Journal of Computational Mathematics, Vol.15, No.4, 1997, 319–326.

AN ITERATIVE PROCEDURE FOR DOMAIN DECOMPOSITION METHOD OF SECOND ORDER ELLIPTIC PROBLEM WITH MIXED BOUNDARY CONDITIONS *1)

Lie-heng Wang
(*State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Chinese Academy of Sciences, Beijing, China*)

Abstract

This paper is devoted to study of an iterative procedure for domain decomposition method of second order elliptic problem with mixed boundary conditions (i.e., Dirichlet condition on a part of boundary and Neumann condition on the another part of boundary). For the pure Dirichlet problem, Marini and Quarteroni [3], [4] considered a similar approach, which is extended to more complex problem in this paper.

1. Introduction

There has been a considerable number of recent developments in non-overlap domain decomposition techniques for second order elliptic problems. We refer especially to Marini and Quarteroni [3], [4] and the references therein. One of motivations for increasing interest in domain decomposition approach is to deal with different type of equations in different parts of the physical domain, such as in the mathematical modeling of elastic composite structures.

In this paper we study an iterative procedure for domain decomposition method of a simple second order elliptic problem with mixed boundary conditions, i.e., Dirichlet condition on a part of boundary and Neumann condition on the another part of boundary. For the pure Dirichlet problems, Marini and Quarteroni [3], [4] considered a similar approach, which is so called the D-N (Dirichlet-Neumann) alternative iteration, while our iterative scheme is so called the N-D (Neumann-Dirichlet) alternative iteration, which is appropriate to the mixed boundary value problems.

The outline of the paper is as follows. In Section 2, we introduce multidomain formulation and iterative scheme for a simple second order elliptic problem with mixed boundary conditions. In section 3, we present an harmonic extension lemma which is important for the analysis of convergence of the iterative scheme. Finally in Section 4, the convergence of the iterative scheme is proved, with different manner from one in [3], [4] in order to appropriate the mixed boundary value problem.

---

*Received April 14, 1995.

1) The Project is supported by the National Natural Science Fundation of China

2. Domain Decomposition Method for Second Order Elliptic Problem with Mixed Boundary Conditions

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. Consider the following boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega,\ u = 0 \quad \text{on } \Gamma_0,\ \partial_\nu u = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0, \tag{2.1}$$

(c.f. Fig. 1), where $f \in L^2(\Omega)$, ν denotes the outward normal unit vector to $\partial\Omega$, $\partial_\nu u$ denotes the outward normal derivative.

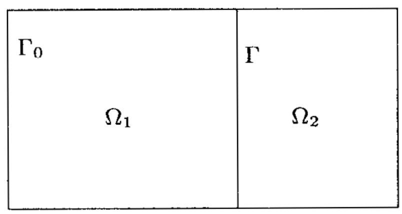

Fig.1

We assume that Ω is partitioned into two non-overlap subdomains Ω_1 and Ω_2, i.e., $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$, and we denote by Γ the common boundary of Ω_1 and Ω_2. It can be easily shown that the problem (2.1) is equivalent to the following split problems:

$$-\Delta u_1 = f \text{ in } \Omega_1,\ u_1 = 0 \text{ on } \Gamma_0,\ \partial_{\nu^1} u_1 = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0),\ \partial_{\nu^1} u_1 = -\partial_{\nu^2} u_2 \text{ on } \Gamma, \tag{2.2}$$

and

$$-\Delta u_2 = f \quad \text{in } \Omega_2,\ \partial_{\nu^2} u_2 = 0 \quad \partial\Omega_2 \setminus \Gamma,\ u_2 = u_1 \quad \text{on}\Gamma, \tag{2.3}$$

where $u_k = u|_{\Omega_k}$ for $k = 1, 2$, ν^k is the outward normal unit vector to $\partial\Omega_k$ (note that $\nu^1 = -\nu^2$ on Γ), and $\partial_{\nu^k} u_k$ $(k = 1, 2)$ is the outward normal derivative.

