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A NEW CLASS OF UNIFORMLY SECOND ORDER ACCURATE DIFFRENCE SCHEMES FOR 2D SCALAR CONSERVATION LAWS*

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Abstract

In this paper, concerned with the Cauchy problem for 2D nonlinear hyperbolic conservation laws,we construct a class of uniformly second order accurate finite difference schemes, which are based on the E-schemes. By applying the convergence theorem of Coquel-Le Floch [1], the family of approximate solutions defined by the scheme is proven to converge to the unique entropy weak L^∞-solution. Furthermore, some numerical experiments on the Cauchy problem for the advection equation and the Riemann problem for the 2D Burgers equation are given and the relatively satisfied result is obtained.

1. Convergence of A Class of Uniformly Second Order Accurate Difference Schemes

In this section, we consider the Cauchy problem for nonlinear hyperbolic scalar conservation laws with two space variables:

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0, \quad u(t,x,y) \in R, t \in (0,T), (x,y) \in R^2, \tag{1.1}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in R^2, \tag{1.2}$$

where f and $g: R \to R$ are Lipschitz continuous functions and the initial data u_0 is a bounded function with compact support.

Let $\Delta t, \Delta x, \Delta y$ be the time, x-space and y-space increments of the discretization respectively. The mesh ratios, $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$, will be kept constants. $\Delta_+ u^n_{i+\frac{1}{2},j} = u^n_{i+1,j} - u^n_{i,j}$, $\Delta_+ u^n_{i,j+\frac{1}{2}} = u^n_{i,j+1} - u^n_{i,j}$.

In [2], the authors have discussed a class of high order accurate schemes constructed from E scheme by the flux limiters. The scheme is in the form ($n \in N$)

$$u^{n+1}_{i,j} = u^n_{i,j} - \lambda_x \Delta_+ f^n_{i+\frac{1}{2},j} - \lambda_y \Delta_+ g^n_{i,j+\frac{1}{2}}, \quad i,j \in Z, \tag{1.3}$$

$$f^n_{i+\frac{1}{2},j} = h\Big(u^n_{i+1,j} - \frac{1}{2}p^n_{i+\frac{1}{2},j},\ u^n_{i,j} + \frac{1}{2}q^n_{i+\frac{1}{2},j}\Big),$$

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$$g^n_{i,j+\frac{1}{2}} = l\Big(u^n_{i,j+1} - \frac{1}{2}r^n_{i,j+\frac{1}{2}},\ u^n_{i,j} + \frac{1}{2}s^n_{i,j+\frac{1}{2}}\Big), \quad i,j \in Z, \tag{1.4}$$

where

$$\begin{aligned}
p^n_{i+\frac{1}{2},j} &= \phi^1(t^n_{i+\frac{3}{2},j})\Delta_+u^n_{i+\frac{3}{2},j}\theta\Big(\frac{|\Delta_+u^n_{i+\frac{3}{2},j}|}{c_1h^{\alpha_1}}\Big)\\
q^n_{i+\frac{1}{2},j} &= \phi^1(w^n_{i+\frac{1}{2},j})\Delta_+u^n_{i-\frac{1}{2},j}\theta\Big(\frac{|\Delta_+u^n_{i-\frac{1}{2},j}|}{c_1h^{\alpha_1}}\Big)\\
r^n_{i,j+\frac{1}{2}} &= \phi^2(t^n_{i,j+\frac{3}{2}})\Delta_+u^n_{i,j+\frac{3}{2}}\theta\Big(\frac{|\Delta_+u^n_{i,j+\frac{3}{2}}|}{c_2h^{\alpha_2}}\Big)\\
s^n_{i,j+\frac{1}{2}} &= \phi^2(w^n_{i,j+\frac{1}{2}})\Delta_+u^n_{i,j-\frac{1}{2}}\theta\Big(\frac{|\Delta_+u^n_{i,j-\frac{1}{2}}|}{c_2h^{\alpha_2}}\Big), \quad n \in N, i,j \in Z, \qquad (1.5)\\
t^n_{i+\frac{1}{2},j} &= \frac{\Delta_+u^n_{i-\frac{1}{2},j}}{\Delta_+u^n_{i+\frac{1}{2},j}}, \quad t^n_{i,j+\frac{1}{2}} = \frac{\Delta_+u^n_{i,j-\frac{1}{2}}}{\Delta_+u^n_{i,j+\frac{1}{2}}}\\
w^n_{i+\frac{1}{2},j} &= \frac{1}{t^n_{i+\frac{1}{2},j}}, \quad w^n_{i,j+\frac{1}{2}} = \frac{1}{t^n_{i,j+\frac{1}{2}}}, \quad n \in N, i,j \in Z\\
\theta(r) &= \begin{cases} 1 & |r| \le 1\\ bh & |r| > 1 \end{cases}, \quad b \ge 0\\
0 &< \alpha^k < 1, \quad c_k > 0, \quad \text{for } k = 1,2
\end{aligned}$$

