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FINITE DIFFERENCE METHOD WITH NONUNIFORM MESHES FOR QUASILINEAR PARABOLIC SYSTEMS*1)

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Abstract

The analysis of the finite difference schemes with nonuniform meshes for the problems of partial differential equations is extremely rare even for very simple problems and even for the method of fully heuristic character. In the present work the boundary value problem for quasilinear parabolic system is solved by the finite difference method with nonuniform meshes. By using of the interpolation formulas for the spaces of discrete functions with unequal meshsteps and the method of a priori estimation for the discrete solutions of finite difference schemes with nonuniform meshes, the absolute and relative convergence of the discrete solutions of the finite defference scheme are proved. The limiting vector function is just the unique generalized solution of the original problem for the parabolic system.

1. In the study of the problem in physics, mechanics, chemical reactions, biology and other practical sciences, the linear and nonlinear parabolic equations and systems are appeared very frequently. Many numerical investigations in scientific and engineering problems especially in the large scale computational problems often contain the numerical solutions of parabolic equations and systems. The method with unequal meshsteps is not avoidable in these computations. Many unexpected and self-contradictory phenomenon raising from the use of unequal meshsteps call our great attention to study the cause and the method of solution.

For the parabolic equations and systems there are various finite difference schemes of approximations with truncation error of different order for the purpose of different usage. There are a great amount of works contributed to the convergence and stability study of finite difference schemes for the linear and nonlinear parabolic equations. All these studies are concerned to the method with equal meshstep.

By use of interpolation formulas for the norms of intermediate quotients for discrete functions and the method of a priori estimation for the discrete solutions of finite difference schemes we get the great success in the studies of the finite difference method with equal meshstep for the solutions of the problems of partial differential equations and systems[1–5]. The study is of rigorous character and avoids the methods of heuristic character. This new method of study is very appropriated for the general difference

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schemes and for the general linear and nonlinear problems of partial differential equations and systems. The results are obtained without any assumptions which are hardly to verify, for example, the existence and uniqueness of the sufficiently smooth solutions for the original problems, the maintenance of the fundamental behavior of the solutions under the treatment of linearization and another ways of simplification.

The consideration for the solutions of partial differential equation by the finite difference method with nonuniform meshes is extremely rare even for very simple problems and even for the method of fully heuristic character. All present situations tell us to know that it is very helpful to study the difference schemes with unequal meshsteps for the problems of partial differential equations and systems by the use of the interpolation formulas for the spaces of discrete functions with unequal meshsteps[6,7], and the method of a priori estimation for the discrete solutions of the difference schemes.

In the present work, we are going to solve the boundary value problems for the quasilinear parabolic systems of partial differential equations of second order by the finite difference schemes with unequal meshsteps. The absolute and relative convergence of discrete solutions for the very general difference schemes for the mentioned problems are proved without any assumption on the existence of the smooth solutions for the original problem.

1. Finite Difference Schemes

2. Let us consider the quasilinear parabolic system of partial differential equations of second order

$$u_t = A(x,t,u)u_{xx} + f(x,t,u,u_x), \tag{1}$$

where $u = (u_1, u_2, \cdots, u_m)$ is a m-dimensional vector unknown function ($m \geq 1$), $u_t = \dfrac{\partial u}{\partial t}$, $u_x = \dfrac{\partial u}{\partial x}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ and $f(x,t,u,p)$ is a m-dimensional vector function of variables $(x,t) \in Q_T = \{0 \leq x \leq l,\ 0 \leq t \leq T\}$ and the vector variables $u, p \in R^m$. Suppose that the $m \times m$ matrix $A(x,t,u)$ satisfies the condition of strong parabolicity:

$$\inf_{(x,t,u)} \inf_{|\xi|=1} (\xi, A(x,t,u)\xi) = \sigma_0 > 0, \tag{2}$$

where (x,t) is any point of a rectangular domain $Q_T = \{0 \leq x \leq l,\ 0 \leq t \leq T\}$ and u, p are the m-dimensional vectors of m-dimensional Euclidean space R^m and σ_0 is a positive constant.

For the simplicity, let us consider the homogeneous boundary value problem for the quasilinear parabolic system (1) of partial differential equations. On the lateral sides $x = 0$ and $x = l$ of the rectangular domain Q_T, the homogeneous boundary conditions are taken to be of the form

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \leq t \leq T, \tag{3}$$

where $l > 0$ and $T > 0$ are given constants. And the initial condition is of the form

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq l, \tag{4}$$

where $\varphi(x) = (\varphi_1(x), \cdots, \varphi_m(x))$ is a given m-dimensional initial vector function.

Suppose that the following conditions are satisfied.

(I) The coefficient matrix $A(x,t,u)$ is a $m \times m$ continuous and positive definite matrix for $(x,t) \in Q_T$ and $u \in R^m$, that is, for any $(x,t) \in Q_T$ and $u \in R^m$, there is a positive constant σ_0, such that

$$(\xi, A(x,t,u)\xi) \geq \sigma_0 |\xi|^2 \tag{5}$$

for any $\xi \in R^m$. This shows that the condition (2) is valid.

(II) For the sake of brevity, let us assume that the m-dimensional vector function $f(x,t,u,p)$ is continuous with respect to $(x,t) \in Q_T$ and $u, p \in R^m$ and is Lipschitz continuous with respect to $u, p \in R^m$. Then there is a constant $K_1 > 0$, such that

$$|f(x,t,u,p)| \leq K_1(|u| + |p| + |f(x,t,0,0)|) \tag{6}$$

for $(x,t) \in Q_T$ and $u, p \in R^m$.

(III) The m-dimensional initial vector function $\varphi(x)$ belongs to $C^{(1)}([0,l])$ and satisfies the homogeneous boundary conditions $\varphi(0) = \varphi(l) = 0$.

(IV) The matrix $A(x,t,u)$ is continuously differentiable with respect to the vector variables $u \in R^m$.

3. We are going to solve this problem by the finite difference method with unequal meshsteps as the following way.

Suppose that the rectangular domain Q_T is divided into small rectangular grids $\bar{Q}_\Delta = \{\bar{Q}^{n+\frac{1}{2}}_{j+\frac{1}{2}} = (x_j \leq x \leq x_{j+1}; t^n \leq t \leq t^{n+1}) | j = 0,1,\cdots,J-1; n = 0,1,\cdots,N-1\}$ by the parallel line $x = x_j (j = 0,1,\cdots,J)$ and $t = t^n$ $(n = 0,1,\cdots,N)$, where

$$0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = l,$$
$$0 = t^0 < t^1 < \cdots <^{N-1} < t^N = T$$

and J and N are two integers. In general the meshsteps

$$h = \{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0,1,\cdots,J-1\}$$
$$\tau = \{\tau^{n+\frac{1}{2}} = t^{n+1} - t^n > 0 | n = 0,1,\cdots,N-1\}$$

are assume to be unequal. Denote by $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ the m-dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j, t^n) | j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ of the grid points.

Let us now construct the finite difference scheme with unequal meshsteps are follows:

$$\frac{v_j^{n+1} - v_j^n}{\tau^{n+\frac{1}{2}}} = A_j^{n+\alpha}\delta^2 v_j^{n+\alpha} + f_j^{n+\alpha}, \quad j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1 \tag{7}$$

corresponding to the quasilinear parabolic system (1) of partial differential equations, where

$$\delta^2 v_j^{n+\alpha} = \frac{\dfrac{v_{j+1}^{n+\alpha} - v_j^{n+\alpha}}{h_{j+\frac{1}{2}}} - \dfrac{v_j^{n+\alpha} - v_{j-1}^{n+\alpha}}{h_{j-\frac{1}{2}}}}{h_j^{(2)}}$$

$$A_j^{n+\alpha} = A(x_j, t^{n+\alpha}, \bar{\delta}^0 v_j^{n+\alpha}), \qquad (8)$$
$$f_j^{n+\alpha} = f(x_j, t^{n+\alpha}, \tilde{\delta}^0 v_j^{n+\alpha}, \tilde{\delta}^1 v_j^{n+\alpha})$$

and

$$\begin{aligned}
\bar{\delta}^0 v_j^{n+\alpha} &= \alpha(\bar{\beta}_{1j}^{n+1} v_{j-1}^{n+1} + \bar{\beta}_{2j}^{n+1} v_j^{n+1} + \bar{\beta}_{3j}^{n+1} v_{j+1}^{n+1}) + (\bar{\beta}_{4j}^{n} v_{j-1}^{n} + \bar{\beta}_{5j}^{n} v_j^{n} \bar{\beta}_{6j}^{n} v_{j+1}^{n}),\\
\tilde{\delta}^0 v_j^{n+\alpha} &= \alpha(\tilde{\beta}_{1j}^{n+1} v_{j-1}^{n+1} + \tilde{\beta}_{2j}^{n+1} v_j^{n+1} + \tilde{\beta}_{3j}^{n+1} v_{j+1}^{n+1}) + (\tilde{\beta}_{4j}^{n} v_{j-1}^{n} + \tilde{\beta}_{5j}^{n} v_{j+1}^{n} + \tilde{\beta}_{6j}^{n} v_{j+1}^{n}),
\end{aligned} \qquad (9)$$

$$\begin{aligned}
\tilde{\delta}^1 v_j^{n+\alpha} &= \tilde{\tilde{\beta}}_{1j}^{n+\alpha} \delta v_{j+\frac{1}{2}}^{n+\alpha} + \tilde{\tilde{\beta}}_{2j}^{n+\alpha} \delta v_{j-\frac{1}{2}}^{n+\alpha} = \tilde{\tilde{\beta}}_{1j}^{n+\alpha} \frac{v_{j+1}^{n+\alpha} - v_j^{n+\alpha}}{h_{j+\frac{1}{2}}} + \tilde{\tilde{\beta}}_{2j}^{n+\alpha} \frac{v_j^{n+\alpha} - v_{j-1}^{n+\alpha}}{h_{j-\frac{1}{2}}},\\
v_j^{n+\alpha} &= \alpha v_j^{n+1} + (1-\alpha) v_j^n, \quad 0 \le \alpha \le 1\\
h_j^{(2)} &= \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}})
\end{aligned}$$

for $j = 1, 2, \cdots, J-1$ and $n = 0, 1, \cdots, N-1$. Here $\bar{\beta}, \tilde{\beta}, \tilde{\tilde{\beta}}$'s are constants depending on the indices $j = 1, 2, \cdots, J-1$ and $n = 0, 1, \cdots, N-1$ and satisfying the restrictions

$$\begin{aligned}
&\alpha(\bar{\beta}_{1j}^{n+1} + \bar{\beta}_{2j}^{n+1} + \bar{\beta}_{3j}^{n+1}) + (\bar{\beta}_{4j}^{n} + \bar{\beta}_{5j}^{n} + \bar{\beta}_{6j}^{n}) = 1,\\
&\alpha(\tilde{\beta}_{1j}^{n+1} + \tilde{\beta}_{2j}^{n+1} + \tilde{\beta}_{3j}^{n+1}) + (\tilde{\beta}_{4j}^{n} + \tilde{\beta}_{5j}^{n} + \tilde{\beta}_{6j}^{n}) = 1,\\
&\tilde{\tilde{\beta}}_{1j}^{n+\alpha} + \tilde{\tilde{\beta}}_{2j}^{n+\alpha} = 1, \quad j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1.
\end{aligned}$$

And the absolute values of the constants $\bar{\beta}, \tilde{\beta}, \tilde{\tilde{\beta}}$'s are uniformly bounded by the constant $K_2 > 0$ with respect to the indices $n = 0, 1, \cdots, N-1$ and $j = 1, 2, \cdots, J-1$. It is noticed that the coefficients $\bar{\beta}, \tilde{\beta}, \tilde{\tilde{\beta}}$'s of the difference schemes can be different for different layers and different grid points in the difference approximation of the system. This means that the difference scheme has the large degree of freedom in general.