We now introduce an iterative procedure which is similar to that in [3], [4]. Let (c.f.[1],[2])

$$H^{\frac{1}{2}}(\Gamma) = \{\mu : \|\mu\|_{\frac{1}{2},\Gamma} < \infty\} \tag{2.4}$$

where

$$\|\mu\|_{\frac{1}{2},\Gamma} = \Big\{\|\mu\|_{0,\Gamma}^2 + \int_\Gamma \int_\Gamma \Big(\frac{\mu(s_x) - \mu(s_y)}{|s_x - s_y|}\Big)^2 ds_x ds_y\Big\}^{\frac{1}{2}}, \tag{2.5}$$

and

$$H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))' \text{–the duarity of the space } H^{\frac{1}{2}}(\Gamma). \tag{2.6}$$

Let $g^0 \in H^{-\frac{1}{2}}(\Gamma)$ be given. For $n \geq 1$ the sequence of functions u_1^n, u_2^n are constructed by iterative scheme with solving the following problems:

$$-\Delta u_1^n = f \text{ in } \Omega_1,\ u_1^n = 0 \text{ on}\Gamma_0,\ \partial_{\nu^1} u_1^n = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0),\ \partial_\nu u_1^n = g^{n-1} \text{ on } \Gamma, \tag{2.7}$$

$$-\Delta u_2^n = f \text{ in } \Omega_2,\ \partial_{\nu^2} u_2^n = 0 \text{ on } \partial\Omega_2 \setminus \Gamma,\ u_2^n = u_1^n \text{ on } \Gamma \tag{2.8}$$

where, for $n \geq 1$, $g^n \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$g^n = -\theta \partial_{\nu^2} u_2^n + (1-\theta) g^{n-1} \quad \text{on } \Gamma. \tag{2.9}$$

In (2.9), θ is a (positive) relaxation parameter that will be determined in order to ensure the convergence of the iterative scheme. Note that here the iterative scheme is N-D (Neumann-Dirichlet) alternative iteration, while Marini-Quarteroni procedure in [3], [4] is D-N (Dirichlet-Neumann) alternative iteration.

3. An Harmonic Extension Lemma

In order to study the convergence of the scheme, we first present scheme satisfied by error functions. Let

$$e_1^n = u_1 - u_1^n, \quad e_2^n = u_2 - u_2^n, \tag{3.1}$$

then from the iteration scheme (2.7)–(2.9), it can be seen that

$$-\Delta e_1^n = 0 \text{ in } \Omega_1,\ e_1^n = 0 \text{ on } \Gamma_0,\ \partial_{\nu^1} e_1^n = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0),$$
$$\partial_{\nu^1} e_1^n = -\partial_{\nu^1} u_2 - g^{n-1} \doteq \psi^{n-1} \text{ on } \Gamma, \tag{3.2}$$
$$-\Delta e_2^n = 0 \text{ in } \Omega_2,\ \partial_{\nu^2} e_2^n = 0\ \partial\Omega_2 \setminus \Gamma,\ e_2^n = e_1^n \text{ on } \Gamma, \tag{3.3}$$

and

$$\psi^n = -\theta \partial_{\nu^2} e_2^n + (1-\theta)\psi^{n-1}. \tag{3.4}$$

By the variational calculus, the equations (3.2) and (3.3) can be written in the variational formulas as follows:

$$\begin{cases} \text{to find } e_1^n \in H^1_{\Gamma_0}(\Omega_1), \quad \text{such that} \\ \displaystyle\int_{\Omega_1} \nabla e_1^n \cdot \nabla v dx = \int_\Gamma \psi^{n-1} \cdot v ds \quad \forall v \in H^1_{\Gamma_0}(\Omega_1); \end{cases} \tag{3.5}$$

$$\begin{cases} \text{to find } e_2^n \in H^1(\Omega_2) : e_2^n = e_1^n \text{ on } \Gamma, \text{ such that} \\ \displaystyle\int_{\Omega_1} \nabla e_2^n \cdot \nabla v dx = 0 \quad \forall v \in H^1_\Gamma(\Omega_2); \end{cases} \tag{3.6}$$

and

$$\psi^n = -\theta \partial_{\nu^2} e_2^n + (1-\theta)\psi^{n-1} \quad \text{on } \Gamma, \tag{3.7}$$

where

$$H^1_{\Gamma_0}(\Omega_1) = \{v \in H^1(\Omega_1) :\ v = 0 \text{ on } \Gamma_0\}, \tag{3.8}$$
$$H^1_\Gamma(\Omega_2) = \{v \in H^1(\Omega_2) :\ v = 0 \text{ on } \Gamma\}. \tag{3.9}$$

We now need a lemma, which is so called harmonic extension lemma. In what follows c, c_1 and c_2 denote the generic constants, which may take different values in different places.