$h(u,v)$, $l(u,v)$ are the numerical flux functions of any two three-point E-schemes. ϕ^1, ϕ^2 are flux limiters.

We list two results of the authors in [2] which will be needed in this paper.

Lemma 1.1. [2] *Suppose that the condition*

$$0 \le \phi^k(r) \le \mu, \quad \phi^k(0) = 0, \quad 0 \le \frac{\phi^k(r)}{r} \le 1, \quad \textit{for } k = 1,2, \tag{1.6}$$

holds true and λ_x, λ_y *satisfy the condition*

$$\lambda_x \max_{u,v}\{|h_0|,|h_1|\} + \lambda_y \max_{u,v}\{|l_0|,|l_1|\} \le \frac{1}{2+\mu}, \tag{1.7}$$

where $h_0 = \partial h(u,v)/\partial v$, $h_1 = \partial h(u,v)/\partial u$, $l_0 = \partial l(u,v)/\partial v$, $l_1 = \partial l(u,v)/\partial u$. *Then the scheme* (1.3)–(1.5) *can be of the form* ($n \in N$)

$$u^{n+1}_{i,j} = u^n_{i,j} + C^n_{i+\frac{1}{2},j}\Delta_+u^n_{i+\frac{1}{2},j} - D^n_{i-\frac{1}{2},j}\Delta_+u^n_{i-\frac{1}{2},j} + E^n_{i,j+\frac{1}{2}}\Delta_+u^n_{i,j+\frac{1}{2}} - F^n_{i,j-\frac{1}{2}}\Delta_+u^n_{i,j-\frac{1}{2}},$$

where

$$C^n_{i+\frac{1}{2},j} \ge 0,\ D^n_{i+\frac{1}{2},j} \ge 0,\ E^n_{i,j+\frac{1}{2}} \ge 0,\ F^n_{i,j+\frac{1}{2}} \ge 0, \quad i,j \in Z, \tag{1.8}$$

$$C^n_{i+\frac{1}{2},j} + D^n_{i-\frac{1}{2},j} + E^n_{i,j+\frac{1}{2}} + F^n_{i,j-\frac{1}{2}} \le 1, \quad i,j \in Z. \tag{1.9}$$

Lemma 1.2. [2] *If the function* $\phi^k (k = 1,2)$ *satisfies* $\phi^k(x) = a^k_1 x + a^k_2$, *where* $a^k_1 \ge 0$, $a^k_2 \ge 0$, $a^k_1 + a^k_2 \equiv 1$, *for* $k = 1,2$, *then the scheme* (1.3)–(1.5) *is uniformly second order accurate in space.*