The corresponding finite difference boundary conditions are of the form

$$v_0^n = 0, \quad v_J^n = 0, \quad n = 0, 1, \cdots, N. \qquad (10)$$

And the corresponding finite difference initial conditions are of the form

$$v_j^0 = \varphi_j, \quad j = 1, 2, \cdots, J-1, \qquad (11)$$

where $\varphi_j = \varphi(x_j)$ $(j = 0, 1, \cdots, J)$ is the value of the m-dimensional vector function $\varphi(x)$ on the grid point $x = x_j$ $(j = 0, 1, \cdots, J)$ and also $\varphi_0 = \varphi_J = 0$.

4. Now we want to prove the existence of the discrete solutions for the finite difference scheme (7), (10) and (11).

The finite difference system (7), (10) and (11) can be regarded as a system of nonlinear equations with $J+1$ unknown m-dimensional vectors v_j^{n+1} $(j = 0, 1, \cdots, J)$, as the m-dimensional vectors $v_j^n (j = 0, 1, \cdots, J)$ are given. So the finite difference system (7), (10) and (11) can be solved step by step with respect to $n = 1, 2, \cdots, N$.

Denote $v = (v_0^{n+1}, v_1^{n+1}, \cdots, v_J^{n+1})$, then v is a $m(J+1)$-dimensional vector or a point of a $m(J+1)$-dimensional Euclidean space $\tilde{R} = R^{m(J+1)}$.

When $\alpha = 0$ the system (7), (10) and (11) is an explicit finite difference scheme for the homogeneous boundary problem (3), (4) of the quasilinear parabolic system (1). So the existence of the discrete solution is evident.

When $0 < \alpha \leq 1$, the finite deference scheme (7), (10) and (11) is implicit and then is a system of nonlinear equation of $m(J+1)$-dimensional vector $v \in \tilde{R}$. In this case, we can prove the solvability of the system by the fixed-point theorem of continuous mapping in the finite-dimensional Euclidean space.

For this purpose, we construct a mapping T_λ: $\tilde{R} \to \tilde{R}$ of the $m(J+1)$-dimensional Euclidean space $\tilde{R}$ into itself with a parameter $0 \leq \lambda \leq 1$ defined by $v = T_\lambda(z)$ as follows:

$$v_j^{n+1} = v_j^n + \lambda\tau^{n+\frac{1}{2}}\bar{A}_j^{n+\alpha}(\alpha\delta^2 z_j + (1-\alpha)\delta^2 v_j^n) + \lambda\tau^{n+\frac{1}{2}}\bar{f}_j^{n+\alpha} \tag{12}$$

for $j = 1, \cdots, J-1$ and the boundary conditions $v_0 = 0$ and $v_J = 0$, where $0 \leq \lambda \leq 1$ is a parameter and $z = (z_0, z_1, \cdots, z_J) \in \tilde{R}$. The notations $\bar{A}_j^{n+\alpha}$ and $\bar{f}_j^{n+\alpha}$ ($j = 0, 1, \cdots, J-1$) are as follows:

$$\bar{A}_j^{n+\alpha} = A(x_j, t^{n+\alpha}, \bar{v}_j^{*n+\alpha}) \quad \bar{f}_j^{n+\alpha} = f(x_j, t^{n+\alpha}, \tilde{v}_j^{*n+\alpha}, \tilde{\delta} v_j^{*n+\alpha})$$

and

$$\bar{v}_j^{*n+\alpha} = \alpha(\bar{\beta}_{1j}^{n+1} z_{j-1} + \bar{\beta}_{2j}^{n+1} z_j + \bar{\beta}_{3j}^{n+1} z_{j+1}) + (\bar{\beta}_{4j}^n v_{j-1}^n + \bar{\beta}_{5j}^n v_j^n + \bar{\beta}_{6j}^n v_{j+1}^n), \tag{13}$$

$$\tilde{v}_j^{*n+\alpha} = \alpha(\tilde{\beta}_{1j}^{n+1} z_{j-1} + \tilde{\beta}_{2j}^{n+1} z_j + \tilde{\beta}_{3j}^{n+1} z_{j+1}) + (\tilde{\beta}_{4j}^n v_{j-1}^n + \tilde{\beta}_{5j}^n v_j^n + \tilde{\beta}_{6j}^n v_{j+1}^n), \tag{14}$$

$$\tilde{\delta} v_j^{*n+\alpha} = \tilde{\beta}_{1j}^{n+\alpha}(\alpha\delta z_{j+\frac{1}{2}} + (1-\alpha)\delta v_{j+\frac{1}{2}}^n) + \tilde{\beta}_{2j}^{n+\alpha}(\alpha\delta z_{j-\frac{1}{2}} + (1-\alpha)\delta v_{j-\frac{1}{2}}^n). \tag{15}$$

Thus the mapping T_λ transforms $z = (z_0, z_1, \cdots, z_J) \in \tilde{R}$ to $v = (v_0^{n+1}, v_1^{n+1}, \cdots, v_J^{n+1}) \in \tilde{R}$, that is $v = T_\lambda(z) \in \tilde{R}$ for any $z \in \tilde{R}$ and the parameter $0 \leq \lambda \leq 1$. This defines a continuous mapping $T_\lambda : \tilde{R} \to \tilde{R}$ of the $m(J+1)$-dimensional Euclidean space $\tilde{R}$ into itself with a parameter $0 \leq \lambda \leq 1$.

When $\lambda = 0$, for any $z \in \tilde{R}$, the image $T_0 : \tilde{R} \to \tilde{R}$ is a fixed point $T_0(z) = (v_0^n, v_1^n, \cdots, v_J^n)$ of the $m(J+1)$-dimensional space $\tilde{R}$. This justifies one of the sufficient conditions of the fixed point theorem of the continuous mapping in a finite dimensional Euclidean space in [5].

Therefore in order to prove the existence of the solution for the nonlinear system (7) and (10), it is sufficient to prove the uniform boundedness of all possible fixed points of the mapping $T_\lambda : \tilde{R} \to \tilde{R}$ with respect to the parameter $0 \leq \lambda \leq 1$. Hence we need to prove the uniform boundedness of all possible solutions of the nonlinear system

$$v_j^{n+1} = v_j^n + \lambda\tau^{n+\frac{1}{2}} A_j^{n+\alpha}\delta^2 v_j^{n+\alpha} + \lambda\tau^{n+\frac{1}{2}} f_j^{n+\alpha}, \quad j = 1, 2, \cdots, J-1 \tag{7_λ}$$

and the boundary conditions $v_0^{n+1} = v_J^{n+1} = 0$ with respect to the parameter $0 \leq \lambda \leq 1$.

5. Taking the scalar product $\delta^2 v_j^{n+\alpha} h_j^{(2)}$ ($j = 1, 2, \cdots, J-1$) with the corresponding vector equations of the system $(7)_\lambda$ for $j = 1, 2, \cdots, J-1$ and then summing up the results of the product for $j = 1, 2, \cdots, J-1$, we then obtain

$$\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, v_j^{n+1})h_j^{(2)} = \sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, v_j^n)h_j^{(2)} + \lambda\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, A_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)} +$$

$$+\lambda\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}. \tag{16}$$

Here (u,v) denote the scalar product of two m-dimensional vectors u and v.

For the first term of the above equality, we have

$$\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, v_j^{n+1})h_j^{(2)} = \sum_{j=1}^{J-1}(\delta v_{j+\frac{1}{2}}^{n+\alpha} - \delta v_{j-\frac{1}{2}}^{n+\alpha}, v_j^{n+1}) = -\sum_{j=1}^{J-1}(\delta v_{j+\frac{1}{2}}^{n+\alpha}, v_{j+1}^{n+1} - v_j^{n+1})$$

$$-(\delta v_{\frac{1}{2}}^{n+\alpha}, v_0^{n+1}) + (\delta v_{J-\frac{1}{2}}^{n+\alpha}, v_J^{n+1}) = -\sum_{j=1}^{J-1}(\delta v_{j+\frac{1}{2}}^{n+\alpha}, \delta v_{j+\frac{1}{2}}^{n+1})h_{j+\frac{1}{2}},$$

where
$$\delta v_{j+\frac{1}{2}}^{n+\alpha} = \frac{v_{j+1}^{n+\alpha} - v_j^{n+\alpha}}{h_{j+\frac{1}{2}}}.$$

Also for the second term of the above equality we have

$$\sum_{j=0}^{J-1}(\delta^2 v_j^{n+\alpha}, v_j^n)h_j^{(2)} = -\sum_{j=0}^{J-1}(\delta v_{j+\frac{1}{2}}^{n+\alpha} - \delta v_{j+\frac{1}{2}}^n)h_{j+\frac{1}{2}}.$$

Since the matrix A is positive definite, then

$$\sum_{j=0}^{J-1}(\delta^2 v_j^{n+\alpha}, A_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)} \geq \sigma_0\|\delta^2 v_h^{n+\alpha}\|_2^2.$$

For the last term of the above equality, we get

$$\sum_{j=0}^{J-1}(\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)} \leq \frac{1}{2}\sigma_0\|\delta^2 v_h^{n+\alpha}\|_2^2 + \frac{1}{2\sigma_0}\sum_{j=0}^{J-1}(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}.$$

From the assumption (III), the last sum of the above inequality can be dominated as follows:

$$\sum_{j=0}^{J-1}(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)} \leq C_1\Big(\sum_{j=1}^{J-1}|\tilde{\delta}^0 v_j^{n+\alpha}|^2 h_j^{(2)} + \sum_{j=1}^{J-1}|\tilde{\delta}^1 v_j^{n+\alpha}|^2 h_j^{(2)} + 1\Big),$$

where C_1 is a constant independent of the meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$. As a convention, we denote the constants independent of the discrete solutions and the unequal meshsteps h and τ by C's with different indices also in subsequent discussion. Here we have

$$\sum_{j=1}^{J-1}|\tilde{\delta}^0 v_j^{n+\alpha}|^2 h_j^{(2)} = \sum_{j=1}^{J-1}|\alpha(\tilde{\beta}_{1j}^{n+1}v_{j-1}^{n+1} + \tilde{\beta}_{2j}^{n+1}v_j^{n+1} + \tilde{\beta}_{3j}^{n+1}v_{j+1}^{n+1})$$

$$+ (\tilde{\beta}_{4j}^n v_{j-1}^n + \tilde{\beta}_{5j}^n v_j^n + \tilde{\beta}_{6j}^n v_{j+1}^n)|h_j^{(2)}$$

$$\leq C_2\Big\{\alpha^2\sum_{j=1}^{J-1}(|v_{j-1}^{n+1}|^2+|v_j^{n+1}|^2+|v_{j+1}^{n+1}|^2)\frac{1}{2}(h_{j+\frac{1}{2}}+h_{j-\frac{1}{2}})$$
$$+\sum_{j=1}^{J-1}(|v_{j-1}^{n}|^2+|v_j^{n}|^2+|v_{j+1}^{n}|^2)\frac{1}{2}(h_{j+\frac{1}{2}}+h_{j-\frac{1}{2}})\Big\}$$
$$\leq 2C_2(1+M_h)\{\alpha^2\|v_h^{n+1}\|_2^2+\|v_h\|_2^2\},$$

where
$$M_h=\max_{j=0,1,\cdots,J-1}\left\{\frac{h_{j-\frac{1}{2}}}{h_{j+\frac{1}{2}}},\frac{h_{j+\frac{1}{2}}}{h_{j-\frac{1}{2}}}\right\}$$
is the maximum ratio of the neighboring meshsteps. Then

$$\sum_{j=1}^{J-1}|\tilde{\delta}^0 v_j^{n+\alpha}|^2h_j^{(2)}\leq C_3\{\alpha^2\|v_h^{n+1}\|_2^2+\|v_h^n\|_2^2\},$$

where C_3 is a constant independent of the meshsteps and dependent on the maximum ratio M_h of the neighboring meshsteps. Similarly, we have

$$\sum_{j=1}^{J-1}|\tilde{\delta}^1 v_j^{n+\alpha}|^2h_j^{(2)}=\sum_{j=1}^{J-1}|\tilde{\beta}_{1j}^{n+\alpha}\delta v_{j+\frac{1}{2}}^{n+\alpha}+\tilde{\beta}_{2j}^{n+\alpha}\delta v_{j-\frac{1}{2}}^{n+\alpha}|^2h_j^{(2)}\leq C_4\{\alpha^2\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2\},$$

where C_4 is a constant independent of the unequal meshsteps and dependent on the maximum ratio M_h of the neighboring meshsteps. Hence we have

$$\sum_{j=1}^{J-1}(f_j^{n+\alpha},f_j^{n+\alpha})h_j^{(2)}\leq C_5\{\alpha^2\|v_h^{n+1}\|_2^2+\alpha^2\|\delta v_h^{n+1}\|_2^2+\|v_h^n\|_2^2+\|\delta v_h^n\|_2^2+1\}.$$

For any $m=1,2,\cdots,J$, we have

$$|v_m^n|=\Big|\sum_{j=0}^{m-1}\delta v_{j+\frac{1}{2}}^n h_{j+\frac{1}{2}}\Big|\leq\sqrt{x_m}\|\delta v_h^n\|_2,$$

where $v_0^n=0$. This gives

$$\|v_h^n\|_\infty\leq\sqrt{l}\|\delta v_h^n\|_2. \tag{17}$$

Also there is

$$\|v_h^n\|_2^2=\sum_{j=0}^{J-1}\frac{1}{2}(|v_j|^2+|v_{j+1}|^2)h_{j+\frac{1}{2}}\leq\sum_{j=0}^{J-1}\frac{1}{2}(x_j+x_{j+1})h_{j+\frac{1}{2}}\|\delta v_h^n\|_2^2\leq\frac{1}{2}l^2\|\delta v_h^n\|_2^2.$$

Thus
$$\|v_h^n\|_2\leq\frac{1}{\sqrt{2}}l\|\delta v_h^n\|_2$$
and also
$$\|v_h^{n+1}\|_2\leq\frac{1}{\sqrt{2}}l\|\delta v_h^{n+1}\|_2.$$
Then we have

$$\sum_{j=1}^{J-1}(f_j^{n+\alpha},f_j^{n+\alpha})h_j^{(2)}\leq C_7\{\alpha^2\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2+1\}.$$

Substituting all above obtained estimates into the equality (16), we then have

$$\sum_{j=0}^{J-1}(\delta v^{n+\alpha}_{j+\frac{1}{2}},\delta v^{n+1}_{j+\frac{1}{2}})h_{j+\frac{1}{2}}+\frac{1}{2}\lambda\tau^{n+\frac{1}{2}}\sigma_0\|\delta^2 v^{n+\alpha}_h\|_2^2$$
$$\leq\sum_{j=0}^{J-1}(\delta v^{n+\alpha}_{j+\frac{1}{2}},\delta v^{n}_{j+\frac{1}{2}})h_{j+\frac{1}{2}}+\lambda\tau^{n+\frac{1}{2}}C_7\frac{1}{2\sigma_0}\{\alpha\|\delta v^{n+1}_h\|_2^2+\|\delta v^n_h\|_2^2+1\}$$

or

$$\alpha\|\delta v^{n+1}_h\|_2^2+(1-2\alpha)\sum_{j=0}^{J-1}(\delta v^{n+1}_{j+\frac{1}{2}},\delta v^{n}_{j+\frac{1}{2}})h_{j+\frac{1}{2}}+\frac{1}{2}\lambda\tau^{n+\frac{1}{2}}\sigma_0\|\delta^2 v^{n+\alpha}_h\|_2^2$$
$$\leq(1-\alpha)\|\delta v^n_h\|_2^2+\frac{1}{2\sigma_0}\lambda\tau^{n+\frac{1}{2}}C_7\{\alpha^2\|\delta v^{n+1}_h\|_2^2+\|\delta v^n_h\|_2^2+1\}.$$

Taking

$$\Big|\sum_{j=0}^{J-1}(\delta v^{n+1}_{j+\frac{1}{2}},\delta v^{n}_{j+\frac{1}{2}})h_{j+\frac{1}{2}}\Big|\leq\frac{\alpha}{4}\|\delta v^{n+1}_h\|_2^2+\frac{1}{\alpha}\|\delta v^n_h\|_2^2,$$

we can rewrite the above inequality as follows:

$$\Big(\alpha-|1-2\alpha|\frac{\alpha}{4}-\frac{1}{2\sigma_0}\lambda\tau^{n+\frac{1}{2}}C_7\alpha^2\Big)\|\delta v^{n+1}_h\|_2^2+\frac{1}{2}\lambda\tau^{n+\frac{1}{2}}\sigma_0\|\delta^2 v^{n+\alpha}_h\|_2^2$$
$$\leq\Big((1-\alpha)-\frac{|1-2\alpha|}{\alpha}+\frac{1}{2\sigma_0}\lambda\tau^{n+\frac{1}{2}}C_7\Big)\|\delta v^n_h\|_2^2+\frac{1}{2\sigma}\lambda\tau^{n+\frac{1}{2}}C_7.$$

Taking again $\tau^{n+\frac{1}{2}}$ or τ so small that

$$\frac{1}{2\sigma_0}\tau^{n+\frac{1}{2}}C_7\leq\frac{1}{4},$$

we see that

$$\|\delta v^{n+1}_h\|_2^2\leq\frac{2}{\alpha}(2+\frac{1}{\alpha})\|\delta v^n_h\|_2^2+\frac{1}{4}$$

is uniformly bounded with respect to $0\leq\lambda\leq1$ for given meshsteps. From (17), we also see that $\|v^{n+1}_h\|_\infty$ is also uniformly bounded with respect to $0\leq\lambda\leq1$ for given meshsteps. This completes the proof of the existence of the solutions for the finite difference system (7) and (10).

Theorem 1. *Suppose that the conditions (I) and (II) are satisfied. For sufficiently small meshsteps* $\tau=\{\tau^{n+\frac{1}{2}}|h=0,1,\cdots,N-1\}$ *independent of* $0\leq\alpha\leq1$, *the finite difference scheme (7), (10) and (11) has at least one discrete solution* $v_\triangle=v^\tau_h=\{v^h_j|j=0,1,\cdots,J;\,n=0,1,\cdots,N\}$, *for any* $0\leq\alpha\leq1$.

2. Estimation for Discrete Solutions

In this section we are going to derive some a priori estimates for the discrete solution $v_\triangle=v^\tau_h=\{v^n_j|j=0,1,\cdots,J;\ n=0,1,\cdots,N\}$ of the finite difference system (7),

(10) and (11) corresponding to the boundary problem (3) and (4) for the quasilinear parabolic system (1) under the assumptions (I), (II) and (III).

6. Similarly, taking the scalar product of the vector $\delta^2 v_j^{n+\alpha} h_j^{(2)} \tau^{n+\frac{1}{2}}$ and the vector equation (7) and then summing up the resulting relations for $j = 1, 2, \cdots, J-1$, we get

$$\sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, v_j^{n+1} - v_j^n) h_j^{(2)} = \tau^{n+\frac{1}{2}} \sum_{j=1}^{j-1} (\delta^2 v_j^{n+\alpha}, A_j^{n+\alpha} \delta^2 v_j^{n+\alpha}) h_j^{(2)} + \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} \sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)}, \quad n = 0, 1, \cdots, N-1.$$

By similar verification as before, there is

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + (2\alpha - 1)\|\delta(v_h^{n+1} - v_h^n)\|_2^2 + 2\tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, A_j^{n+\alpha} v_j^{n+\alpha}) h_j^{(2)}$$
$$= 2\tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)}. \tag{18}$$

The last term of the above equality can be dominanted as follows:

$$2\tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)} \le \tau^{n+\frac{1}{2}} \sigma_0 \|\delta^2 v_h^{n+\alpha}\|_2^2 + \frac{1}{2\sigma_0} \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (f_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)}.$$

And also we have the estimate

$$\frac{1}{2\sigma_0} \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (f_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)} \le \frac{1}{2\sigma_0} \tau^{n+\frac{1}{2}} C_7 \{\|\delta v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2 + 1\}.$$

Hence we have

$$2\tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha}) h_j^{(2)} \le \sigma_0 \tau^{n+\frac{1}{2}} \|\delta^2 v_h^{n+\alpha}\|_2^2 + \frac{1}{2\sigma_0} \tau^{n+\frac{1}{2}} C_8 \{\|\delta v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2 + 1\}.$$

7. Let us consider the case $2\alpha - 1 \ge 0$ or $\frac{1}{2} \le \alpha \le 1$, that is, the finite difference scheme (7), (10) and (11) is implicit or called strongly implicit. Then (18) becomes

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \sigma_0 \tau^{n+\frac{1}{2}} \|\delta^2 v_h^{n+\alpha}\|_2^2 \le C_8 \tau^{n+\frac{1}{3}2} \{\|\delta v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2 + 1\}. \tag{19}$$

Then we have

$$\|\delta v_h^{n+1}\|_2^2 \le \Big(\frac{1 + C_8 \tau^{n+\frac{1}{2}}}{1 - C_8 \tau^{n+\frac{1}{2}}}\Big) \|\delta v_h^n\|_2^2 + \frac{C_8 \tau^{n+\frac{1}{2}}}{1 - C_8 \tau^{n+\frac{1}{2}}}.$$

From this iterative formula, we obtain

$$\|\delta v_h^{n+1}\|_2^2 \le \Big[\prod_{k=0}^{n} \Big(\frac{1 + C_8 \tau^{k+\frac{1}{2}}}{1 - C_8 \tau^{k+\frac{1}{2}}}\Big)\Big] \|\delta v_h^0\|_2^2 + \sum_{k=0}^{n} \frac{C_8 \tau^{k+\frac{1}{2}}}{1 - C_8 \tau^{k+\frac{1}{2}}} \prod_{j=k+1}^{n} \Big(\frac{1 + C_8 \tau^{j+\frac{1}{2}}}{1 - C_8 \tau^{j+\frac{1}{2}}}\Big).$$

We take the meshstep $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ so small, that is, the maximum meshstep $\tau^* = \max_{n=0,1,\cdots,N-1} \tau^{n+\frac{1}{2}}$ so small that $C_8\tau^* < \frac{1}{2}$. Thus we have $\prod_{k=0}^{n} \left(\frac{1+C_8\tau^{k+\frac{1}{2}}}{1-C_8\tau^{k+\frac{1}{2}}}\right) \le e^{3C_8 t^{n+1}}$ and $\sum_{k=0}^{n} \frac{C_8\tau^{k+\frac{1}{2}}}{1-C_8\tau^{k+\frac{1}{2}}} \prod_{j=k+1}^{n} \left(\frac{1+C_8\tau^{j+\frac{1}{2}}}{1-C_8\tau^{j+\frac{1}{2}}}\right) \le 2C_8 t^{n+1} e^{3C_8 t^{n+1}}$. Therefore we get the estimate

$$\|\delta v_h^n\|_2^2 \le e^{3C_8 t^n} \|\delta v_h^0\|_2^2 + 2C_8 t^{n+1} e^{3C_8 t^n}.$$

Lemma 1. *Under the conditions (I), (II), (III) for the sufficiently small meshsteps $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ and $\frac{1}{2} \le \alpha \le 1$, there is the estimate*

$$\max_{n=0,1,\cdots,N} \|\delta v_h^n\|_2 \le K_3, \tag{20}$$

of the discrete solution $v_\triangle = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ of the finite difference scheme (7), (10) and (11) for the homogeneous boundary problem (3) and (4) of the quasilinear parabolic system (1), where K_3 is a constant independent of the meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$.

Corollary. *Under the conditions of Lemma 1, there are the estimates*

$$\max_{n=0,1,\cdots,N} (\|v_h^n\|_2, \|v_h^n\|_\infty) \le K_4 \tag{21}$$

of the discrete solution $v_\triangle = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ of the difference scheme (7), (10) and (11) for the boundary problem (3) and (4) of the system (1), where K_4 is a constent independent of the unequal meshsteps.

Summing up (19) for $n = s, s+1, \cdots, m$, we have

$$\|\delta v_h^{m+1}\|_2^2 - \|\delta v_h^s\|_2^2 + \sigma_0 \sum_{n=s}^{m} \|\delta^2 v_h^{n+\alpha}\|_2^2 \tau^{n+\frac{1}{2}}$$
$$\le C_8 \sum_{n=s}^{m} (\|\delta v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2)\tau^{n+\frac{1}{2}} + C_8(t^m - t^s).$$

Then for any $0 \le s < m \le N-1$

$$\sum_{n=s}^{m} \|\delta^2 v_h^{n+\alpha}\|_2^2 \tau^{n+\frac{1}{2}} \le \frac{1}{\sigma_0}\{K_3^2(1 + 2C_8 T) + C_8 T\}.$$

Lemma 2. *Under the conditions of Lemma 1, there is the estimate*

$$\sum_{n=s}^{n} \|\delta^2 v_h^{n+\alpha}\|_2^2 \tau^{n+\frac{1}{2}} \le K_5, \quad 0 \le s < m \le N-1 \tag{22}$$

for the discrete solution $v_\triangle = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ of the finite difference scheme (7), (10) and (11), where K_5 is a constant independent of the meshsteps.

From the finite difference scheme (7), we have

$$\sum_{n=s}^{m}\left\|\frac{v_h^{n+1}-v_h^n}{\tau^{n+\frac{1}{2}}}\right\|_2^2\tau^{n+\frac{1}{2}}=\sum_{n=s}^{m}\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\tau^{n+\frac{1}{2}}}\right|h_j^{(2)}\tau^{n+\frac{1}{2}}$$
$$\le 2\sum_{n=s}^{m}\sum_{j=1}^{J-1}|A_j^{n+\alpha}\delta^2 v_j^{n+\alpha}|^2 h_j^{(2)}\tau^{n+\frac{1}{2}}+2\sum_{n=s}^{m}\sum_{j=1}^{J-1}|f_j^{n+\alpha}|^2 h_j^{(2)}\tau^{n+\frac{1}{2}}.$$

Since the coefficient matrix A depends only on x, t and u, then $A_j^{n+\alpha}$ ($j=1,\cdots,J-1$; $n=0,1,\cdots,N-1$) is uniformly bounded with respect to the meshsteps h and τ. Again from the behavior (II) of the vector function f, it is clearly that

$$\sum_{n=s}^{m}\sum_{j=1}^{J-1}|f_j^{n+\alpha}|^2 h_j^{(2)}\tau^{n+\frac{1}{2}}\le C_9\sum_{n=s}^{m}\sum_{j=1}^{J-1}\{|\tilde{\delta}^0 v_j^{n+\alpha}|^2+|\tilde{\delta}^1 v_j^{n+\alpha}|^2+1\}h_j^{(2)}\tau^{n+\frac{1}{2}}\le C_{10}.$$

This implies the following lemma.

Lemma 3. *Under the conditions of Lemma 1, there is the estimate*

$$\sum_{n=s}^{m}\left\|\frac{v_h^{n+1}-v_h^n}{\tau^{n+\frac{1}{2}}}\right\|_2^2\tau^{n+\frac{1}{2}}\le K_6, \tag{23}$$

$0\le s<m\le N-1$, *for the discrete solution* $v_\Delta=v_h^\tau=\{v_j^n|j=0,1,\cdots,J;\ n=0,1,\cdots,N\}$ *of the finite difference scheme* (7), (10) *and* (11), *where* K_6 *is a constant independent of the meshsteps* h *and* τ.

Hence we have the following theorem for the estimates of the discrete solutions v_Δ of the finite difference scheme (7), (10) and (11) corresponding to the homogeneous boundary problem (3) and (4) for the quasilinear parabolic system (1) of partial differential equations.

Theorem 2. *Under the conditions (I), (II) and (III), for the sufficiently small meshsteps* $\tau=\{\tau^{n+\frac{1}{2}}|n=0,1,\cdots,N-1\}$ *and* $\frac{1}{2}\le\alpha\le 1$, *there are the estimates*

$$\max_{n=0,1,\cdots,N}(\|v_h^n\|_2^2+\|\delta v_h^n\|_2^2)+\sum_{n=0}^{N-1}(\|v_h^{n+\alpha}\|_2^2+\|\delta v_h^{n+\alpha}\|_2^2+\|\delta^2 v_h^{n+\alpha}\|_2^2)\tau^{n+\frac{1}{2}}$$
$$+\sum_{n=0}^{N-1}\left\|\frac{v_h^{n+1}-v_h^n}{\tau^{n+\frac{1}{2}}}\right\|_2^2\tau^{n+\frac{1}{2}}\le K_7\left\{\|\varphi_h\|_2^2+\|\delta\varphi_h\|_2^2+\sum_{n=0}^{N-1}\|\bar{f}_h^{n+\alpha}\|_2^2\tau^{n+\frac{1}{2}}\right\} \tag{24}$$

for the discrete solution $v_\Delta=v_h^\tau=\{v_j^n|j=0,1,\cdots,J;\ n=0,1,\cdots,N\}$ *of the strongly implicit difference scheme* (7), (10) *and* (11) *corresponding to the homogeneous boundary problem* (3) *and* (4) *for the quasilinear parabolic system* (1), *where* K_7 *is the constant independent of the meshsteps and here* $\bar{f}(x,t)\equiv f(x,t,0,0)$.

8. Now let us consider the explicit and weakly implicit finite difference scheme (7), (10) and (11), that is the case of $0\le\alpha<\frac{1}{2}$.

⟨1⟩ For this purpose let us at first consider a mapping of the Euclidean space R^* of $m(J+1)(N+1)$-dimension into itself. For any $z_\Delta=z_h^\tau=\{z_j^n|j=0,1,\cdots,J;$

$n = 0, 1, \cdots, N\}$, we define $v_\triangle = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ as the solution of the following finite difference system

$$\frac{v_j^{n+1} - v_j^n}{\tau^{n+\frac{1}{2}}} = \overset{*}{A}{}_j^{n+\alpha} \delta^2 v_j^{n+\alpha} + f_j^{n+\alpha} \tag{7*}$$

and the homogeneous boundary conditions (10), where $\overset{*}{A}{}_j^{n+\alpha}$ $(j = 1, \cdots, J-1)$ are obtained from $A_j^{n+\alpha}$ $(j = 1, \cdots, J-1)$ as $v_\triangle$ is replaced by $z_\triangle$ respectively. For any given $z_\triangle \in R^*$, the solution $v_\triangle$ of (7)*, (10) and (11) exists by means of Theorem 1 in the previous section.

⟨2⟩ The solution $v_\triangle$ of the difference system (7)*, (10) and (11) is unique. For the justification of the uniqueness of the solution, let us suppose that for given vectors $v_j^n (j = 0, 1, \cdots, J)$, there are two solutions v_j^{n+1} and $\tilde{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$. Hence then we also have

$$\frac{\tilde{v}_j^{n+1} - v_j^n}{\tau^{n+\frac{1}{2}}} = \overset{*}{A}{}_j^{n+\alpha} (\alpha \delta^2 \tilde{v}_j^{n+1} + (1-\alpha)\delta^2 v_j^n) + \tilde{f}_j^{n+\alpha}$$

where $\tilde{f}_j^{n+\alpha}$ $(j = 1, \cdots, J-1)$ are obtained from $f_j^{n+\alpha}$ $(j = 1, \cdots, J-1)$ as the vectors v_j^{n+1} $(j = 0, 1, \cdots, J)$ are replaced by $\tilde{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$ respectively.

Denote $w_j^{n+1} = v_j^{n+1} - \tilde{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$. Subtracting one difference system by the another, we get

$$\begin{aligned} w_j^{n+1} =\alpha = \overset{*}{A}{}_j^{n+\alpha} \tau^{n+\frac{1}{2}} \delta^2 w_j^{n+1} + \tau^{n+\frac{1}{2}} (f_0)_h^{n+\alpha} \alpha (\tilde{\beta}_{1j}^{n+1} w_{j-1}^{n+1} + \tilde{\beta}_{2j}^{n+1} w_j^{n+1} + \tilde{\beta}_{3j}^{n+1} w_{j+1}^{n+1}) \\ + \tau^{n+\frac{1}{2}} (f_1)_j^{n+\alpha} \alpha (\tilde{\beta}_{1j}^{n+\alpha} \delta w_{j+\frac{1}{2}}^{n+1} + \tilde{\beta}_{2j}^{n+\alpha} \delta w_{j-\frac{1}{2}}^{n+1}), \end{aligned}$$

where

$$f_0 = \int_0^1 \frac{\partial f}{\partial u}(x, t, \tau u + (1-\tau)\tilde{u}, \tau p + (1-\tau)\tilde{p}) d\tau,$$

$$f_1 = \int_0^1 \frac{\partial f}{\partial p}(x, t, \tau u + (1-\tau)\tilde{u}, \tau p + (1-\tau)\tilde{p}) d\tau,$$

are $m \times m$ matrices and then are all bounded.

Making the scalar product of the vectors $\delta^2 w_j^{n+1} h_j^{(2)}$ and the above equations and summing up the resulting products for $j = 1, \cdots, J-1$, we obtain

$$\begin{aligned} \sum_{j=1}^{J-1} (\delta^2 w_j^{n+1}, w_h^{n+1}) h_j^{(2)} = \alpha \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 w_j^{n+1}, \overset{*}{A}{}_j^{n+\alpha} \delta w_j^{n+1}) h_j^{(2)} \\ + \alpha \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 w_j^{n+1}, (f_0)_j^{n+\alpha} [\tilde{\beta}_{1j}^{n+1} w_{j-1}^{n+1} + \tilde{\beta}_{2j}^{n+1} w_j^{n+1} + \tilde{\beta}_{3j}^{n+1} w_{j+1}^{n+1}]) h_j^{(2)} \\ + \alpha \tau^{n+\frac{1}{2}} \sum_{j=1}^{J-1} (\delta^2 w_j^{n+1}, (f_1)_j^{n+\alpha} [\tilde{\beta}_{1j}^{n+\alpha} \delta w_{j+\frac{1}{2}}^{n+1} + \tilde{\beta}_{2j}^{n+\alpha} \delta w_{j-\frac{1}{2}}^{n+1}]) h_j^{(2)}, \end{aligned}$$

where $(f_0)_j^{n+\alpha}$ and $(f_1)_j^{n+\alpha}$ $(j = 1, \cdots, J-1)$ are bounded.

By a similar method of derivation as before, we can obtain the inequality

$$\|\delta w_h^{n+1}\|_2^2 + \frac{1}{2}\alpha\sigma_0\|\delta^2 w_h^{n+1}\|_2^2\tau^{n+\frac{1}{2}} \le \alpha C_{11}\tau^{n+\frac{1}{2}}\|\delta w_h^{n+1}\|_2^2.$$

Let us choose the meshstep $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ or τ^* so small that $\tau^* C_{11} < 1$. Then the above inequality implies that $\|\delta w_h^{n+1}\|2 = 0$. This shows that $v_j^{n+1} = \tilde{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$ and the solution v_Δ of the difference system $(7)^*$, (10) and (11) is unique.

Thus the mapping of Euclidean space R^* into itself is defined. Denote this mapping of Euclidean space R^* into itself by $H : R^* \to R^*$, that is, for any $z_\Delta \in R^*$, there is $v_\Delta = H z_\Delta$.

⟨3⟩ Let $G = \left\{ z_\Delta | \max_{n=0,1,\cdots,N; j=0,1,\cdots,J} |z_j^n| \le D \right\}$ be a bounded set of the Euclidean space R^*, where D is a constant to be determined. Evidently G is convex. We want to determined the constant D in order that the image of the bounded convex set under the mapping H lies inside the set G itself, that is $H(G) \subset G$.

Now making the scalar product of the vector $\delta^2 v_j^{n+\alpha} h_j^{(2)} \tau^{n+\frac{1}{2}}$ and the vector equation $(7)^*$ and summing up the resulting products for $j = 1, \cdots, N-1$, we have

$$\begin{aligned}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, v_j^{n+1} - v_j^n)h_j^{(2)} =& \tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, \overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)} \\ &+ \tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}\end{aligned}$$

or

$$\begin{aligned}&\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + (2\alpha-1)\|\delta(v_h^{n+1} - v_h^n)\|_2^2 + 2\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, \overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)} \\ &\quad = -2\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 v_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}.\end{aligned}$$

The difference term in the above equality can be calculated as the following way:

$$\begin{aligned}\|\delta(v_h^{n+1} - v_h^n)\|_2^2 &= \sum_{j=0}^{J-1}\left|\frac{(v_{j+1}^{n+1} - v_{j+1}^n) - (v_j^{n+1} - v_j^n)}{h_{j+\frac{1}{2}}}\right|^2 h_{j+\frac{1}{2}} \\ &\le \left(\frac{\tau^{n+\frac{1}{2}}}{h_*}\right)^2\sum_{j=0}^{J-1}\left|\frac{v_{j+1}^{n+1} - v_{j+1}^n}{\tau^{n+\frac{1}{2}}} - \frac{v_j^{n+1} - v_j^n}{\tau^{n+\frac{1}{2}}}\right|^2 h_{j+\frac{1}{2}} \\ &\le 4\left(\frac{\tau^{n+\frac{1}{2}}}{h_*}\right)^2\sum_{j=0}^{J-1}\left|\frac{v_j^{n+1} - v_j^n}{\tau_{j+\frac{1}{2}}}\right| h_j^{(2)} \\ &\le 4\left(\frac{\tau^{n+\frac{1}{2}}}{h_*}\right)^2\sum_{j=0}^{J-1}|\overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha} + f_j^{n+\alpha}|^2 h_j^{(2)}\end{aligned}$$

$$\leq 4\Big(\frac{\tau^{n+\frac{1}{2}}}{h_*}\Big)^2\sum_{j=0}^{J-1}\Big\{(1+\bar\varepsilon)(\overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha}, \overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)}$$
$$+\Big(1+\frac{1}{\bar\varepsilon}\Big)(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}\Big\}$$
$$\leq 4\Big(\frac{\tau^{n+\frac{1}{2}}}{h_*}\Big)^2\sum_{j=0}^{J-1}\Big\{(1+\bar\varepsilon)\frac{\rho^2(\overset{*}{A}{}_j^{n+\alpha})}{\sigma(\overset{*}{A}{}_j^{n+\alpha})}(\delta^2 v_j^{n+\alpha}, \overset{*}{A}{}_j^{n+\alpha}\delta^2 v_j^{n+\alpha})h_j^{(2)}$$
$$+\Big(1+\frac{1}{\bar\varepsilon}\Big)(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}\Big\},$$

where $h_* = \min_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}$ and the symbols $\rho(\overset{*}{A}{}_j^{n+\alpha})$ and $\sigma(\overset{*}{A}{}_j^{n+\alpha})$ are defined as

$$\rho(A) = \sup_{\xi\in R^n}\frac{|A\xi|}{|\xi|}, \quad \sigma(A) = \inf_{\xi\in R^m}\frac{(\xi, A\xi)}{|\xi|^2}$$

respectively, here $\rho(A)$ is the radius of the spectrum of the matrix A and $\sigma(A)$ measures the definiteness of the matrix A. Evidently, $\rho(A)\geq\sigma(A)\geq\sigma_0>0$. In the case of $0\leq\alpha<\frac{1}{2}$, let us suppose that the meshsteps $\tau=\{\tau^{n+\frac{1}{2}}|n=0,1,\cdots,N-1\}$ and $h=\{h_{j+\frac{1}{2}}|j=0,1,\cdots,J-1\}$ are not chosen independently, but can be chosen with restriction

$$4\Big(\frac{1}{2}-\alpha\Big)\frac{\tau^{n+\frac{1}{2}}}{h_*^2}\max_{(x,t)\in Q_T,|u|\leq 6K_2D}\frac{\rho^2(A(x,t,u))}{\sigma(A(x,t,u))}\leq 1-\varepsilon,$$

where $\varepsilon=1$ for $1\geq\alpha\geq\frac{1}{2}$; $0<\varepsilon<1$ for $0\leq\alpha<\frac{1}{2}$, and

$$\alpha[|\bar\beta_{1j}^{n+1}|+|\bar\beta_{2j}^{n+1}|+|\bar\beta_{3j}^{n+1}|]+[|\bar\beta_{4j}^{n}|+|\bar\beta_{5j}^{n}|+|\bar\beta_{6j}^{n}|]\leq 6K_2.$$

Then

$$(1-2\alpha)\|\delta(v_h^{n+1}-v_h^n)\|_2^2 \leq 2(1+\bar\varepsilon)(1-\varepsilon)\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 v_h^{n+\alpha}\|_2^2$$
$$+2\Big(1+\frac{1}{\bar\varepsilon}\Big)(1-\varepsilon)\frac{\tau^{n+\frac{1}{2}}}{\sigma_0}\sum_{j=1}^{J-1}(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}.$$

Hence in the case of $0\leq\alpha<\frac{1}{2}$, we have

$$\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+2\sigma_0\tau^{n+\frac{1}{2}}(1-\tilde\varepsilon-(1+\bar\varepsilon)(1-\varepsilon))\|\delta^2 v_h^{n+\alpha}\|_2^2$$
$$\leq\frac{\tau^{n+\frac{1}{2}}}{\sigma_0}\Big(\frac{1}{4\tilde\varepsilon}+2\Big(1+\frac{1}{\varepsilon}\Big)(1-\varepsilon)\Big)\sum_{j=1}^{J-1}(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}.$$

Then taking $\tilde\varepsilon=\frac{1}{4}\varepsilon$ and $\bar\varepsilon=\frac{1}{2}\varepsilon$, we get

$$\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+\frac{1}{2}\varepsilon\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 v_h^{n+\alpha}\|_2^2\leq\frac{5}{\varepsilon}\frac{\tau^{n+\frac{1}{2}}}{\sigma_0}\sum_{j=1}^{J-1}(f_j^{n+\alpha}, f_j^{n+\alpha})h_j^{(2)}. \quad (25)$$

By the same way of estimation, we can prove that there exists a constant $D(\varepsilon)$ such that

$$\max_{j=0,1,\cdots,J;N=0,1,\cdots,N} |v_j^n| \leq D(\varepsilon)$$

for given $0 < \varepsilon < 1$, where $D(\varepsilon)$ is independent of the discrete function $v_\triangle$ and $z_\triangle$ and also independent of the meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$. Taking this $D(\varepsilon)$ for given $0 < \varepsilon < 1$, we see that the mapping H maps the bounded convex set G into itself. By the Brouwer fixed point theorem, there exist at least one fixed point of the mapping $H : G \to G$ in the bounded convex set G. Hence for the fixed point $v_\triangle$, there is $v_\triangle = Hv_\triangle \in G$. For these fixed point, the inequality

$$4\Big(\frac{1}{2} - \alpha\Big)\frac{\tau^{n+\frac{1}{2}}}{h_*^2}\frac{\rho^2(A_j^{n+\alpha})}{\sigma(A_j^{n+\alpha})} \leq 1 - \varepsilon \tag{26}$$

holds for $j = 1, 2, \cdots, J-1$; $n = 0, 1, \cdots, N-1$ and $0 \leq \alpha < \dfrac{1}{2}$.

From the above discussion, we have the following statement for the restriction of meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$.

(V) For any given small constant $0 < \varepsilon < 1$, there is a constant $D(\varepsilon)$ which is determined by ε and the data of the original problem or is dependent on ε and on the coefficient matrix $A(x, t, u)$ and the free vector term $f(x, t, u, p)$ of the system (1) and the initial vector function $\varphi(x)$. The meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ are so chosen that they satisfy the relation

$$4\Big(\frac{1}{2} - \alpha\Big)\frac{\tau^*}{h_*^2}\max_{(x,t)\in Q_T, |u| \leq 6K_2 D(\varepsilon)} \frac{\rho^2(A(x,t,u))}{\sigma(A(x,t,u))} \leq 1 - \varepsilon, \tag{27}$$

where $1 - 2\alpha > 0$.

It is clear that the fixed points are the discrete solutions of the finite difference scheme (7), (10) and (11). Hence by an analogous argument, we obtain the following theorem of estimation for the discrete solutions of the explicit ($\alpha = 0$) and the weakly implicit $\Big(0 < \alpha < \dfrac{1}{2}\Big)$ difference scheme (7), (10) and (11).

Theorem 3. *Suppose that the conditions (I), (II) and (III) are satisfied. Suppose that* $0 \leq \alpha < \dfrac{1}{2}$ *and the meshsteps* $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ *and* $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ *satisfy the condition of restriction (V). The weakly implicit* $\Big(0 < \alpha < \dfrac{1}{2}\Big)$ *and the explicit* ($\alpha = 0$) *finite difference schemes (7), (10) and (11) have at least one discrete solution* $v_\triangle = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ *and there are the estimates*

$$\max_{0,1,\cdots,N}(\|v_h^n\|_2^2 + \|\delta v_h^n\|_2^2) + \sum_{j=0}^{N-1}(\|v_h^{n+\alpha}\|_2^2 + \|\delta v_h^{n+\alpha}\|_2^2 + \|\delta^2 v_h^{n+\alpha}\|_2^2)\tau^{n+\frac{1}{2}}$$
$$+ \sum_{j=0}^{N-1}\Big\|\frac{v_h^{n+1} - v_h^n}{\tau^{n+\frac{1}{2}}}\Big\|_2^2 \tau^{n+\frac{1}{2}}$$

$$\leq K(\varepsilon)\Big\{\|\varepsilon_h\|_2^2+\|\delta\varepsilon_h\|_2^2+\sum_{n=0}^{N-1}\|\bar{f}_h^{n+\alpha}\|_2^2\tau^{n+\frac{1}{2}}\Big\}, \tag{28}$$

where $K(\varepsilon)$ is a constant independent of the meshsteps h and τ and dependent on $0<\varepsilon<1$ given in the condition (V) and here $\bar{f}(x,t)\equiv f(x,t,0,0)$.

9. Now we turn to prove the uniqueness of the discrete solution $v_\Delta=v_h^\tau=\{v_j^n|j=0,1,\cdots,J;n=0,1,\cdots,N\}$ for the nonlinear difference system (7), (10) and (11). Suppose that for the given $v_j^n(j=0,1,\cdots,J)$ there are two solutions v_j^{n+1} and $\bar{v}_j^{n+1}$ $(j=0,1,\cdots,J)$. Then we have

$$\frac{v_j^{n+1}-v_j^n}{\tau^{n+\frac{1}{2}}}=A_j^{n+\alpha}\delta^2v_j^{n+\alpha}+f_j^{n+\alpha},\quad (j=1,\cdots,J-1);\quad v_0^{n+1}=v_J^{n+1}=0$$

and

$$\frac{\bar{v}_j^{n+1}-v_j^n}{\tau^{n+\frac{1}{2}}}=\bar{A}_j^{n+\alpha}\delta^2\bar{v}_j^{n+\alpha}+\bar{f}_j^{n+\alpha},\quad (j=1,\cdots,J-1);\quad \bar{v}_0^{n+1}=\bar{v}_J^{n+1}=0,$$

where $\bar{A}_j^{n+\alpha}$ and $\bar{f}_j^{n+\alpha}$ $(j=1,2,\cdots,J-1)$ are obtained from $A_j^{n+\alpha}$ and $f_j^{n+\alpha}$ $(j=1,2,\cdots,J-1)$ respectively by the replacement of v_j^{n+1} $(j=0,1,\cdots,J)$ by the corresponding $\bar{v}_j^{n+1}$ $(j=0,1,\cdots,J)$. The difference $w_j=v_j^{n+1}-\bar{v}_j^{n+1}$ satisfies

$$\begin{aligned} w_j=&\alpha\tau^{n+\frac{1}{2}}A_j^{n+\alpha}\delta^2w_j+\tau^{n+\frac{1}{2}}(A_j^{n+\alpha}-\bar{A}_j^{n+\alpha})\delta^2\bar{v}_j^{n+\alpha}\\ &\tau^{n+\frac{1}{2}}(f_j^{n+\alpha}-\bar{f}_j^{n+\alpha}),\quad (j=1,2,\cdots,J-1),\\ w_0=&w_J=0. \end{aligned} \tag{29}$$

From the assumptions (I), (II), (IV) and the estimates in Theorem 2 and 3, we have the following estimates

$$\begin{aligned} &\|f_h^{n+\alpha}-\bar{f}_h^{n+\alpha}\|_2^2\leq\alpha^2(\varepsilon_1\|\delta^2w_h\|_2^2+C_{11}(\varepsilon_1)\|w_h\|_2^2),\\ \max_{n=0,1,\cdots,N-1}&\|A_h^{n+\alpha}-\bar{A}_h^{n+\alpha}\|_\infty^2\leq\alpha^2(\varepsilon_2\|\delta w_h\|_2^2+C_{12}(\varepsilon_2)\|w_h\|_2^2),\\ &\|\delta^2\bar{v}_h^{n+\alpha}\|_2^2\tau^{n+\frac{1}{2}}\leq C_{13}, \end{aligned} \tag{30}$$

where C's are independent of the unequal meshsteps.

Now firstly making the scalar product of the vectors $\delta^2w_jh_j^{(2)}$ with the vector equation (29) and summing up the resulting products for $j=1,2,\cdots,J-1$, we have

$$\begin{aligned} \sum_{j=1}^{J-1}(\delta^2w_j,w_j)h_j^{(2)}=&\alpha\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2w_j,A_j^{n+\alpha}\delta^2w_j)h_j^{(2)}\\ &+\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2w_j,(A_j^{n+\alpha}-\bar{A}_j^{n+\alpha})\delta^2\bar{v}_j^{n+\alpha})h_j^{(2)}\\ &+\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2w_j,(f_j^{n+\alpha}-\bar{f}_j^{n+\alpha}))h_j^{(2)}. \end{aligned} \tag{31}$$

Using the estimates (30), we have

$$\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 w_j,(A_j^{n+\alpha}-\bar{A}_j^{n+\alpha})\delta^2\bar{v}_j^{n+\alpha})h_j^{(2)}\le\tau^{n+\frac{1}{2}}\Big\{\alpha\varepsilon_3\|\delta^2 w_h\|_2^2$$
$$+\frac{1}{4\alpha\varepsilon_3}\|A_h^{n+\alpha}-\bar{A}_h^{n+\alpha}\|_\infty^2\|\delta^2\bar{v}_h^{n+\alpha}\|_2^2\Big\}\le\tau^{n+\frac{1}{2}}\alpha\varepsilon_3\|\delta^2 w_h\|_2^2$$
$$+\frac{\alpha}{4\varepsilon_3}(\varepsilon_2\|\delta w_h\|_2^2+C_{11}(\varepsilon_2)\|w_h\|_2^2)\cdot C_{13}\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 w_j,(f_j^{n+\alpha}-\bar{f}_j^{n+\alpha}))h_j^{(2)}$$
$$\le\tau^{n+\frac{1}{2}}\Big\{\alpha\varepsilon_3\|\delta^2 w_h\|_2^2+\frac{\alpha}{4\varepsilon_3}(\varepsilon_1\|\delta^2 w_h\|_2^2+C_{12}(\varepsilon_1)\|w_h\|_2^2)\Big\}.$$

Then (31) becomes

$$\Big(1-\frac{4\varepsilon_2 C_{13}}{4\varepsilon_3}\Big)\|\delta w_h\|_2^2+\alpha\tau^{n+\frac{1}{2}}\Big[\sigma_0-2\varepsilon_3-\frac{\varepsilon_1}{4\varepsilon_3}\Big]\|\delta^2 w_h\|_2^2$$
$$\le\frac{\alpha}{4\varepsilon_3}(C_{13}C_{12}(\varepsilon_2)+\tau^{n+\frac{1}{2}}C_{11}(\varepsilon_1))\|w_h\|_2^2.$$

Taking $\varepsilon-1=\varepsilon_2=\varepsilon_3^2$, $\varepsilon_3\le\min\Big(\frac{2}{C_{13}},\frac{2\sigma_0}{9}\Big)$, there is

$$\|\delta w_h\|_2^2+\alpha\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h\|_2^2\le C_{14}\|w_h\|_2^2. \tag{32}$$

Secondly taking the scalar product of the vector $w_j h_j^{(2)}$ and the vector equation (29) and then summing up the resulting relations for $j=1,2,\cdots,J-1$, we obtain

$$\sum_{j=1}^{J-1}|w_j|^2 h_j^{(2)}=\alpha\tau^{n+\frac{1}{2}}\sum_{j-=1}^{J-1}(w_j,A_j^{n+\alpha}\delta^2 w_j)h_j^{(2)}+\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(w_j,(A_j^{n+\alpha}-\bar{A}_j^{n+\alpha})\delta^2\bar{v}_j^{n+\alpha})h_j^{(2)}$$
$$+\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(w_j,f_j^{n+\alpha}-\bar{f}_j^{n+\alpha})h_j^{(2)}. \tag{33}$$

Using the estimates (30), we have

$$\alpha\tau^{n+\frac{1}{2}}\Big|\sum_{j=1}^{J-1}(w_j,A_j^{n+\alpha}\delta^2 w_j)h_j^{(2)}\Big|\le\alpha\tau^{n+\frac{1}{2}}(\bar{\varepsilon}\|\delta^2 w_h\|_2^2+C_{15}(\bar{\varepsilon})\|w_h\|_2^2),$$
$$\tau^{n+\frac{1}{2}}\Big|\sum_{j=1}^{J-1}(w_j,(A_j^{n+\alpha}-\bar{A}_j^{n+\alpha})\delta^2\bar{v}_j^{n+\alpha})h_j^{(2)}\Big|\le\frac{\bar{\varepsilon}}{\alpha}\|A_h^{n+\alpha}-\bar{A}_h^{n+\alpha}\|_\infty^2(\|\delta^2\bar{h}_h^{n+\alpha}\|_2^2\tau^{n+\frac{1}{2}})$$
$$+\alpha\tau^{n+\frac{1}{2}}C_{15}(\bar{\varepsilon})\|w_h\|_2^2\le\alpha\bar{\varepsilon}C_{13}[\varepsilon_2\|\delta w_h\|_2^2+C_{12}(\varepsilon_2)\|w_h\|_2^2]+\alpha\tau^{n+\frac{1}{2}}C_{15}(\bar{\varepsilon})\|w_h\|_2^2,$$
$$\tau^{n+\frac{1}{2}}\Big|\sum_{j=1}^{J-1}(w_j,f_j^{n+\alpha}-\bar{f}_j^{n+\alpha})h_j^{(2)}\Big|\le\tau^{n+\frac{1}{2}}\frac{\bar{\varepsilon}}{\alpha}\|f_h^{n+\alpha}-\bar{f}_h^{n+\alpha}\|_2^2$$
$$+\alpha\tau^{n+\frac{1}{2}}C_{15}(\bar{\varepsilon})\|w_h\|_2^2\le\tau^{n+\frac{1}{2}}\alpha\bar{\varepsilon}[\varepsilon_1\|\delta^2 w_h\|_2^2+C_{11}(\varepsilon_1)\|w_h\|_2^2]+\alpha\tau^{n+\frac{1}{2}}C_{15}(\bar{\varepsilon})\|w_h\|_2^2.$$

Then (33) becomes

$$\|w_h\|_2^2(1-\alpha\bar{\varepsilon}C_{13}C_{12}(\varepsilon_2)-\alpha\bar{\varepsilon}\tau^{n+\frac{1}{2}}C_{11}(\varepsilon_1)-3\alpha\tau^{n+\frac{1}{2}}C_{15}(\bar{\varepsilon}))$$
$$\leq \bar{\varepsilon}\tau^{n+\frac{1}{2}}\alpha(1+\varepsilon_1)\|\delta^2 w_h\|_2^2+\alpha\bar{\varepsilon}C_{13}\varepsilon_2\|\delta w_h\|_2^2.$$

Taking

$$\bar{\varepsilon}\leq\frac{1}{4C_{12}C_{13}},\quad \tau^{n+\frac{1}{2}}\leq\frac{1}{4(C_{11}\bar{\varepsilon}+3C_{15}(\bar{\varepsilon}))},$$

we get

$$\|w_h\|_2^2\leq 4\alpha\bar{\varepsilon}\tau^{n+\frac{1}{2}}\|\delta^2 w_h\|_2^2+2C_{13}\bar{\varepsilon}\|\delta w_h\|_2^2. \tag{34}$$

Substituting (34) into (32), we have

$$\|\delta w_h\|_2^2+\alpha\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h\|_2^2\leq 4\alpha\bar{\varepsilon}\tau^{n+\frac{1}{2}}C_{14}\|\delta^2 w_h\|_2^2+2C_{13}C_{14}\bar{\varepsilon}\|\delta w_h\|_2^2$$

or

$$(1-2C_{13}C_{14}\bar{\varepsilon})\|\delta w_h\|_2^2+\alpha\tau^{n+\frac{1}{2}}(\sigma_0-4C_{14}\bar{\varepsilon})\|\delta^2 w_h\|_2^2\leq 0.$$

Taking

$$\bar{\varepsilon}\leq\min\Big(\frac{1}{4C_{12}C_{13}},\frac{1}{4C_{13}C_{14}},\frac{\sigma_0}{C_{14}}\Big),$$

we have

$$\|\delta w_h\|_2^2+\alpha\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h\|_2^2\leq 0.$$

Hence we obtain $w_h=0$, that is $v_h^{n+1}=\bar{v}_h^{n+1}$. This completes the proof of the uniqueness for the discrete solution of the difference scheme (7), (10) and (11).

Theorem 4. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are satisfied and the restriction condition* (V) *for the unequal meshsteps is valid for the cases of the explicit* ($\alpha=0$) *and weakly implicit* $\left(0<\alpha<\frac{1}{2}\right)$ *finite difference schemes. As the meshsteps* $\tau=\{\tau^{n+\frac{1}{2}}|n=0,1,\cdots,N-1\}$ *or* τ^* *is sufficiently small, the discrete solution* $v_\triangle=v_h^\tau=\{v_j^n|j=0,1,\cdots,J;n=0,1,\cdots,N\}$ *of the difference scheme* (7), (10) *and* (11) *is unique for either the case of strongly implicit or the case of explicit and weakly implicit difference schemes.*

3. Convergence Theorems

10. In this section we are going to establish convergence theorems of the implicit and explicit finite difference scheme (7), (10) and (11) on the basis of the obtained estimates and the convergence properties of the discrete solutions $v_\triangle=v_h^\tau=\{v_j^n|j=0,1,\cdots,J;$ $n=0,1,\cdots,N\}$.

Let us define the piecewise constant functions

$$v_h^\tau(x,t)=v_j^{n+1},\quad \bar{v}_h^\tau=v_j^{n+\alpha},\quad \tilde{v}_h^\tau(x,t)=\frac{v_j^{n+1}-v_j^n}{\tau^{n+\frac{1}{2}}}$$

for $(x,t)\in Q_j^{n+\frac{1}{2}}=\{x^{(1)}_{j-\frac{1}{2}}\leq x\leq x^{(1)}_{j+\frac{1}{2}};\ t^n<t\leq t^{n+1}\}$ ($j=0,1,\cdots,J;\ n=0,1,\cdots,N-1$), where $h_{-\frac{1}{2}}=h_{J+\frac{1}{2}}=0$, $x_0=x_{-1}$, $x_J=x_{J+1}$. Define the piecewise constant functions

$$v_h^{'\tau}(x,t)=\delta v_j^{n+1},\quad \bar{v}_h^{'\tau}(x,t)=\delta v_j^{n+\alpha}$$

for $(x,t) \in Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \{x_j \le x \le x_{j+1};\ t^n < t \le t^{n+1}\}$ $(j = 0,1,\cdots,J-1;\ n = 0,1,\cdots,N-1)$. Again define

$$\overset{''}{v}{}_h^\tau(x,t) = \delta^2 v_j^{n+1}, \quad \overset{''}{\bar{v}}{}_h^\tau(x,t) = \delta^2 v_j^{n+\alpha}$$

for $(x,t) \in Q_j^{n+\frac{1}{2}}$ $(j = 0,1,\cdots,J-1;\ n = 0,1,\cdots,N-1)$.

Let us define $A_h^\tau(x,t) = A_j^{n+\alpha}$ and $f_h^\tau(x,t) = f_j^{n+\alpha}$ in $Q_j^{n+\frac{1}{2}}$ for $j = 0,1,\cdots,J-1$ and $n = 0,1,\cdots,N-1$.

As $i \to \infty$ and the sequences of meshsteps $\tau_{(i)}$ and $h_{(i)}$ tend to zero, the sequence of piecewise constant functions $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ uniformly converges to $u(x,t)$ in Q_T and the sequences of piecewise constant functions corresponding to the $\bar{\delta}^0 v_j^{n+\alpha}$ and $\tilde{\delta}^0 v_j^{n+\alpha}$ in the expressions of $A_j^{n+\alpha}$ and $f_j^{n+\alpha}$ respectively are also uniformly convergent to $u(x,t)$ in Q_T as $i \to \infty$ and then $\tau_{(i)}$ and $h_{(i)} \to 0$. Hence as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$, the sequence of piecewise constant functions $\{A_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is also uniformly convergent t $A(x,t,u(x,t))$ in Q_T.

As $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$, the sequence of piecewise constant functions $\{\overset{'}{\bar{v}}{}_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is strongly convergent to $u_x(x,t)$ in $L_2(Q_T)$. Then the sequence $\overset{'}{\bar{v}}{}_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ also converges to $u_x(x,t)$ pointwisely almost everywhere in Q_T. In this case the sequences of the piecewise constant functions corresponding to the $\tilde{\delta}' v_j^{n+\alpha}$ in the expressions of $f_j^{n+\alpha}$ is also convergent to $u_x(x,t)$ pointwisely almost everywhere in Q_T. Therefore the sequence $\{f_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is also pointwisely almost everywhere convergent to $f(x,t,u(x,t),\ u_x(x,t))$ in Q_T. On the other hand $f_h^\tau(x,t)$ is uniformly bounded in $L_2(Q_T)$ with respect to the meshsteps τ and h. So the sequence $\{f_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is weakly convergent to the m-dimensional vector function $f(x,t,u(x,t),\ u_x(x,t))$ as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$.

Also the sequence of piecewise constant functions $\{\overset{''}{v}{}_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is weakly convergent to $u_{xx}(x,t)$ in $L_2(Q_T)$ and the sequence $\{\tilde{v}_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is weakly convergent to $u_t(x,t)$ in Q_T as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$.

Here the limiting m-dimensional vector function $u(x,t)$ belongs to the functional space $W_2^{(2,1)}(Q_T)$.

11. Let $\Phi(x,t)$ be any smooth test function. Define the piecewise constant function $\Phi_h^\tau(x,t)$ corresponding to the discrete function $\{\Phi_j^n = \Phi(x_j,t^n)|j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ as before, that $\Phi_h^\tau(x,t) = \Phi_j^n$ in $Q_j^{n+\frac{1}{2}}$ for $j = 0,1,\cdots,J$ and $n = 0,1,\cdots,N-1$.

Here we have evidently the identity

$$\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \Phi_j^n \Big[\frac{v_j^{n+1} - v_j^n}{\tau^{n+\frac{1}{2}}} - A_j^{n+\alpha}\delta^2 v_j^{n+\alpha} - f_j^{n+\alpha}\Big] h_j^{(2)} \tau^{n+\frac{1}{2}} = 0.$$

This is equivalent to the integral identity

$$\int_0^T \int_{x^{(1)}_{\frac{1}{2}}}^{x^{(1)}_{J-\frac{1}{2}}} \Phi_h^\tau(x,t)[\tilde{v}_h^\tau(x,t) - A_h^\tau(x,t)\bar{v}_h''^\tau(x,t) - f_h^\tau(x,t)]dxdt = 0.$$

When $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$, $\{\Phi_{h_{(i)}}^{\tau(i)}(x,t)\}$ and $A_{h_{(i)}}^{\tau(i)}(x,t)\}$ are uniformly convergent to $\Phi(x,t)$ and $A(x,t,u(x,t))$ respectively in Q_T and $\{\tilde{v}_{h_{(i)}}^{\tau(i)}(x,t)\}$, $\bar{v}_{h_{(i)}}''^{\tau(i)}(x,t)\}$ and $\{f_{h_{(i)}}^{\tau(i)}(x,t)\}$ are weakly convergent to $u_t(x,t)$, $u_{xx}(x,t)$ and $f(x,t,u(x,t))$, $u_x(x,t))$ respectively in $L_2(Q_T)$.

Here we have

$$\begin{aligned}
&\int_0^T \int_{x^{(1)}_{\frac{1}{2}}}^{x^{(1)}_{J-\frac{1}{2}}} \Phi_h^\tau(x,t)\tilde{v}_h^\tau(x,t)dxdt - \int\int_{Q_T} \Phi(x,t)\tilde{v}_h^\tau(x,t)dxdt \\
\leq &\left|\int\int_{Q_T} (\Phi_h^\tau(x,t) - \Phi(x,t))\tilde{v}_h^\tau(x,t)dxdt\right| \\
&+ \left|\int_0^T \int_0^{x^{(1)}_{\frac{1}{2}}} \Phi_h^\tau(x,t)\tilde{v}_h^\tau(x,t)dxdt\right| + \left|\int_0^T \int_{x^{(1)}_{J-\frac{1}{2}}}^{l} \Phi_h^\tau(x,t)\tilde{v}_h^\tau(x,t)dxdt\right|
\end{aligned}$$

By means of

$$\left|\int\int_{Q_T} (\Phi_h^\tau(x,t) - \Phi(x,t))\tilde{v}_h^\tau(x,t)dxdt\right| \leq \sqrt{lT}\|\Phi_h^\tau - \Phi\|_{L_\infty(Q_T)}\|\tilde{v}_h^\tau\|_{L^2(Q_T)},$$

$$\left|\int_0^T \int_0^{x^{(1)}_{\frac{1}{2}}} \Phi_h^\tau(x,t)\tilde{v}_h^\tau(x,t)dxdt\right| \leq \sqrt{\frac{1}{2}h_{\frac{1}{2}}T}\|\Phi_h^\tau\|_{L_\infty(Q_T)}\|\tilde{v}_h^\tau\|_{L^2(Q_T)},$$

$$\left|\int_0^T \int_{x^{(1)}_{J-\frac{1}{2}}}^{l} \Phi_h^\tau(x,t)\tilde{v}_h^\tau(x,t)dxdt\right| \leq \sqrt{\frac{1}{2}h_{J-\frac{1}{2}}T}\|\Phi_h^\tau\|_{L_\infty(Q_T)}\|\tilde{v}_h^\tau\|_{L^2(Q_T)},$$

we see that as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$

$$\int_0^T \int_{x^{(1)}_{\frac{1}{2}}}^{x^{(1)}_{J-\frac{1}{2}}} \Phi_{h_{(i)}}^{\tau(i)}(x,t)\tilde{v}_{h_{(i)}}^{\tau(i)}(x,t)dxdt \to \int\int_{Q_T} \Phi(x,t)u_t(x,t)dxdt.$$

Similarly, we can verify that as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$,

$$\int_0^T \int_{x^{(1)}_{\frac{1}{2}}}^{x^{(1)}_{J-\frac{1}{2}}} \Phi_{h_{(i)}}^{\tau(i)}(x,t)f_{h_{(i)}}^{\tau(i)}(x,t)dxdt \to \int\int_{Q_T} \Phi(x,t)f(x,t,u(x,t),u_x(x,t))dxdt,$$

$$\int_0^T \int_{x^{(1)}_{\frac{1}{2}}}^{x^{(1)}_{J-\frac{1}{2}}} \Phi_{h_{(i)}}^{\tau(i)}(x,t)A_{h_{(i)}}^{\tau(i)}(x,t)\bar{v}_{h_{(i)}}''^{\tau(i)}(x,t)dxdt \to \int\int_{Q_T} \Phi(x,t)A(x,t,u(x,t))u_{xx}(x,t)dxdt.$$

Thus for any smooth test function $\Phi(x,t)$, we obtain m-dimensional integral identity

$$\int\int_{Q_T} \Phi(x,t)[u_t(x,t) - A(x,t,u(x,t))u_{xx}(x,t) - f(x,t,u(x,t)),u_x(x,t)]dxdt = 0.$$

This means that the m-dimensional vector function $u(x,t)$ satisfies the quasilinear parabolic system (1) of partial differential equations in generalized sense. It is clear that $u(x,t)$ belongs to the Banach space $W_2^{(2,1)}(Q_T)$.

Since the sequence $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is uniformly convergent to $u(x,t)$ in the rectangular domain Q_T, the limiting m-dimensional vector function $u(x,t)$ satisfies the homogeneous boundary conditions (3) and the initial condition (4) in classical sense. This means that the m-dimensional vector function $u(x,t) \in W_2^{(2,1)}(Q_T)$ is just the generalized solution of the boundary problem with the homogeneous boundary conditions (3) and the initial condition (4) for the quasilinear parabolic system (1) of partial differential equations.

Hence we have proved that there exists a sequence of meshsteps $\tau_{(i)}$ and $h_{(i)}$ ($i = 1, 2, \cdots$) such that as $i \to \infty$ and then $\tau_{(i)} \to 0$ and $h_{(i)} \to 0$, the discrete solutions $v_{h_{(i)}}^{\tau_{(i)}}$ of the finite difference scheme (7), (10) and (11) converge to the m-dimensional vector function $u(x,t) \in W_2^{(2,1)}(Q_T)$, which is just the generalized solution of the problem (3) and (4) for the quasilinear parabolic system (1). The sense of convergence is as follows: $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ converges uniformly to $u(x,t)$ in Q_T and $\{\bar{v}_{h_{(i)}}^{'\tau_{(i)}}(x,t)\}$, $\{\bar{v}_{h_{(i)}}^{''\tau_{(i)}}(x,t)\}$ and $\{\tilde{v}_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ converge weakly to $u_x(x,t)$, $u_{xx}(x,t)$ and $u_t(x,t)$ respectively in $L_2(Q_T)$.

12. Theorem 5. *Under the conditions (I), (II), (III) and (IV), the homogeneous boundary problem with the boundary conditions* (3) *and the initial condition* (4) *for the quasilinear parabolic system* (1) *of partial differential equations has a unique generalized solution* $u(x,t) \in W_2^{(2,1)}(Q_T)$, *which is a m-dimensional vector function, satisfying the quasilinear parabolic system* (1) *in generalized sense and satisfying the boundary conditions* (3) *and the initial condition* (4) *in classical sense.*

The uniqueness of the generalized solution can be justified by usual way.

By means of the uniqueness of the generalized solution of the homogeneous boundary problem (3) and (4) for the quasilinear parabolic system (1) of partial differential equations, we then can obtain the absolute convergence theorem for the strongly implicit finite difference scheme (7), (10) and (11) and the relative convergence theorem for the weakly implicit and explicit finite difference schemes (7), (10) and (11) as follows:

Theorem 6. *Under the conditions* $(I), (II), (III)$ *and* (IV) *as the meshsteps* $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ *and* $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ *tend to zero, the m-dimensional discrete vector solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ *of the strongly implicit finite difference scheme* (7), (10) *and* (11) *with* $\frac{1}{2} \le \alpha \le 1$ *converges to the unique generalized solution* $u(x,t) \in W_2^{(2,1)}(Q_T)$ *of the boundary problem* (3) *and* (4) *for the quasilinear parabolic system* (1) *of partial differential equations.*

Theorem 7. *Suppose that the conditions* $(I), (II), (III)$ *and* (IV) *are satisfied. When the meshsteps* $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ *and* $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ *tend to zero with the condition of restriction* (V), *the m-dimensional discrete vector solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$ *of the weakly implicit* $\left(0 < \alpha < \frac{1}{2}\right)$ *and the explicit* $(\alpha = 0)$ *finite difference schemes* (7), (10) *and* (11) *converges* *to the unique generalized vector solution* $u(x,t) \in W_2^{(2,1)}(Q_T)$ *of the boundary problem* (3) *and* (4) *for the quasilinear parabolic system* (1) *of partial differential equations.*

The convergence of discrete vector functions v_Δ to the vector solution $u(x,t) \in W_2^{(2,1)}(Q_T)$ as the unequal meshsteps $h = \{h_{j+\frac{1}{2}} | j = 0, 1, \cdots, J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} | n = 0, 1, \cdots, N-1\}$ tend to zero means that for any convergent to zero sequence $\{h_{(i)}, \tau_{(i)}\}$ of meshsteps as $i \to \infty$, the corresponding sequence $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ converges to $u(x,t) \in W_2^{(2,1)}(Q_T)$ in the above mentioned sense. