psi_h^n\|^2_{H^1(\Omega)} + \|\mathbf{P}^n - \mathbf{P}_h^n\|^2_{H^1(\Omega)}\Big] \le C(h^2 + \Delta t^2). \tag{3.5}$$

where the constant C is independent of h and Δt.

The proof of this theorem will be given in next section. We remark that both the energy error estimate (3.5) and the L^2 error estimate (3.4) are optimal with respect to the rate of convergence. We also remark that the regularity assumptions in the theorem indeed can be proved under suitable regularity and compatible assumptions on the initial data.

In the remainder of this section, we introduce the elliptic projection operator that will be used in next section. Given $(\psi, \mathbf{P}) \in \mathcal{H}^2_{\mathrm{qp}}(\Omega) \cap \mathbf{H}^2_{\mathrm{per}}(\Omega)$, denote by $\mathbf{A} = \mathbf{P} + G$, we define $(\psi_h, \mathbf{P}_h) = R_h(\psi, \mathbf{P}) \in \mathcal{V}_h \times \mathbf{V}_h$ by requiring that:

$$\int_\Omega \Big[\Big(\frac{i}{\kappa}\nabla(\psi_h - \psi) + \mathbf{A}(\psi_h - \psi)\Big)\Big(-\frac{i}{\kappa}\nabla\bar{\omega}_h + \mathbf{A}\bar{\omega}_h\Big) + (\psi_h - \psi)\bar{\omega}_h\Big]\mathrm{dx} - \frac{i}{\kappa}\int_{\partial\Omega}\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\cdot \mathbf{n}\Big]\bar{\omega}_h \mathrm{ds} = 0 \ \ \forall \omega_h \in \mathcal{V}_h, \tag{3.6}$$

$$\int_\Omega \Big[\mathrm{div}\,(\mathbf{P}_h - \mathbf{P})\mathrm{div}\,\mathbf{Q}_h + \mathrm{curl}\ (\mathbf{P}_h - \mathbf{P})\mathrm{curl}\ \mathbf{Q}_h + (\mathbf{P}_h - \mathbf{P})\mathbf{Q}_h\Big]\mathrm{dx} = 0 \ \forall \mathbf{Q}_h \in \mathbf{V}_h. \tag{3.7}$$

It is easy to see that the solution $(\psi_h, \mathbf{P}_h)$ of (3.6)–(3.7) is uniquely existent. Thus the elliptic projection operator $R_h : \mathcal{H}^2_{\mathrm{qp}}(\Omega) \times \mathbf{H}^2_{\mathrm{per}}(\Omega) \to \mathcal{V}_h \times \mathbf{V}_h$ is well-defined.

Let $f = \Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2\psi + \psi \in \mathcal{L}^2(\Omega)$. Then $\psi_h \in \mathcal{V}_h$ can be viewed as the finite element approximation of the solution $\psi \in \mathcal{H}^2_{\mathrm{qp}}(\Omega)$ of the following elliptic problem:

$$\Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2\psi + \psi = f \ \text{ in } \Omega,$$

$$\psi(x+\mathbf{t}_k) = \psi(x)e^{i\kappa g_k(x)} \quad \text{on } \Gamma_{-k},\ k=1,2,$$
$$\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\Big|_{x+\mathbf{t}_k}\cdot \mathbf{n}_{+k} = \Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)e^{i\kappa g_k}\Big]_x \cdot \mathbf{n}_{+k} \quad \text{on } \Gamma_{-k},\ k=1,2.$$

The term of the boundary integral in (3.6) results from the fact that $\mathcal{V}_h \not\subset \mathcal{H}^1_{\rm qp}(\Omega)$.

Note that $\mathbf{A} = \mathbf{P} + G$ and $\mathbf{P}$ is periodic. We can show the following error estimate by the method in [11, §4.1]:

$$\|\psi - \psi_h\|_{H^j(\Omega)} \le Ch^{2-j}\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)} \quad j=0,1, \tag{3.8}$$

where $\mathcal{P}:\mathbb{R}\to\mathbb{R}$ is a quadratic polynomial. From this estimate, the finite element inverse estimates, the finite element interpolation theory and the embedding theorems, we have

$$\begin{aligned}\|\psi_h\|_{L^\infty(\Omega)} &\le \|\psi_h - I_h\psi\|_{L^\infty(\Omega)} + \|I_h\psi\|_{L^\infty(\Omega)} \le Ch^{-1}\|\psi_h - I_h\psi\|_{L^2(\Omega)} + \|I_h\psi\|_{L^\infty(\Omega)} \\ &\le Ch^{-1}(\|\psi_h - \psi\|_{L^2(\Omega)} + \|\psi - I_h\psi\|_{L^2(\Omega)}) + \|I_h\psi\|_{L^\infty(\Omega)} \\ &\le C\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\|\nabla\psi_h\|_{L^4(\Omega)} &\le \|\nabla(\psi_h - I_h\psi)\|_{L^4(\Omega)} + \|\nabla I_h\psi\|_{L^4(\Omega)} \\ &\le Ch^{-1/2}\|\nabla(\psi_h - I_h\psi)\|_{L^2(\Omega)} + C\|\nabla\psi\|_{L^4(\Omega)} \\ &\le Ch^{-1/2}\|\nabla(\psi_h - \psi)\|_{L^2(\Omega)} + \|\nabla(\psi - I_h\psi)\|_{L^2(\Omega)}) + C\|\nabla\psi\|_{L^4(\Omega)} \\ &\le C\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)},\end{aligned} \tag{3.10}$$

where $\mathcal{P}:\mathbb{R}\to\mathbb{R}$ is a quadratic polynomial.

By applying the standard finite element approximation theory in [6] to (3.7) we can prove the following estimate:

$$\|\mathbf{P} - \mathbf{P}_h\|_{H^j(\Omega)} \le Ch^{2-j}\|\mathbf{P}\|_{H^2(\Omega)} \quad j=0,1. \tag{3.11}$$

Then, with the same reasoning leading to (3.9)–(3.10), we also obtain

$$\|\mathbf{P}_h\|_{L^\infty(\Omega)} + \|\nabla\mathbf{P}_h\|_{L^4(\Omega)} \le C\|\mathbf{P}\|_{H^2(\Omega)}. \tag{3.12}$$

4. The Error Estimates

In this section we show the error estimates in Theorem 3.1. Throughout this section we always denote C a generic constant independent of h and Δt which generally has different values at any two different places.

To begin with, we note the following identity

$$\sum_{i,j=1}^{2}\int_\Omega \Big|\frac{\partial Q_i}{\partial x_j}\Big|^2 \mathrm{dx} = \int_\Omega \Big(|\mathrm{div}\,\mathbf{Q}|^2 + |\mathrm{curl}\,\mathbf{Q}|^2\Big)\mathrm{dx} \quad \forall \mathbf{Q}\in \mathbf{H}^1_{\rm per}(\Omega) \tag{4.1}$$

which can be easily proved by using Green's formula. The following stability estimates for the solutions of Problem (DP) can be proved by the method in [3, §4].

Lemma 4.1. *Let Δt be sufficiently small. Then the solution $(\psi_h^n, \mathbf{P}_h^n)$ of Problem (DP) fulfills the following estimates*

$$\max_{1\le n\le M}(\|\psi_h^n\|_{L^2(\Omega)}^2 + \|\mathbf{P}_h^n\|_{L^2(\Omega)}^2) + \sum_{n=1}^{M}\Delta t(\|\psi_h^n\|_{L^4(\Omega)}^4 + \|\mathbf{P}_h^n\|_{L^4(\Omega)}^4) + \sum_{n=1}^{M}\Delta t(\|\nabla\psi_h^n\|_{L^2(\Omega)}^2 + \|\nabla\mathbf{P}_h^n\|_{L^2(\Omega)}^2) \le C.$$

Lemma 4.2. *Let $(\psi, \mathbf{P}) \in \mathcal{H}_{\rm qp}^2(\Omega) \times \mathbf{H}_{\rm per}^2(\Omega)$ and $(\psi_h, \mathbf{P}_h) = R_h(\psi, \mathbf{P})$ defined according to (3.6)–(3.7). Then we have*

$$\Big|\int_{\partial\Omega}\psi_h\bar{\psi}(\mathbf{Q}_h\cdot\mathbf{n})\mathrm{ds}\Big| \le Ch^2\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)}^2\|\mathbf{Q}_h\|_{H^1(\Omega)} \quad \forall\mathbf{Q}_h\in\mathbf{V}_h,$$

where $\mathcal{P}:\mathbb{R}\to\mathbb{R}$ is a quadratic polynomial and C is the constant independent of $\psi, \mathbf{P}, \mathbf{Q}_h$ and h.

Proof. Since $\psi\in\mathcal{H}_{\rm qp}^2(\Omega)$, $\psi_h\in\mathcal{V}_h$ and $\mathbf{Q}_h$ is periodic, we have

$$\begin{aligned}\int_{\partial\Omega}\psi_h\bar{\psi}(\mathbf{Q}_h\cdot\mathbf{n})\mathrm{ds} &= -\sum_{k=1}^{2}\int_{\Gamma_{-k}}[(\psi_h\bar{\psi})(x+\mathbf{t}_k) - (\psi_h\bar{\psi})(x)](\mathbf{Q}_h(x)\cdot\mathbf{n}_{-k})\mathrm{ds}\\ &= -\sum_{k=1}^{2}\int_{\Gamma_{-k}}[\psi_h(x+\mathbf{t}_k) - \psi_h(x)e^{i\kappa g_k(x)}]\\ &\quad\cdot\bar{\psi}(x)e^{-i\kappa g_k(x)}(\mathbf{Q}_h(x)\cdot\mathbf{n}_{-k})\mathrm{ds}.\end{aligned}$$

On the other hand, it is proved in [11, pp. 111-112] that

$$\|\psi_h(x+\mathbf{t}_k) - \psi_h(x)e^{i\kappa g_k(x)}\|_{L^2(\Gamma_{-k})} \le Ch^2\|\psi_h\|_{H^1(\Gamma_{-k})}.$$

Thus

$$\begin{aligned}\Big|\int_{\partial\Omega}\psi_h\bar{\psi}(\mathbf{Q}_h\cdot\mathbf{n})\mathrm{ds}\Big| &\le Ch^2\|\psi\|_{L^\infty(\Omega)}\|\psi_h\|_{H^1(\partial\Omega)}\|\mathbf{Q}_h\|_{L^2(\partial\Omega)}\\ &\le Ch^2\|\psi\|_{L^\infty(\Omega)}(\|\psi_h - I_h\psi\|_{H^1(\partial\Omega)} + \|I_h\psi\|_{H^1(\partial\Omega)})\|\mathbf{Q}_h\|_{H^1(\Omega)}\\ &\le Ch^2\|\psi\|_{L^\infty(\Omega)}(Ch^{-1/2}\|\psi_h - I_h\psi\|_{H^1(\Omega)} + C\|\psi\|_{H^1(\partial\Omega)})\|\mathbf{Q}_h\|_{H^1(\Omega)}\\ &\le Ch^2\|\psi\|_{H^2(\Omega)}(Ch^{1/2}\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)} + C\|\psi\|_{H^2(\Omega)})\|\mathbf{Q}_h\|_{H^1(\Omega)}\\ &\le Ch^2\mathcal{P}(\|\mathbf{P}\|_{H^2(\Omega)})\|\psi\|_{H^2(\Omega)}^2\|\mathbf{Q}_h\|_{H^1(\Omega)},\end{aligned}$$

where in the first inequality we have used the Cauchy-Schwarz's inequality; in the second inequality we have used the triangle inequality and the trace theorem; in the third inequality we have used the following finite element inverse estimate

$$\|\zeta_h\|_{H^1(\partial\Omega)} \le Ch^{-1/2}\|\zeta_h\|_{H^1(\Omega)} \quad \forall\zeta_h\in\mathcal{V}_h$$

and the stability estimate of the interpolation operator; and in the fourth inequality we have used the triangle inequality, the trace theorem; the estimate (3.8) and the finite element interpolation theory. This completes the proof. □

Given $f \in L^2(Q_T)$, denote by

$$[\![f]\!]^n = \frac{1}{\Delta t} \int_{I^n} f(\cdot, t)\mathrm{dt} \tag{4.2}$$

the average of f on the interval $I^n = (t^{n-1}, t^n]$ for $n = 1, \cdots, M$. Then it is easy to see that

$$\|[\![f]\!]^n - f(\cdot, t)\|^2_{L^2(\Omega)} \le \Delta t \int_{I^n} \left\|\frac{\partial f}{\partial t}\right\|^2_{L^2(\Omega)} \mathrm{dt} \quad \text{a.e. } t \in I^n. \tag{4.3}$$

Now we are in the position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. At first we note that from the regularity assumptions of the theorem we have $\psi \in C(0,T;\mathcal{H}^2_{\mathrm{qp}}(\Omega))$ and $\mathbf{P} \in C(0,T;\mathbf{H}^2_{\mathrm{per}}(\Omega))$. For $n = 1,2\cdots,M$, since $(\psi^n, \mathbf{P}^n) \in \mathcal{H}^2_{\mathrm{qp}}(\Omega) \times \mathbf{H}^2_{\mathrm{per}}(\Omega)$, we let $(\widetilde{\psi}^n_h, \tilde{\mathbf{P}}^n_h) = R_h(\psi^n, \mathbf{P}^n)$ denote the elliptic projection defined according to (3.6)–(3.7). Then it follows from (3.8)–(3.13) and the regularity assumptions of the theorem that

$$\max_{1\le n\le M} \left(\|\psi^n - \widetilde{\psi}^n_h\|_{H^j(\Omega)} + \|\mathbf{P}^n - \tilde{\mathbf{P}}^n_h\|_{H^j(\Omega)}\right) \le Ch^{2-j} \quad \text{for } j = 0, 1 \tag{4.4}$$

and

$$\|\widetilde{\psi}^n_h\|_{L^\infty(\Omega)} + \|\nabla\widetilde{\psi}^n_h\|_{L^4(\Omega)} + \|\tilde{\mathbf{P}}^n_h\|_{L^\infty(\Omega)} + \|\nabla\tilde{\mathbf{P}}^n_h\|_{L^4(\Omega)} \le C. \tag{4.5}$$

For convenience we write $\mathbf{A}^n = \mathbf{P}^n + G$, $\mathbf{A}^n_h = \mathbf{P}^n_h + G$ and $\tilde{\mathbf{A}}^n_h = \tilde{\mathbf{P}}^n_h + G$, for $n = 1, \cdots, M$. Let $\zeta^n_h = \psi^n_h - \widetilde{\psi}^n_h$ and $\mathbf{E}^n_h = \mathbf{P}^n_h - \tilde{\mathbf{P}}^n_h = \mathbf{A}^n_h - \tilde{\mathbf{A}}^n_h$, then we know from (2.12)–(2.18), (3.2)–(3.3) and (3.6)–(3.7) that

$$\begin{aligned}
&\eta \int_\Omega \partial\zeta^n_h \bar{\omega}_h \mathrm{dx} + \frac{1}{\kappa^2}\int_\Omega \nabla\zeta^n_h \nabla\bar{\omega}_h \mathrm{dx}\\
=&\eta \int_\Omega \partial(\psi^n - \widetilde{\psi}^n_h)\bar{\omega}_h \mathrm{dx} - i\eta\kappa \int_\Omega ([\![\operatorname{div} \mathbf{P}\psi]\!]^n - \operatorname{div} \mathbf{P}^n\psi^n)\bar{\omega}_h \mathrm{dx}\\
&- \frac{i}{\kappa}\int_\Omega \Big[[\![\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)]\!]^n - \Big(\frac{i}{\kappa}\nabla\psi^n + \mathbf{A}^n\psi^n\Big)\Big]\nabla\bar{\omega}_h \mathrm{dx}\\
&+ \int_\Omega \Big[[\![\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\mathbf{A}]\!]^n - \Big(\frac{i}{\kappa}\nabla\psi^n + \mathbf{A}^n\psi^n\Big)\mathbf{A}^n\Big]\bar{\omega}_h \mathrm{dx}\\
&+ \frac{i}{\kappa}\int_{\partial\Omega} \Big[[\![\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\cdot\mathbf{n}]\!]^n - \Big(\frac{i}{\kappa}\nabla\psi^n + \mathbf{A}^n\psi^n\Big)\cdot\mathbf{n}\Big]\bar{\omega}_h \mathrm{ds}\\
&+ \int_\Omega [[\![(|\psi|^2 - 1)\psi]\!]^n - (|\psi^n|^2 - 1)\psi^n]\bar{\omega}_h \mathrm{dx}\\
&- i\eta\kappa \int_\Omega \Big(\operatorname{div} \mathbf{P}^n\psi^n - \operatorname{div} \mathbf{P}^n_h\psi^n_h\Big)\bar{\omega}_h \mathrm{dx} - \frac{i}{\kappa}\int_\Omega \Big(\mathbf{A}^n\widetilde{\psi}^n_h - \mathbf{A}^n_h\psi^n_h\Big)\nabla\bar{\omega}_h \mathrm{dx}\\
&+ \frac{i}{\kappa}\int_\Omega \Big(\mathbf{A}^n\nabla\widetilde{\psi}^n_h - \mathbf{A}^n_h\nabla\psi^n_h\Big)\bar{\omega}_h \mathrm{dx} + \int_\Omega \Big(|\mathbf{A}^n|^2\widetilde{\psi}^n_h - |\mathbf{A}^n_h|^2\psi^n_h\Big)\bar{\omega}_h \mathrm{dx}\\
&+ \int_\Omega \Big[(|\psi^n|^2 - 1)\psi^n - (|\psi^n_h|^2 - 1)\psi^n_h\Big]\bar{\omega}_h \mathrm{dx} + \int_\Omega (\widetilde{\psi}^n_h - \psi^n)\bar{\omega}_h \mathrm{dx}\\
=:& \,(\mathrm{I})_1 + \cdots + (\mathrm{I})_{12}
\end{aligned} \tag{4.6}$$

$$\int_\Omega \partial\mathbf{E}^n_h \mathbf{Q}_h \mathrm{dx} + \int_\Omega \Big(\operatorname{div} \mathbf{E}^n_h \operatorname{div} \mathbf{Q}_h + \operatorname{curl}\mathbf{E}^n_h \operatorname{curl}\mathbf{Q}_h\Big)\mathrm{dx}$$

$$
\begin{aligned}
&= \int_\Omega \partial(\mathbf{P}^n - \tilde{\mathbf{P}}_h^n)\mathbf{Q}_h \mathrm{dx} + \int_\Omega \Big[\mathrm{div}\,([\![\mathbf{P}]\!]^n - \mathbf{P}^n)\mathrm{div}\,\mathbf{Q}_h + \mathrm{curl}([\![\mathbf{P}]\!]^n - \mathbf{P}^n)\mathrm{curl}\ \mathbf{Q}_h\Big]\mathrm{dx} \\
&\quad + \int_\Omega \Re\Big[\Big[\!\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\bar{\psi}\Big]\!\Big]^n - \Big(\frac{i}{\kappa}\nabla\psi^{n-1} + \mathbf{A}^{n-1}\psi^{n-1}\Big)\bar{\psi}^{n-1}\Big]\mathbf{Q}_h\mathrm{dx} \\
&\quad + \int_\Omega \Re\Big[\frac{i}{\kappa}\Big(\nabla\psi^{n-1}\bar{\psi}^{n-1} - \nabla\psi_h^{n-1}\bar{\psi}_h^{n-1}\Big)\mathbf{Q}_h\Big]\mathrm{dx} \\
&\quad + \int_\Omega \Big(\mathbf{A}^{n-1}|\psi^{n-1}|^2 - \mathbf{A}_h^{n-1}|\psi_h^{n-1}|^2\Big)\mathbf{Q}_h\mathrm{dx} + \int_\Omega (\tilde{\mathbf{P}}_h^n - \mathbf{P}^n)\mathbf{Q}_h\mathrm{dx} \\
&=: (\mathrm{II})_1 + \cdots + (\mathrm{II})_6
\end{aligned}
\tag{4.7}
$$

It is not difficult to estimate the terms $(\mathrm{I})_1 - (\mathrm{I})_6$ and $(\mathrm{II})_1 - (\mathrm{II})_3$ by using (4.3)–(4.4), Lemma 4.3 at the end of this section, and the fact that $\psi \in C(0,T;\mathcal{H}^2_{\mathrm{qp}}(\Omega))$ and $\mathbf{P} \in C(0,T;\mathbf{H}^2_{\mathrm{per}}(\Omega))$ to obtain

$$
\sum_{j=1}^{6} |(I)_j| + \sum_{j=1}^{3} |(II)_j| \le C\Big(\frac{h^2}{\sqrt{\Delta t}} + \sqrt{\Delta t}\Big)\Theta_n^{1/2}(\|\omega_h\|_{H^1(\Omega)} + \|\mathbf{Q}_h\|_{H^1(\Omega)}), \tag{4.8}
$$

where

$$
\Theta_n = \Delta t + \int_{I^n} \Big(\Big\|\frac{\partial\psi}{\partial t}\Big\|^2_{H^2(\Omega)} + \Big\|\frac{\partial \mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\Big)\mathrm{dt}\,.
$$

By applying (4.4)–(4.5), Lemma 4.1 and the regularity assumptions of the theorem, we can prove the estimates

$$
\begin{aligned}
\sum_{j=8}^{12} |(I)_j| \le\, & Ch^2\|\omega_h\|_{H^1(\Omega)} + C\|\psi_h^n\|_{L^4(\Omega)}\|\mathbf{E}_h^n\|_{L^4(\Omega)}\|\nabla\omega_h\|_{L^2(\Omega)} \\
& + C\|\mathbf{A}_h^n\|_{L^4(\Omega)}\|\nabla\zeta_h^n\|_{L^2(\Omega)}\|\omega_h\|_{L^4(\Omega)} + C(\|\zeta_h^n\|_{H^1(\Omega)} + \|\mathbf{E}_h^n\|_{H^1(\Omega)})\|\omega_h\|_{L^4(\Omega)} \\
& + C(\|\mathbf{A}_h^n\|^2_{L^4(\Omega)} + \|\psi_h^n\|^2_{L^4(\Omega)})\|\zeta_h^n\|_{L^4(\Omega)}\|\omega_h\|_{L^4(\Omega)}
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
\sum_{j=5}^{6} |(\mathrm{II})_j| \le\, & Ch^2\|\mathbf{Q}_h\|_{L^2(\Omega)} + C\|\psi_h^{n-1}\|^2_{L^4(\Omega)}\|\mathbf{E}_h^{n-1}\|_{L^4(\Omega)}\|\mathbf{Q}_h\|_{L^4(\Omega)} \\
& + C\|\zeta_h^{n-1}\|_{L^4(\Omega)}\|\mathbf{Q}_h\|_{L^4(\Omega)}
\end{aligned}
\tag{4.10}
$$

by some standard but tedious argument. Here we omit the details.

Now it remains to estimate $(\mathrm{I})_7$ and $(\mathrm{II})_4$. At first we decompose the term $(\mathrm{I})_7$ as follows:

$$
\begin{aligned}
(\mathrm{I})_7 = & - i\eta\kappa \int_\Omega \mathrm{div}\,(\mathbf{P}^n - \tilde{\mathbf{P}}_h^n)\widetilde{\psi}_h^n\bar{\omega}_h\mathrm{dx} - i\eta\kappa \int_\Omega \mathrm{div}\,\mathbf{P}^n(\psi^n - \widetilde{\psi}_h^n)\bar{\omega}_h\mathrm{dx} \\
& + i\eta\kappa \int_\Omega \mathrm{div}\,\mathbf{E}_h^n\widetilde{\psi}_h^n\bar{\omega}_h\mathrm{dx} + i\eta\kappa \int_\Omega \mathrm{div}\,\mathbf{P}_h^n\zeta_h^n\bar{\omega}_h\mathrm{dx} =: (\mathrm{III})_1 + \cdots + (\mathrm{III})_4.
\end{aligned}
$$

The last three terms can be bounded by using (4.4)–(4.5) to get

$$
\sum_{j=2}^{4} |(\mathrm{III})_j| \le Ch^2\|\omega_h\|_{H^1(\Omega)} + C\|\mathrm{div}\,\mathbf{E}_h^n\|_{L^2(\Omega)}\|\omega_h\|_{L^2(\Omega)}
$$

$$+ C\|\mathrm{div}\,\mathbf{P}_h^n\|_{L^2(\Omega)}\|\zeta_h^n\|_{L^4(\Omega)}\|\omega_h\|_{L^4(\Omega)}.$$

To estimate $(\mathrm{III})_1$, we note first that $\widetilde{\psi}_h^n\bar{\omega}_h$ and $\mathbf{P}^n - \tilde{\mathbf{P}}_h^n$ is periodic. Thus, by Green's formula, we get

$$\begin{aligned}(\mathrm{III})_1 &= i\eta\kappa\int_\Omega(\mathbf{P}^n - \tilde{\mathbf{P}}_h^n)\nabla\widetilde{\psi}_h^n\bar{\omega}_h\mathrm{dx} + i\eta\kappa\int_\Omega(\mathbf{P}^n - \tilde{\mathbf{P}}_h^n)\widetilde{\psi}_h^n\nabla\bar{\omega}_h\mathrm{dx}\\ &\le Ch^2\|\nabla\widetilde{\psi}_h^n\|_{L^4(\Omega)}\|\omega_h\|_{L^4(\Omega)} + Ch^2\|\widetilde{\psi}_h^n\|_{L^\infty(\Omega)}\|\nabla\omega_h\|_{L^2(\Omega)} \le Ch^2\|\omega_h\|_{H^1(\Omega)},\end{aligned}$$

where we have used (4.4)–(4.5). In summary, we have

$$\begin{aligned}|(I)_7| \le& Ch^2\|\omega_h\|_{H^1(\Omega)} + C\|\mathrm{div}\,\mathbf{E}_h^n\|_{L^2(\Omega)}\|\omega_h\|_{L^2(\Omega)}\\ &+ C\|\mathrm{div}\,\mathbf{P}_h^n\|_{L^2(\Omega)}\|\zeta_h^n\|_{L^4(\Omega)}\|\omega_h\|_{L^4(\Omega)}.\end{aligned}\tag{4.11}$$

Similarly, we may decompose $(II)_4$ as follows:

$$\begin{aligned}(II)_4 =& \int_\Omega\Re\Big[\frac{i}{\kappa}\nabla(\psi^{n-1} - \widetilde{\psi}_h^{n-1})\bar{\psi}^{n-1}\mathbf{Q}_h\Big]\mathrm{dx} + \int_\Omega\Re\Big[\frac{i}{\kappa}\nabla\widetilde{\psi}_h^{n-1}(\bar{\psi}^{n-1} - \bar{\widetilde{\psi}}_h^{n-1})\mathbf{Q}_h\Big]\mathrm{dx}\\ &- \int_\Omega\Re\Big[\frac{i}{\kappa}\nabla\zeta_h^{n-1}\bar{\widetilde{\psi}}_h^{n-1}\mathbf{Q}_h\Big]\mathrm{dx} - \int_\Omega\Re\Big[\frac{i}{\kappa}\nabla\psi_h^{n-1}\bar{\zeta}_h^{n-1}\mathbf{Q}_h\Big]\mathrm{dx}\\ =&: (\mathrm{IV})_1 + \cdots + (\mathrm{IV})_4\end{aligned}$$

and bound the last three terms by

$$\begin{aligned}\sum_{j=2}^4|(\mathrm{IV})_j| \le& Ch^2\|\mathbf{Q}_h\|_{H^1(\Omega)} + C\|\nabla\zeta_h^{n-1}\|_{L^2(\Omega)}\|\mathbf{Q}_h\|_{L^2(\Omega)}\\ &+ C\|\nabla\psi_h^{n-1}\|_{L^2(\Omega)}\|\zeta_h^{n-1}\|_{L^4(\Omega)}\|\mathbf{Q}_h\|_{L^4(\Omega)}.\end{aligned}$$

To estimate $(\mathrm{IV})_1$, we apply again Green's formula to obtain

$$\begin{aligned}(\mathrm{IV})_1 =& \int_\Omega\Re\Big[-\frac{i}{\kappa}(\psi^{n-1} - \widetilde{\psi}_h^{n-1})\mathrm{div}\,(\bar{\psi}^{n-1}\mathbf{Q}_h)\Big]\mathrm{dx}\\ &+ \int_{\partial\Omega}\Re\Big[\frac{i}{\kappa}(\psi^{n-1} - \widetilde{\psi}_h^{n-1})\bar{\psi}^{n-1}(\mathbf{Q}_h\cdot\mathbf{n})\Big]\mathrm{dx} \le Ch^2\|\mathbf{Q}_h\|_{H^1(\Omega)},\end{aligned}$$

by using (4.4)–(4.5), the regularity assumption $\psi \in C(0,T;\mathcal{H}^2_{\mathrm{qp}}(\Omega))$, Lemma 4.2 and the fact that $\psi^{n-1}\bar{\psi}^{n-1}$ is periodic.

Therefore, we have

$$\begin{aligned}|(\mathrm{II})_4| \le& Ch^2\|\mathbf{Q}_h\|_{H^1(\Omega)} + C\|\nabla\zeta_h^{n-1}\|_{L^2(\Omega)}\|\mathbf{Q}_h\|_{L^2(\Omega)}\\ &+ C\|\nabla\psi_h^{n-1}\|_{L^2(\Omega)}\|\zeta_h^{n-1}\|_{L^4(\Omega)}\|\mathbf{Q}_h\|_{L^4(\Omega)}.\end{aligned}\tag{4.12}$$

Now letting $\omega_h = \Delta t\zeta_h^n \in \mathcal{V}_h$ in (4.6) and $\mathbf{Q}_h = \Delta t\mathbf{E}_h^n \in \mathbf{V}_h$ in (4.7), taking the real part of (4.6), adding the obtained equation with (4.7) together, and applying the identity (4.1) and the estimates (4.8)–(4.12), we can conclude that

$$\Big(\frac{\eta}{2}\|\zeta_h^n\|^2_{L^2(\Omega)} + \|\mathbf{E}_h^n\|^2_{L^2(\Omega)}\Big) - \Big(\frac{\eta}{2}\|\zeta_h^{n-1}\|^2_{L^2(\Omega)} + \|\mathbf{E}_h^{n-1}\|^2_{L^2(\Omega)}\Big)$$

$$+\Delta t\Big(\frac{1}{\kappa^2}\|\nabla\zeta_h^n\|^2_{L^2(\Omega)}+\|\nabla\mathbf{E}_h^n\|^2_{L^2(\Omega)}\Big)$$
$$\le C(h^4+\Delta t^2)\Theta_n+C\Delta t(\|\zeta_h^n\|^2_{L^2(\Omega)}+\|\mathbf{E}_h^n\|^2_{L^2(\Omega)})+\frac{1}{4\kappa^2}\Delta t\|\nabla\zeta_h^n\|^2_{L^2(\Omega)}$$
$$+\frac{1}{4\kappa^2}\Delta t\|\nabla\zeta_h^{n-1}\|^2_{L^2(\Omega)}+\frac{1}{4}\Delta t\|\nabla\mathbf{E}_h^n\|^2_{L^2(\Omega)}+C\Delta t(1+\Upsilon_n)^{1/2}$$
$$\cdot\Big(\|\zeta_h^n\|^2_{L^4(\Omega)}+\|\zeta_h^{n-1}\|^2_{L^4(\Omega)}+\|\mathbf{E}_h^n\|^2_{L^4(\Omega)}+\|\mathbf{E}_h^{n-1}\|^2_{L^4(\Omega)}\Big), \tag{4.13}$$

where

$$\Upsilon_n=\|\psi_h^n\|^4_{L^4(\Omega)}+\|\mathbf{A}_h^n\|^4_{L^4(\Omega)}+\|\psi_h^{n-1}\|^4_{L^4(\Omega)}+\|\nabla\psi_h^{n-1}\|^2_{L^2(\Omega)}+\|\operatorname{div}\mathbf{P}_h^n\|^2_{L^2(\Omega)}.$$

By using the following special case of the multiplicative inequality in [16, pp. 62-63]:

$$\|v\|_{L^4(\Omega)}\le C\|v\|^{1/2}_{H^1(\Omega)}\|v\|^{1/2}_{L^2(\Omega)}\quad\forall v\in H^1(\Omega)$$

and the well-known Yong's inequality $ab\le\epsilon a^2+\frac{1}{4\epsilon}b^2$ $\forall a,b>0$ and $\forall\epsilon>0$, we can bound the last term at the right-hand side of (4.14) by

$$\frac{1}{8\kappa^2}\Delta t\Big(\|\nabla\zeta_h^n\|^2_{L^2(\Omega)}+\|\nabla\zeta_h^{n-1}\|^2_{L^2(\Omega)}\Big)+\frac{1}{8}\Delta t\Big(\|\nabla\mathbf{E}_h^n\|^2_{L^2(\Omega)}+\|\nabla\mathbf{E}_h^{n-1}\|^2_{L^2(\Omega)}\Big)$$
$$+C\Delta t(1+\Upsilon_n)\Big(\|\zeta_h^n\|^2_{L^2(\Omega)}+\|\zeta_h^{n-1}\|^2_{L^2(\Omega)}+\|\mathbf{E}_h^n\|^2_{L^2(\Omega)}+\|\mathbf{E}_h^{n-1}\|^2_{L^2(\Omega)}\Big).$$

Note that by the Lemma 4.1 and the regularity assumptions of the theorem, we have

$$\sum_{n=1}^{M}\Theta_n\le C,\quad\sum_{n=1}^{M}\Delta t\Upsilon_n\le C.$$

Thus, by the discrete Gronwall's inequality, we obtain

$$\max_{1\le n\le M}\Big(\|\zeta_h^n\|^2_{L^2(\Omega)}+\|\mathbf{E}_h^n\|^2_{L^2(\Omega)}\Big)+\sum_{n=1}^{M}\Delta t\Big(\|\nabla\zeta_h^n\|^2_{L^2(\Omega)}+\|\nabla\mathbf{E}_h^n\|^2_{L^2(\Omega)}\Big)\le C(h^4+\Delta t^2).$$

Now the desired estimate (3.4)–(3.5) follows from the triangle inequality and (4.4). This completes the proof of the theorem. □

Lemma 4.3. *Assume that the solution $(\psi,\mathbf{P})$ of Problem (P) satisfies that*

$$\psi\in H^1(0,T;\mathcal{H}^2_{\rm qp}(\Omega))\quad\textit{and}\quad\mathbf{P}\in H^1(0,T;\mathbf{H}^2_{\rm per}(\Omega)).$$

For $n=1,\cdots,M$, let $(\tilde\psi_h^n,\,\tilde{\mathbf{P}}_h^n)=R_h(\psi^n,\mathbf{P}^n)$ defined according to (3.6)–(3.7). Then we have

$$\|\partial(\psi^n-\tilde\psi_h^n)\|_{L^2(\Omega)}\le C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big(\Big\|\frac{\partial\psi}{\partial t}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\Big)\mathrm{dt}\Big]^{1/2}, \tag{4.14}$$

$$\|\partial(\mathbf{P}^n-\tilde{\mathbf{P}}_h^n)\|_{L^2(\Omega)}\le C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\mathrm{dt}\Big]^{1/2}, \tag{4.15}$$

where the constant C is independent of h and Δt.

Proof. The estimate (4.15) is obvious and well-known. We only prove (4.14). First we note that from the regularity assumptions of the lemma we have

$$\max_{1\le n\le M}\Big(\|\psi^n\|_{H^2(\Omega)}+\|\mathbf{P}^n\|_{H^2(\Omega)}\Big)\le C. \tag{4.16}$$

From (3.6) we know that for any $n=1,\cdots,M$, $\widetilde{\psi}_h^n\in\mathcal{V}_h$ satisfies

$$\begin{aligned}&\int_\Omega\Big[\Big(\frac{i}{\kappa}\nabla(\widetilde{\psi}_h^n-\psi^n)+\mathbf{A}^n(\widetilde{\psi}_h^n-\psi^n)\Big)\Big(-\frac{i}{\kappa}\nabla\bar{\omega}_h+\mathbf{A}^n\bar{\omega}_h\Big)+(\widetilde{\psi}_h^n-\psi^n)\bar{\omega}_h\Big]\mathrm{dx}\\&\quad-\frac{i}{\kappa}\int_{\partial\Omega}\Big[\Big(\frac{i}{\kappa}\nabla\psi^n+\mathbf{A}^n\psi^n\Big)\cdot\mathbf{n}\Big]\bar{\omega}_h\mathrm{ds}=0\quad\forall\omega_h\in\mathcal{V}_h.\end{aligned}\tag{4.17}$$

Let $\zeta^n=\partial\psi^n$ and $\zeta_h^n=\partial\widetilde{\psi}_h^n$. By subtracting the relation (4.17) for n and $n-1$ and doing some easy calculations, we obtain

$$\begin{aligned}&\int_\Omega\Big[\Big(\frac{i}{\kappa}\nabla(\zeta_h^n-\zeta^n)+\mathbf{A}^n(\zeta_h^n-\zeta^n)\Big)\Big(-\frac{i}{\kappa}\nabla\bar{\omega}_h+\mathbf{A}^n\bar{\omega}_h\Big)+(\zeta_h^n-\zeta^n)\bar{\omega}_h\Big]\mathrm{dx}\\&\quad-\frac{i}{\kappa}\int_{\partial\Omega}\Big[\Big(\frac{i}{\kappa}\nabla\zeta^n+\mathbf{A}^n\zeta^n\Big)\cdot\mathbf{n}\Big]\bar{\omega}_h\mathrm{ds}\\&=\frac{i}{\kappa}\int_\Omega\partial\mathbf{A}^n(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\nabla\bar{\omega}_h\mathrm{dx}-\frac{i}{\kappa}\int_\Omega\partial\mathbf{A}^n\nabla(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\bar{\omega}_h\mathrm{dx}\\&\quad-\int_\Omega\partial(|\mathbf{A}^n|^2)(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\bar{\omega}_h\mathrm{dx}+\frac{i}{\kappa}\int_{\partial\Omega}(\partial\mathbf{A}^n\cdot\mathbf{n})\psi^{n-1}\bar{\omega}_h\mathrm{ds}\\&=\frac{2i}{\kappa}\int_\Omega\partial\mathbf{A}^n(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\nabla\bar{\omega}_h\mathrm{dx}+\frac{i}{\kappa}\int_\Omega(\partial\mathrm{div}\,\mathbf{A}^n)(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\bar{\omega}_h\mathrm{dx}\\&\quad-\int_\Omega\partial\mathbf{A}^n(\mathbf{A}^n+\mathbf{A}^{n-1})(\widetilde{\psi}_h^{n-1}-\psi^{n-1})\bar{\omega}_h\mathrm{dx}+\frac{2i}{\kappa}\int_{\partial\Omega}(\partial\mathbf{A}^n\cdot\mathbf{n})\psi^{n-1}\bar{\omega}_h\mathrm{ds}\\&=:(\mathrm{V})_1+\cdots+(\mathrm{V})_4\quad\forall\omega_h\in\mathcal{V}_h,\end{aligned}\tag{4.18}$$

where in the last equality we have used the Green's formula and the fact that $\partial\mathbf{A}^n$ and $\widetilde{\psi}_h^{n-1}\bar{\omega}_h$ are periodic.

Remember that $\mathbf{A}^n=\mathbf{P}^n+G$ and thus $\partial\mathbf{A}^n=\partial\mathbf{P}^n$. By applying the Hölder's inequality, the estimate (4.4) and the embedding theorems, we can obtain that

$$\begin{aligned}\sum_{j=1}^{3}|(\mathrm{V})_j|\le&C\|\partial\mathbf{P}^n\|_{L^\infty(\Omega)}\|\widetilde{\psi}_h^{n-1}-\psi^{n-1}\|_{L^2(\Omega)}\|\nabla\omega_h\|_{L^2(\Omega)}\\&+C\|\partial(\mathrm{div}\,\mathbf{P}^n)\|_{L^4(\Omega)}\|\widetilde{\psi}_h^{n-1}-\psi^{n-1}\|_{L^2(\Omega)}\|\omega_h\|_{L^4(\Omega)}\\&+C\|\partial\mathbf{P}^n\|_{L^\infty(\Omega)}(\|\mathbf{A}^n\|_{L^\infty(\Omega)}+\|\mathbf{A}^{n-1}\|_{L^\infty(\Omega)})\|\widetilde{\psi}_h^{n-1}-\psi^{n-1}\|_{L^2(\Omega)}\|\omega_h\|_{L^2(\Omega)}\\\le&Ch^2\|\partial\mathbf{P}^n\|_{H^2(\Omega)}\|\omega_h\|_{H^1(\Omega)}\\\le&C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\mathrm{dt}\Big]^{1/2}\|\omega_h\|_{H^1(\Omega)}.\end{aligned}\tag{4.19}$$

Since $\psi^{n-1}\in\mathcal{H}^2_{\mathrm{qp}}(\Omega)$, $\omega_h\in\mathcal{V}_h$ and $\partial\mathbf{A}^n=\partial\mathbf{P}^n$ is periodic, we have

$$(\mathrm{V})4=-\frac{2i}{\kappa}\sum_{k=1}^{2}\int_{\Gamma_{-k}}\Big[(\psi^{n-1}\bar{\omega}_h)(x+\mathbf{t}_k)-(\psi^{n-1}\bar{\omega}_h)(x)\Big](\partial\mathbf{P}^n\cdot\mathbf{n}_{-k})\mathrm{ds}$$

$$= -\frac{2i}{\kappa}\sum_{k=1}^{2}\int_{\Gamma_{-k}}\left[\bar{\omega}_h(x+\mathbf{t}_k)-\bar{\omega}_h(x)e^{-i\kappa g_k(x)}\right]\psi^{n-1}e^{i\kappa g_k(x)}(\partial\mathbf{P}^n\cdot\mathbf{n}_{-k})\mathrm{ds}\,.$$

Again we have the following estimate in [11, pp. 111–112]

$$\|\omega_h(x+\mathbf{t}_k)-\omega_h(x)e^{i\kappa g_k(x)}\|_{L^2(\Gamma_{-k})}\le Ch^2\|\omega_h\|_{H^1(\Gamma_{-k})}.$$

Thus we have

$$\begin{aligned}|(\mathrm{V})_4| &\le Ch^2\|\psi^{n-1}\|_{L^\infty(\partial\Omega)}\|\partial\mathbf{P}^n\cdot\mathbf{n}_{-k}\|_{L^2(\partial\Omega)}\|\omega_h\|_{H^1(\partial\Omega)}\\ &\le Ch^2\|\partial\mathbf{P}^n\|_{H^1(\Omega)}\|\omega_h\|_{H^1(\partial\Omega)}\\ &\le C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\mathrm{dt}\Big]^{1/2}\|\omega_h\|_{H^1(\partial\Omega)}\end{aligned}\tag{4.20}$$

where we have used (4.16) and the embedding theorem.

Now starting from the relation (4.18) and the estimates (4.19)–(4.20), we can argue as in [11, §4.1] and apply (4.16) to get

$$\begin{aligned}\|\zeta_h^n-\zeta^n\|_{L^2(\Omega)} &\le Ch^2\mathcal{P}(\|\mathbf{P}^n\|_{H^2(\Omega)})\|\zeta^n\|_{H^2(\Omega)}+C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\mathrm{dt}\Big]^{1/2}\\ &\le C\frac{h^2}{\sqrt{\Delta t}}\Big[\int_{I^n}\Big(\Big\|\frac{\partial\psi}{\partial t}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\mathbf{P}}{\partial t}\Big\|^2_{H^2(\Omega)}\Big)\mathrm{dt}\Big]^{1/2},\end{aligned}$$

where $\mathcal{P}(\cdot):\mathbb{R}\to\mathbb{R}$ is a quadratic polynomial. This completes the proof. □

5. A Numerical Example

We consider the most interesting periodicity structure, namely that corresponding to an equilateral triangular lattice having one fluxoid associated with each lattice cell. Thus we have that $n=1, r_1=r_2$, and $\theta=\pi/3$. From (2.19) we have that the basic parallelogram Ω is generated by the vectors $\mathbf{t}_1=(r_1,0)$ and $\mathbf{t}_2=(r_2\cos(\pi/3), r_2\sin(\pi/3))$ with

$$r_1=r_2=\sqrt{\frac{\pi}{\kappa\alpha\sin(\theta)}}.$$

We take the dimensionless constant $\eta=12$ as in [4], the Ginzburg-Landau constant $\kappa=5/3$, and the constant $\alpha=5/3$ which corresponds to the average magnetic field $\bar{B}=10/3$. The initial conditions are set to be $\psi_0=\rho(x_1,x_2)(0.6+0.8i)$ and $\mathbf{P}_0=0$, where $\rho(x_1,x_2)=1$ for (x_1,x_2) in the circle centered at the center of Ω with radius 0.1, $\rho(x_1,x_2)=0$ for (x_1,x_2) outside the circle centered at the center of Ω with radius 0.2, and in between, ρ is smooth.

The triangulation over Ω is obtained by first subdividing Ω into a uniform parallelogram grid having 20 intervals in each direction and then dividing each small parallelogram into two triangles by joining the lower left corner and the upper right corner of the parallelogram. The time step size Δt is set to be 0.05. Numerical computations were performed on a PC 586 by using the software package "Finite Element Program

Automatic Generator" by Guoping Liang. In our computations we observed that the solution achieves a steady state after 150 time steps. The level curves of the density of superconducting electron carriers at the 50th and 150th time step are given in the Fig.2 and Fig.3, respectively. The maximum and minimum value of the level curves are 0.1 and zero. In the computations the solution in only a single lattice Ω was computed, this solution was extended, using periodicity relations to obtain the solution outside Ω.

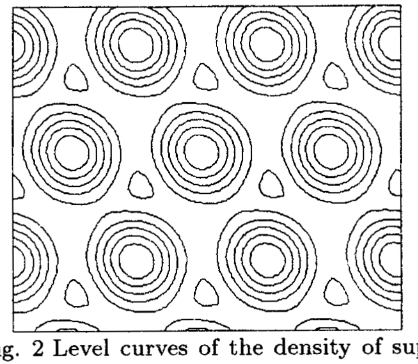

Fig. 2 Level curves of the density of superconducting electron pairs at the 50th time step.

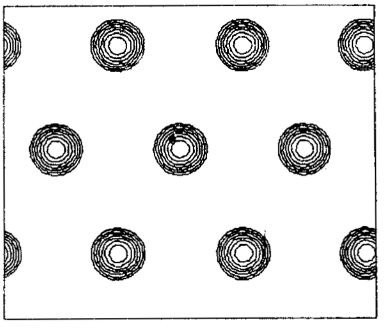

Fig. 3 Level curves of the density of superconducting electron pairs at the 150th time step.

6. Concluding Remarks

In this paper we showed that the TDGL model for type-II superconductors supports the solutions whose associated observables are periodic with respect to some lattice not necessarily rectangular. After fixing the Lorentz gauge, we obtained the unique existence of strong solutions and some regularity results for the solutions. We also proposed a semi-implicit finite element scheme solving the system of nonlinear partial differential equations and proved the optimal error estimates for the numerical method both in the L^2 and energy norm. This justifies the reliability of our numerical results presented in §5. The method in §4 to obtain the optimal error estimate in L^2 norm can be extended to show that the numerical scheme in [3] also preserves the optimal error estimate in L^2 norm.

It is proved in [13] that under the Coloumb gauge, the solution of the gauge periodic TDGL model converges to one of the solutions of the stationary periodic Ginzburg-Landau model studied in [1], [10], [11], [13] as the time $t \to \infty$. We can prove the same results for the gauge periodic TDGL model under Lorentz gauge studied in this paper by combining the methods in [13] and [14]. Thus the numerical method in §3 provides another possibility to solve the stationary periodic Ginzburg-Landau model which is different from the method studied in [11], where a system of nonlinear elliptic equations with multiple solutions was solved directly.

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