Substitute the formula (1.5) by

$$
\begin{aligned}
p^n_{i+\frac{1}{2},j} &= \Delta_+ u^n_{i+\frac{3}{2},j}\theta_1\Big(\frac{|\Delta_+ u^n_{i+\frac{3}{2},j}|}{c_1 h^{\alpha_1}}\Big)\theta_2\Big(\frac{|\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}|}{c_2 h^{\alpha_2}}\Big),\\
q^n_{i+\frac{1}{2},j} &= \Delta_+ u^n_{i-\frac{1}{2},j}\theta_1\Big(\frac{|\Delta_+ u^n_{i-\frac{1}{2},j}|}{c_1 h^{\alpha_1}}\Big)\theta_2\Big(\frac{|\Delta_+ u^n_{i-\frac{1}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}|}{c_2 h^{\alpha_2}}\Big),\\
r^n_{i,j+\frac{1}{2}} &= \Delta_+ u^n_{i,j+\frac{3}{2}}\theta_1\Big(\frac{|\Delta_+ u^n_{i,j+\frac{3}{2}}|}{d_1 h^{\beta_1}}\Big)\theta_2\Big(\frac{|\Delta_+ u^n_{i,j+\frac{3}{2}} - \Delta_+ u^n_{i,j+\frac{1}{2}}|}{d_2 h^{\beta_2}}\Big),\\
s^n_{i,j+\frac{1}{2}} &= \Delta_+ u^n_{i,j-\frac{1}{2}}\theta_1\Big(\frac{|\Delta_+ u^n_{i,j-\frac{1}{2}}|}{d_1 h^{\beta_1}}\Big)\theta_2\Big(\frac{|\Delta_+ u^n_{i,j-\frac{1}{2}} - \Delta_+ u^n_{i,j+\frac{1}{2}}|}{d_2 h^{\beta_2}}\Big),
\end{aligned}
\tag{1.10}
$$

where

$$
\theta_1(r) = \begin{cases} 1 & |r| \le 1 \\ bh & |r| > 1 \end{cases}, \quad \theta_2(r) = \begin{cases} 1 & |r| \le 1 \\ 0 & |r| > 1 \end{cases},
$$
$$
b \ge 0,\ c_1, c_2, d_1, d_2 > 0,\ 0 < \alpha_1, \beta_1 < 1,\ 1 \le \alpha_2, \beta_2 < 2.
$$

From Lemma 1.2, it is easy to know that the scheme (1.3), (1.4), (1.10) is also uniformly second order accurate in space, since the item θ_2 in (1.10) doesn't exert any influence on accuracy of the scheme.

We consider approximate solutions $u^h: R_+ \times R^2 \to R$ to problem (1.1), (1.2) which are piecewise constant, i.e.

$$
u^h(t,x,y) = u^n_{i,j}, \quad \text{for} \quad t \in [t_n, t_{n+1}), \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}). \tag{1.11}
$$

For $n = 0$, $u^0_{i,j} (i, j \in Z)$ is defined from the initial data u_0 by

$$
u^0_{i,j} = \frac{1}{h_x h_y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_0(x,y)\,dx\,dy, \quad i, j \in Z. \tag{1.12}
$$

To prove the convergence of scheme (1.3), (1.4), (1.10), the following two lemmas are critical.

Lemma 1.3. *The scheme* (1.3), (1.4), (1.10) *is stable in L^∞-norm, i.e., there exists a constant $C > 0$ independent of h, h_x and h_y such that*

$$
|u^n_{i,j}| \le C, \quad n \in N, \quad i, j \in Z \tag{1.13}
$$

provided that λ_x and λ_y satisfy the condition

$$
\lambda_x \max_{u,v}\{|h_0|, |h_1|\} + \lambda_y \max_{u,v}\{|l_0|, |l_1|\} \le \frac{1}{3} \tag{1.14}
$$