Lemma 3.1. *Let $\Omega \in R^2$ be a domain with boundary $\partial\Omega$, which is partitioned into two non-overlap subdomain Ω_1 and Ω_2 with common boundary $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (c.f.Fig. 1). Let $\phi \in H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))'$ be given, such that*

$$\int_\Gamma \phi ds = 0. \tag{3.10}$$

Assume that the functions $N_1(\phi)$ and $N_2(\phi)$ are the solutions of the following problems respectively:

$$\begin{cases} -\Delta N_1(\phi) = 0 \text{ in } \Omega_1, \\ \partial_{\nu^1} N_1(\phi) = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0),\ N_1(\phi) = 0 \text{ on } \Gamma_0,\ \partial_{\nu^1} N_1(\phi) = \phi \text{ on } \Gamma, \end{cases} \tag{3.11}$$

and

$$-\Delta N_2(\phi) = 0 \text{ in } \Omega_2,\ \partial_{\nu^2} N_2(\phi) = 0 \text{ on } \partial\Omega_2 \setminus \Gamma,\ \partial_{\nu^2} N_2(\phi) = -\phi \text{ on } \Gamma. \tag{3.12}$$

Then there exist $c_1, c_2 = const. > 0$ independent of $\phi \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$c_1 |N_1(\phi)|_{1,\Omega_1} \le |N_2(\phi)|_{1,\Omega_2} \le c_2 |N_2(\phi)|_{1,\Omega_1} \quad \forall \phi \in H^{-\frac{1}{2}}(\Gamma). \tag{3.13}$$

Proof. (i) We first prove that $|N_1(\phi)|_{1,\Omega_1}$ is equivalent to $\|\phi\|_{-\frac{1}{2},\Gamma}$. To do this, with use of the trace theorem (c.f. [2]), from (3.11) it can be seen that

$$\begin{aligned} |N_1(\phi)|^2_{1,\Omega_1} &= \int_{\Omega_1} \nabla N_1(\phi) \cdot \nabla N_1(\phi) dx = \int_\Gamma \phi \cdot N_1(\phi) ds \le \|\phi\|_{-\frac{1}{2},\Gamma} \|N_1(\phi)\|_{\frac{1}{2},\Gamma} \\ &\le \|\phi\|_{-\frac{1}{2},\Gamma} \|N_1(\phi)\|_{\frac{1}{2},\partial\Omega_1} \le c \|\phi\|_{-\frac{1}{2},\Gamma} \|N_1(\phi)\|_{1,\Omega_1}, \end{aligned}$$

from which by Poincare inequality, we have

$$|N_1(\phi)|_{1,\Omega_1} \le c \|\phi\|_{-\frac{1}{2},\Gamma}. \tag{3.14}$$

Next, we have

$$\|\phi\|_{-\frac{1}{2},\Gamma} = \operatorname*{Sup}_{\mu \in H^{\frac{1}{2}}(\Gamma)} \frac{\int_\Gamma \phi \cdot \mu ds}{\|\mu\|_{\frac{1}{2},\Gamma}}. \tag{3.15}$$

For any given $\mu \in H^{\frac{1}{2}}(\Gamma)$, there exists $w_1 \in H^1(\Omega_1)$, such that

$$\begin{cases} -\Delta w_1 = 0 \text{ in } \Omega_1, \\ w_1 = \mu \text{ on } \Gamma,\ \partial_{\nu^1} w_1 = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0),\ w_1 = 0 \text{ on } \Gamma_0, \end{cases} \tag{3.16}$$

and by a priori estimate (c.f.[2]) of the solution of the problem (3.16), we have

$$\|w_1\|_{1,\Omega_1} \le c \|\mu\|_{\frac{1}{2},\Gamma}. \tag{3.17}$$

So

$$\int_\Gamma \phi \cdot \mu ds = \int_{\Omega_1} \nabla N_1(\phi) \cdot \nabla w_1 dx \le |N_1(\phi)|_{1,\Omega_1} |w_1|_{1,\Omega_1} \le c |N_1(\phi)|_{1,\Omega_1} \|\mu\|_{\frac{1}{2},\Gamma},$$

and then

$$\frac{\int_\Gamma \phi \cdot \mu ds}{\|\mu\|_{\frac{1}{2},\Gamma}} \le c |N_1(\phi)|_{1,\Omega_1} \quad \forall \mu \in H^{\frac{1}{2}}(\Gamma). \tag{3.18}$$

Thus

$$\|\phi\|_{-\frac{1}{2},\Gamma} \leq c|N_1(\phi)|_{1,\Omega_1}. \tag{3.19}$$

Combining (3.14) and (3.19) implies that $|N_1(\phi)|_{1,\Omega_1}$ is equivalent to $\|\phi\|_{-\frac{1}{2},\Gamma}$.

(ii) We now prove that $|N_2(\phi)|_{1,\Omega_2}$ is eqivalent to $\|\phi\|_{-\frac{1}{2},\Gamma}$. To do this, from (3.12), by the trace theorem (c.f.[2]) and Friedrichs inequality, it can be seen that for any given $\xi=$ const.

$$\begin{aligned}|N_2(\phi)|^2_{1,\Omega_2} &= \int_{\Omega_2} \nabla N_2(\phi)\cdot\nabla N_2(\phi)dx = \int_{\Omega_2} nablaN_2(\phi)\cdot\nabla(N_2(\phi)-\xi)dx\\ &= -\int_\Gamma \phi\cdot(N_2(\phi)-\xi)ds \leq \|\phi\|_{-\frac{1}{2},\Gamma}\|\gamma_0(N_2(\phi)-\xi)\|_{\frac{1}{2},\Gamma}\\ &\leq c\|\phi\|_{-\frac{1}{2},\Gamma}\|N_2(\phi)-\xi\|_{1,\Omega_2} \leq c\|\phi\|_{-\frac{1}{2},\Gamma}\Big\{|N_2(\phi)-\xi|^2_{1,\Omega_2}\\ &\quad + \Big(\int_{\Omega_2}(N_2(\phi)-\xi)dx\Big)^2\Big\}^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Let $\xi = \dfrac{1}{|\Omega_2|}\displaystyle\int_{\Omega_2} N_2(\phi)dx$ in (3.20), then $|N_2(\phi)|^2_{1,\Omega_2} \leq c\|\phi\|_{-\frac{1}{2},\Gamma}|N_2(\phi)|_{1,\Omega_2}$, from which

$$|N_2(\phi)|_{1,\Omega_2} \leq c\|\phi\|_{-\frac{1}{2},\Gamma}. \tag{3.21}$$

Next, by the similar way in (i), it can be seen that

$$\|\phi\|_{-\frac{1}{2},\Gamma} \leq c|N_2(\phi)|_{1,\Omega_2}. \tag{3.22}$$

Combining (3.21) and (3.22) implies that $|N_2(\phi)|_{1,\Omega_2}$ is equivalent to $\|\phi\|_{-\frac{1}{2},\Gamma}$. Therefore the Lemma is proved.

From the Lemma 3.1 we have

Lemma 3.2. *There exist σ, $\tau =$ const. > 0, such that*

$$\sigma = \mathrm{Sup}\Big\{\frac{|N_1(\phi)|^2_{1,\Omega_1}}{|N_2(\phi)|_{1,\Omega_2}},\ \phi\in H^{-\frac{1}{2}}(\Gamma)\Big\}, \quad \tau = \mathrm{Sup}\Big\{\frac{|N_2(\phi)^2_{1,\Omega_2}}{|N_1(\phi)|^2_{1,\Omega_1}},\ \phi\in H^{-\frac{1}{2}}(\Gamma)\Big\}. \tag{3.23}$$

4. Convergence Analysis of Iterative Scheme

In order to prove the convergence of the iterative scheme (2.7)–(2.9), it is sufficient to prove that

$$\|e_1^n\|_{1,\Omega_1},\ \|e_2^n\|_{1,\Omega_2} \longrightarrow 0, \quad n\longrightarrow\infty \tag{4.1}$$

for an appropriate parameter θ. To do this, we first rewrite the equations (3.2)–(3.4), deleted the supscript n: for any given $\psi \in H^{-\frac{1}{2}}(\Gamma)$,

$$\begin{cases} -\Delta e_1(\psi) = 0 \text{ in } \Omega_1,\\ e_1(\psi) = 0 \text{ on } \Gamma_0,\ \partial_{\nu^1} e_1(\psi) = 0 \text{ on } \partial\Omega_1\setminus(\Gamma\cup\Gamma_0),\ \partial_{\nu^1} e_1(\psi) = \psi \text{ on } \Gamma, \end{cases} \tag{4.2}$$

$$\begin{cases} -\Delta e_2(\psi) = 0 \text{ in } \Omega_2, \\ \partial_{\nu^2} e_2(\psi) = 0 \text{ on } \partial\Omega_2 \setminus \Gamma, \ e_2(\psi) = e_1(\psi) \text{ on } \Gamma. \end{cases} \tag{4.3}$$

Let us introduce the following two operators T:

$$T : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \ \psi \in H^{-\frac{1}{2}}(\Gamma) \to T\psi = -\partial_{\nu^2} e_2(\psi) \text{ on } \Gamma, \tag{4.4}$$

and

$$T_\theta : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \ \psi \in H^{-\frac{1}{2}}(\Gamma) \to T_\theta\psi = \theta T\psi + (1-\theta)\psi. \tag{4.5}$$

Then it can be seen that

$$e_1(\psi) = N_1(\psi), \quad e_2(\psi) = N_2(T\psi). \tag{4.6}$$

We have

Theorem 4.1. *There exists a positive constant $\theta^* \in (0,1]$, such that*

$$\begin{cases} \forall\theta \in (0,\theta^*), \exists K(\theta) < 1 : \\ \{|e_1(T_\theta\psi)|^2_{1,\Omega_1} + |e_2(T_\theta\psi)|^2_{1,\Omega_2}\}^{\frac{1}{2}} \le K(\theta)\{|e_1(\psi)|^2_{1,\Omega_1} + |e_2(\psi)|^2_{1,\Omega_2}\}^{\frac{1}{2}} \ \forall\psi \in H^{-\frac{1}{2}}(\Gamma). \end{cases} \tag{4.7}$$

Proof. By the notations as above, it can be seen that

$$e_1(T_\theta\psi) = \theta e_1(T\psi) + (1-\theta)e_1(\psi), \tag{4.8}$$

$$e_2(T_\theta\psi) = \theta e_2(T\psi) + (1-\theta)e_2(\psi), \tag{4.9}$$

and then

$$\begin{aligned} |e_1(T_{\theta\psi})|^2_{1,\Omega_1} &= \int_{\Omega_1} |\nabla e_1(T_\theta\psi)|^2 dx = \theta^2 |e_1(T\psi)|^2_{1,\Omega_1} \\ &+ (1-\theta)^2 |e_1(\psi)|^2_{1,\Omega_1} + 2\theta(1-\theta)\int_{\Omega_1} \nabla e_1(T\psi)\cdot\nabla e_1(\psi)dx. \end{aligned} \tag{4.10}$$

From (4.2), (4.3) we have

$$\begin{aligned} \int_{\Omega_1} \nabla e_1(T\psi)\cdot\nabla e_1(\psi)dx &= \int_\Gamma \partial_{\nu^1} e_1(T\psi)\cdot e_1(\psi)ds = \int_\Gamma T\psi\cdot e_1(\psi)ds \\ &= -\int_\Gamma \partial_{\nu^2} e_2(\psi)\cdot e_2(\psi)ds = -|e_2(\psi)|^2_{1,\Omega_2}. \end{aligned} \tag{4.11}$$

Thus

$$|e_1(T_\theta\psi)|^2_{1,\Omega_1} = \theta^2|e_1(T\psi)|^2_{1,\Omega_1} + (1-\theta)^2|e_1(\psi)|^2_{1,\Omega_1} - 2\theta(1-\theta)|e_2(\psi)|^2_{1,\Omega_2}. \tag{4.12}$$

By the similar way, we have

$$|e_2(T_\theta\psi)|^2_{1,\Omega_2} = \theta^2|e_2(T\psi)|^2_{1,\Omega_2} + (1-\theta)^2|e_2(\psi)|^2_{1,\Omega_2} - 2\theta(1-\theta)|e_1(T\psi)|^2_{1,\Omega_1}. \tag{4.13}$$

From the Lemma 3.2, we have

$$|e_1(T\psi)|^2_{1,\Omega_1} = |N_1(T\psi)|^2_{1,\Omega_1} \le \sigma|N_2(T\psi)|^2_{1,\Omega_2} = \sigma|e_2(\psi)|^2_{1,\Omega_2},$$

and

$$|e_2(\psi)|^2_{1,\Omega_2} = |N_2(T\psi)|^2_{1,\Omega_2} \le \tau|N_1(T\psi)|^2_{1,\Omega_1} = \tau|e_1(T\psi)|^2_{1,\Omega_1},$$

from which, we have

$$\frac{1}{\tau}|e_2(\psi)|^2_{1,\Omega_2} \le |e_1(T\psi)|^2_{1,\Omega_1} \le \sigma|e_2(\psi)|^2_{1,\Omega_2}. \tag{4.14}$$

We now turn to estimate $|e_2(T\psi)|^2_{1,\Omega_2}$. From (4.2) and (4.3), we have

$$\begin{aligned} |e_2(T\psi)|^2_{1,\Omega_2} &= \int_{\Omega_2} \nabla e_2(T\psi)\cdot\nabla e_2(T\psi)dx = \int_\Gamma \partial_{\nu^2} e_2(T\psi)e_2(T\psi)ds \\ &= -\int_\Gamma T(T\psi)e_1(T\psi)ds = -\int_{\Omega_1} \nabla e_1(T(T\psi))\cdot\nabla e_1(T\psi)dx \\ &\le |e_1(T(T\psi))|_{1,\Omega_1}|e_1(T\psi)|_{1,\Omega_1}. \end{aligned} \tag{4.15}$$

By the Lemma 3.2, we have

$$|e_1(T(T\psi))|^2_{1,\Omega_1} = |N_1(T(T\psi))|^2_{1,\Omega_1} \le \sigma|N_2(T(T\psi))|^2_{1,\Omega_2} = \sigma|e_2(T\psi)|^2_{1,\Omega_2}. \tag{4.16}$$

Combinig (4.14)–(4.16), it can be seen that

$$|e_2(T\psi)|_{1,\Omega_2} \le \sqrt{\sigma}|e_1(T\psi)|_{1,\Omega_1} \le \sigma|e_2(\psi)|_{1,\Omega_2}. \tag{4.17}$$

Summarizing (4.12), (4.13), (4.14) and (4.17), it can be seen that

$$\begin{aligned} |e_1(T_\theta\psi)|^2_{1,\Omega_1} + |e_2(T_\theta\psi)|^2_{1,\Omega_2} &\le (1-\theta)^2|e_1(\psi)|^2_{1,\Omega_1} \\ &+ \Big\{(1-\theta)^2 + 2\Big(1+\sigma+\frac{1}{\tau}\Big)\theta\Big(\theta - \frac{1}{1+\frac{\sigma\tau}{1+\tau}}\Big)\Big\}|e_2(\psi)|^2_{1,\Omega_2} \end{aligned} \tag{4.18}$$

Let

$$\theta_1^* = \frac{1}{1+\frac{\sigma\tau}{1+\tau}}, \quad 0 < \theta_1^* < 1, \tag{4.19}$$

then for $0 < \theta \le \theta_1^*$,

$$|e_1(T_\theta\psi)|^2_{1,\omega_1} + |e_2(T_\theta\psi)|^1_{1,\Omega_2} \le K_1(\theta)\{|e_1(\psi)|^2_{1,\Omega_1} + |e_2(\psi)|^2_{1,\Omega_2}\}, \tag{4.20}$$

where

$$0 < K_1(\theta) = (1-\theta)^2 < 1. \tag{4.21}$$

Next let

$$\theta_2^* = \min\Big\{1, \frac{1}{1+\frac{\sigma-\frac{1}{2}}{2+\frac{1}{\tau}}}\Big\}, \tag{4.22}$$

then it can be easily seen that

$$\theta_1^* < \theta_2^* \le 1, \tag{4.23}$$

and for $\theta_1^* < \theta < \theta_2^*$,

$$\begin{aligned} (1-\theta)^2 &\le (1-\theta)^2 + 2\Big(1+\sigma+\frac{1}{\tau}\Big)\theta\Big(\theta - \frac{1}{1+\frac{\sigma\tau}{1+\tau}}\Big) \\ &= 1 + \theta\Big(3+2\sigma+\frac{2}{\tau}\Big)\Big(\theta - \frac{1}{1+\frac{\sigma-\frac{1}{2}}{2+\frac{1}{\tau}}}\Big) = K_2(\theta) < 1, \end{aligned} \tag{4.24}$$

from which we have that, for $\theta_1^* \le \theta < \theta_2^*$,

$$|e_1(T_\theta\psi)|^2_{1,\Omega_1} + |e_2(T_\theta\psi)|^2_{1,\Omega_2} \le K_2(\theta)\{|e_1(\psi)|^2_{1,\Omega_1} + |e_2(\psi)|^2_{1,\Omega_2}\}. \tag{4.25}$$

Combining (4.20) and (4.25), in order to complete the proof of the Theorem 4.1 it is sufficient to take

$$\theta^* = \theta_2^*, \tag{4.26}$$

and

$$0 < K(\theta) = \max(K_1(\theta), K_2(\theta)) < 1. \tag{4.27}$$

We conclude this section with the following convergence result.

Theorem 4.2. *For any given* $g^0 \in H^{-\frac{1}{2}}(\Gamma)$, *the iterative scheme* (2.7)–(2.9) *is convergence, as* $n \to \infty$, *with parameter* $\theta \in (0, \theta^*)$, *where*θ^* *is given by* (4.26).

Proof. By the Theorem 4.1, for each $n \le 1$, we have

$$|e_1^n|^2_{1,\Omega_1} + |e_2^n|^2_{1,\Omega_2} \le K(\theta)\{|e_1^{n-1}|^2_{1,\Omega_1} + |e_2^{n-1}|^2_{1,\Omega}\}. \tag{4.28}$$

Then

$$|e_1^n|^2_{1,\Omega_1} + |e_1^n|^2_{1,\Omega_2} \le K^n(\theta)\{|e_1^0|^2_{1,\Omega_1} + |e_2^0|^2_{1,\Omega_2}\} \longrightarrow 0, \ n \longrightarrow \infty. \tag{4.29}$$

By the Poincare inequality, since $e_1^n = 0$ on Γ_0,

$$\|e_1^n\|^2_{1,\Omega_1} \le c|e_1^n|^2_{1,\Omega} \longrightarrow 0, \quad n \longrightarrow \infty; \tag{4.30}$$

from which, we have

$$\|e_1^n\|_{\frac{1}{2},\Gamma} \le c\|e_1^n\|_{1,\Omega_1} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.31}$$

By the Friedrichs inequality, since $e_2^n = e_1^n$ on Γ

$$\begin{aligned}\|e_2^n\|^2_{1,\Omega_2} &\le c\Big\{|e_2^n|^2_{1,\Omega_2} + \Big(\int_\Gamma e_2^n ds\Big)^2\Big\}\\ &\le c\{|e_2^n|^2_{1,\Omega_2} + \|e_1^n\|^2_{\frac{1}{2},\Gamma}\} \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{4.32}$$

Combining (4.30) and (4.32) implies the convergence of the iterative scheme (2.7)–(2.9).

References

[1] J.H. Bramble, J.E. Pasciak and A.H. Schatz, The construction of precoditioners for elliptic problems by substructuring, I, *Math. Compu.*, **47** (1986), 103–134.

[2] J.L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer-Verlag, 1972.

[3] L.D. Marini and A. Quarteroni, An iterative procedure for domain decomposition methods: A finite element approach. In: Glowinski R., Golub G.H., Meurant G.A., Periaux J. (eds.) Domain Decomposition Methods for Partial Differential Equations, Philadelphia, SIAM 1988, 129–143,.

[4] L.D. Marini and A. Quarteroni, A relaxation procedure for domain decomposition methods using finite elements, *Numer. Math.*, **55** (1989), 575–598.