Proof. Set

$$
\phi_1(r) = \begin{cases} 1 & r > 1 \\ r & 0 \le r \le 1 \\ 0 & r < 0 \end{cases}, \qquad \phi_2(r) = \begin{cases} 0 & r > 1 \\ 1 - r & 0 \le r \le 1 \\ 1 & r < 0 \end{cases}, \tag{1.15}
$$

obviously the functions $\phi_1(r)$ and $\phi_2(r)$ satisfy

$$
\phi_1(r) + \phi_2(r) \equiv 1. \tag{1.16}
$$

We introduce the notation

$$\bar{f}^n_{i+\frac{1}{2},j} = h\Big(u^n_{i+1,j} - \frac{1}{2}\bar{p}^n_{i+\frac{1}{2},j}, u^n_{i,j} + \frac{1}{2}\bar{q}^n_{i+\frac{1}{2},j}\Big),$$
$$\bar{g}^n_{i,j+\frac{1}{2}} = l\Big(u^n_{i,j+1} - \frac{1}{2}\bar{r}^n_{i,j+\frac{1}{2}}, u^n_{i,j} + \frac{1}{2}\bar{s}^n_{i,j+\frac{1}{2}}\Big), \tag{1.17}$$

where

$$\bar{p}^n_{i+\frac{1}{2},j} = \phi_1(t^n_{i+\frac{3}{2},j})p^n_{i+\frac{1}{2},j}, \quad \bar{q}^n_{i+\frac{1}{2},j} = \phi_1\Big(\frac{1}{t^n_{i+\frac{1}{2},j}}\Big)q^n_{i+\frac{1}{2},j},$$
$$\bar{r}^n_{i,j+\frac{1}{2}} = \phi_1(t^n_{i,j+\frac{3}{2}})r^n_{i,j+\frac{1}{2}}, \quad \bar{s}^n_{i,j+\frac{1}{2}} = \phi_1\Big(\frac{1}{t^n_{i,j+\frac{1}{2}}}\Big)s^n_{i,j+\frac{1}{2}}.$$

Furthermore,we denote

$$K^n_{i,j} = \lambda_x(\Delta_+\bar{f}^n_{i+\frac{1}{2},j} - \Delta_+ f^n_{i+\frac{1}{2},j}) + \lambda_y(\Delta_+\bar{g}^n_{i,j+\frac{1}{2}} - \Delta_+ g^n_{i,j+\frac{1}{2}}), \tag{1.18}$$

thus the scheme (1.3), (1.4), (1.10) can be rewritten as follows:

$$u^{n+1}_{i,j} = u^n_{i,j} - \lambda_x\Delta_+\bar{f}^n_{i+\frac{1}{2},j} - \lambda_y\Delta_+\bar{g}^n_{i,j+\frac{1}{2}} + K^n_{i,j}. \tag{1.19}$$

Since the function ϕ_1 satisfies $0 \le \phi_1(r)/r \le 1$, $0 \le \phi_1(r) \le 1$, $\phi_1(0) = 0$ and the condition (1.14) holds true,similarly discussing as Lemma 1, we know that the scheme (1.3), (1.4), (1.10) can be of the following form

$$\begin{aligned} u^{n+1}_{i,j} =& u^n_{i,j} + C^n_{i+\frac{1}{2},j}\Delta_+ u^n_{i+\frac{1}{2},j} - D^n_{i-\frac{1}{2},j}\Delta_+ u^n_{i-\frac{1}{2},j} \\ &+ E^n_{i,j+\frac{1}{2}}\Delta_+ u^n_{i,j+\frac{1}{2}} - F^n_{i,j-\frac{1}{2}}\Delta_+ u^n_{i,j-\frac{1}{2}} + K^n_{i,j}, \end{aligned} \tag{1.20}$$

where $C^n_{i+\frac{1}{2},j}$, $D^n_{i-\frac{1}{2},j}$, $E^n_{i,j+\frac{1}{2}}$, $F^n_{i,j-\frac{1}{2}}$ satisfy (1.8) and (1.9).

In order to estimate the term $K^n_{i,j}$, we set

$$B^n_{i,j} = \lambda_x|\Delta_+\bar{f}^n_{i+\frac{1}{2},j} - \Delta_+ f^n_{i+\frac{1}{2},j}|, \quad C^n_{i,j} = \lambda_y|\Delta_+\bar{g}^n_{i,j+\frac{1}{2}} - \Delta_+ g^n_{i,j+\frac{1}{2}}|,$$

thus

$$|K^n_{i,j}| \le B^n_{i,j} + C^n_{i,j}. \tag{1.21}$$

First, we analyze the term $B^n_{i,j}$. From (1.4) and (1.17), the following inequality can be deduced ($M = \lambda_x/2 \max\limits_{u,v}\{|h_0|, |h_1|\}$)

$$B^n_{i,j} \le M\{|p^n_{i+\frac{1}{2},j} - \bar{p}^n_{i+\frac{1}{2},j}| + |q^n_{i+\frac{1}{2},j} - \bar{q}^n_{i+\frac{1}{2},j}| + |p^n_{i-\frac{1}{2},j} - \bar{p}^n_{i-\frac{1}{2},j}| + |q^n_{i-\frac{1}{2},j} - \bar{q}^n_{i-\frac{1}{2},j}|\}. \tag{1.22}$$

Noticing (1.16) and the definition of θ_1, we get

$$|p^n_{i+\frac{1}{2},j} - \bar{p}^n_{i+\frac{1}{2},j}| \le C_0\Big|\phi_2(t^n_{i+\frac{3}{2},j})\Delta_+ u^n_{i+\frac{3}{2},j}\theta_2\Big(\frac{|\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}|}{c_2 h^{\alpha_2}}\Big)\Big|, \tag{1.23}$$

where C_0 is a positive constant independent of h, h_x and h_y.

We deduce from (1.15) and the definition of function θ_2:

$$|p^n_{i+\frac{1}{2},j} - \bar{p}^n_{i+\frac{1}{2},j}| \leq \begin{cases} 0 & t^n_{i+\frac{3}{2},j} > 1 \text{ or } |\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}| > c_2 h^{\alpha_2} \\ C_0|\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}| & 0 \leq t^n_{i+\frac{3}{2},j} \leq 1 \& |\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}| \leq c_2 h^{\alpha_2} \\ C_0|\Delta_+ u^n_{i+\frac{3}{2},j}| & t^n_{i+\frac{3}{2},j} < 0 \& |\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}| \leq c_2 h^{\alpha_2} \end{cases} \quad (1.24)$$

We further remark that the term $\Delta_+ u^n_{i+\frac{3}{2},j}$ and the term $\Delta_+ u^n_{i+\frac{1}{2},j}$ must have the different symbol if $t^n_{i+\frac{3}{2},j} < 0$, thus $|\Delta_+ u^n_{i+\frac{3}{2},j}| \leq |\Delta_+ u^n_{i+\frac{3}{2},j} - \Delta_+ u^n_{i+\frac{1}{2},j}|$, if $t^n_{i+\frac{3}{2},j} < 0$. Taking the above inequality into account,we get from (1.24) $|p^n_{i+\frac{1}{2},j} - \bar{p}^n_{i+\frac{1}{2},j}| \leq C_0 c_2 h^{\alpha_2}$. Similarly for the other terms of $B^n_{i,j}$, thus we deduce from (1.22) that there exists a constant $C_0^1 > 0$, independent of h, h_x and h_y such that $B^n_{i,j} \leq C_0^1 h^{\alpha_2}$, $n \in N$, $i, j \in Z$.

For the term $C^n_{i,j}$, we can similarly discuss and obtain the similar result: $C^n_{i,j} \leq C_0^2 h^{\beta_2}$, $n \in N$, $i, j \in Z$, C_0^2 is a positive constant independent of h, h_x and h_y.

Thus, from (1.21) we know that there exists a constant $C > 0$ independent of h, h_x and h_y such that

$$|K^n_{i,j}| \leq C h^{\min(\alpha_2,\beta_2)}, \quad n \in N, \quad i, j \in Z. \quad (1.25)$$

Now we can prove the uniformly bounded property of the scheme (1.3), (1.4), (1.10).

From (1.20) and (1.25), we can get

$$\begin{aligned} |u^{n+1}_{i,j}| \leq & (1 - C^n_{i+\frac{1}{2},j} - D^n_{i-\frac{1}{2},j} - E^n_{i,j+\frac{1}{2}} - F^n_{i,j-\frac{1}{2}})|u^n_{i,j}| + C^n_{i+\frac{1}{2},j}|u^n_{i+1,j}| + D^n_{i-\frac{1}{2},j}|u^n_{i-1,j}| \\ & + E^n_{i,j+\frac{1}{2}}|u^n_{i,j+1}| + F^n_{i,j-\frac{1}{2}}|u^n_{i,j-1}| + C h^{\min(\alpha_2,\beta_2)}. \end{aligned}$$

Taking the supremum and by induction, we find

$$\sup_{i,j \in Z} |u^n_{i,j}| \leq \sup_{i,j \in Z} |u^0_{i,j}| + C n h^{\min(\alpha_2,\beta_2)}, \quad n \in N.$$

Because of $n.h \leq T$ and $1 \leq \alpha_2$, $\beta_2 < 2$, the result follows.

Lemma 1.4. *Suppose that the condition* (1.14) *holds true, then there exist constants* $M_1, M_2 > 0$, *independent of* h, h_x *and* h_y *such that*

$$|f^n_{i+\frac{1}{2},j} - h^n_{i+\frac{1}{2},j}| \leq M_1 h^{\alpha_1}, \quad n \in N, i, j \in Z, \quad (1.26)$$

$$|g^n_{i,j+\frac{1}{2}} - l^n_{i,j+\frac{1}{2}}| \leq M_2 h^{\beta_1}, \quad n \in N, i, j \in Z, \quad (1.27)$$

where $h^n_{i+\frac{1}{2},j} = h(u^n_{i+1,j}, u^n_{i,j})$, $l^n_{i,j+\frac{1}{2}} = l(u^n_{i,j+1}, u^n_{i,j})$, $n \in N$, $i, j \in Z$.

Proof. We only need prove the inequality (1.26). The proof of (1.27) is similar.

Let $M_0 = 1/2 \max_{u,v}\{|h_0|, |h_1|\}$, so the formula (1.4) and (1.10) deduce

$$|f^n_{i+\frac{1}{2},j} - h^n_{i+\frac{1}{2},j}| \le M_0\left[\left|\Delta_+ u^n_{i+\frac{3}{2},j}\theta_1\Big(\frac{|\Delta_+ u^n_{i+\frac{3}{2},j}|}{C_1 h^{\alpha_1}}\Big)\right| + \left|\Delta_+ u^n_{i-\frac{1}{2},j}\theta_1\Big(\frac{|\Delta_+ u^n_{i-\frac{1}{2},j}|}{C_1 h^{\alpha_1}}\Big)\right|\right]. \tag{1.28}$$

Finally Lemma 1.3, the definition of function θ_1 and the fact $0 < \alpha_1 < 1$ yield the following inequality $|f^n_{i+\frac{1}{2},j} - h^n_{i+\frac{1}{2},j}| \le M_1 h^{\alpha_1}$, where $M_1 > 0$, a constant which is independent of h, h_x, and h_y.

This completes the proof.

On the basis of the uniform L^∞-estimate (1.13) and the inequality (1.27), (1.28), using Theorem 4.3 of Coquel-Le Floch [1], we can easily obtain the convergence of scheme (1.3), (1.4), (1.10). The conclusion is summarized in the following theorem.

Theorem 1.1. *Consider the family of approximate solutions $\{u^h\}_{h>0}$ constructed by the scheme* (1.3), (1.4), (1.10) *and* (1.11), (1.12). *Assume*

$$2/3 < \alpha_1 < 1, \ 2/3 < \beta_1 < 1. \tag{1.29}$$

Suppose u is the unique entropy weak L^∞-solution to problem (1.1), (1.2). Then under the condition (1.14) and the CFL stability condition

$$\lambda_x \sup_u |f'(u)| \le \frac{1}{4}, \quad \lambda_y \sup_u |g'(u)| \le \frac{1}{4}, \tag{1.30}$$

the approximate solutions u^h converge in the L^1 strong topology to the unique entropy solution u, as $h \to 0$.

We can use the Runge-Kutta method adapted to nonlinear hyperbolic equations by Shu-Osher [3] etc.to improve the order of accuracy in time of the scheme (1.3), (1.4), (1.10), so a class of schemes with uniformly high order accuracy both in space and time be obtained. The previous property of convergence has been proven to be preserved under the definite CFL condition by Coquel Le-Floch [4].

For example:

$$\begin{cases} \tilde{u}^n_{i,j} = u^n_{i,j} - \lambda_x \Delta_+ f_{i+\frac{1}{2},j}(u^n) - \lambda_y \Delta_+ g_{i,j+\frac{1}{2}}(u^n) \\ u^{n+1}_{i,j} = 1/2 u^n_{i,j} + 1/2[\tilde{u}^n_{i,j} - \lambda_x \Delta_+ f_{i+\frac{1}{2},j}(\tilde{u}^n) - \lambda_y \Delta_+ g_{i,j+\frac{1}{2}}(\tilde{u}^n)], \end{cases} \quad n \in N \quad i,j \in Z \tag{1.31}$$

where $f_{i+\frac{1}{2},j}$, $g_{i,j+\frac{1}{2}}$ are given by (1.4), (1.10). It is easy to verify that the scheme (1.31) is uniformly second order accurate both in space and time. If the condition (1.14), (1.29) and the CFL stability condition (1.30) hold true, then the family of approximate solutions constructed by the scheme (1.31) converges to the unique entropy L^∞-solution.

The previous scheme and convergence result can be easily extended to the case of an equation with more space dimensions, since the theory of Coquel Le-Floch is applicable for an equation with an arbitrary number of space variables.

2. Numerical Tests

We select the Cauchy problem for 2D linear advection equation and the Riemann problem for 2D Burgers equation to test the new uniformly second order accurate scheme introduced in section 1.

Example 1: Consider the 2D linear advection equation

$$u_t + (\alpha(y)u)_x + (\beta(x)u)_y = 0 \tag{2.1}$$

with

$$\alpha(y) = -(y - y_0)\omega, \quad \beta(x) = (x - x_0)\omega \tag{2.2}$$

The exect solution of (2.1), (2.2) consists in the rotation of the initial values round (x_0, y_0) with the angular velocity ω. We will present one numerical test. As initial value we choose a cone. The problem has been used by C.D. Munz[5] etc. to test numerical schemes. We choose the angular velocity ω to be 1 and $x_0 = 20$, $y_0 = 20$. The region of computation is $[0, 40] \times [0, 40]$. At time $t = 2\pi$ the initial value has carried out one full rotation and returned to its initial position. The approximation of the initial value on our grid is shown in Fig. 1. We present numerical calculations on the scheme (1.31), where the functions h and l are both chosen as the numerical flux of the first order upwind scheme. Select the time step $h = 0.01$. Fig.2 shows the numerical result obtained after one full rotation corresponding to 628 time steps.

Fig.2 indicates the shape of the cone is well preserved,the top of the cone is not clipped and the numerical dissipation is weak. The result of the cone shows the uniform high order accurate property of the scheme (1.31).

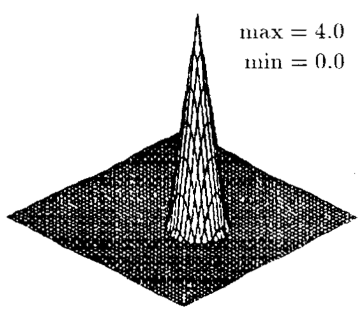

Fig. 1. The initial value of a cone

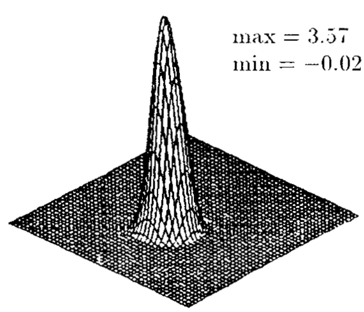

Fig. 2. The numerical result of the cone after one full rotation

Example 2: We solve a Riemann problem for the 2D Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0 \tag{2.3}$$

of the type

$$u(x, y, 0) = \begin{cases} u_1 & x > 0, \quad y > 0 \\ u_2 & x < 0, \quad y > 0 \\ u_3 & x < 0, \quad y < 0 \\ u_4 & x > 0, \quad y < 0 \end{cases}$$

Depending on the orders of the $u_i's$, there are eight essentially different solution types. See [6] for details. We used scheme (1.31) with 30×30 grid points for two cases. We choose Case 1. $u_1 = -0.2$, $u_2 = -1.0$, $u_3 = 0.5$, u_4=0.8 and choose Case 2. $u_1 = -1.0$, $u_2 = -0.2$, $u_3 = 0.8$, u_4=0.5. The numerical fluxes h, l are the same as Example 1. Fig.3-4 show the approximations of the initial values on our grid.The numerical results of two cases at $t = 1.0$ are displayed in Fig.5-6 respectively. We observe that the scheme has good resolution for above two cases.

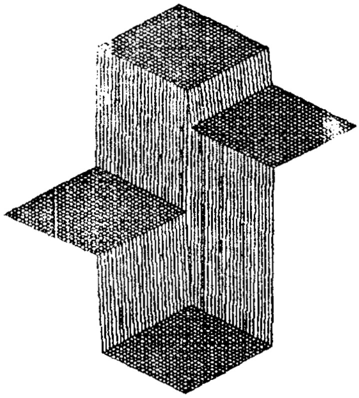

Fig. 3. The initial value of case 1

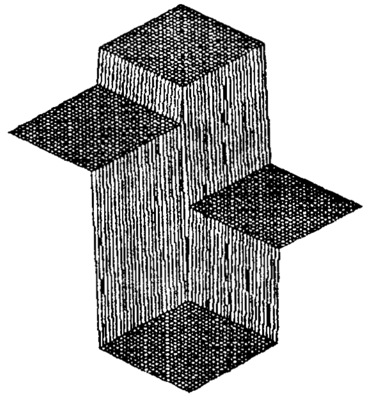

Fig. 4. The initial value of case 2

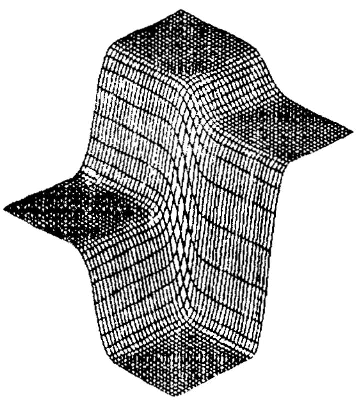

Fig. 5. The numerical result of case 1, at $t = 1.0$

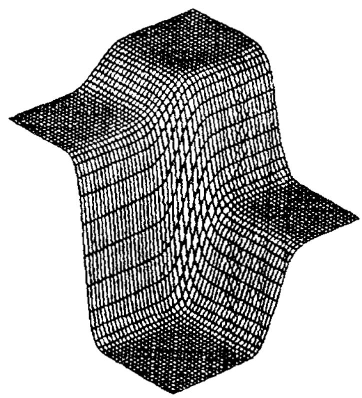

Fig. 6. The numerical result of case 2, at $t = 1.0$

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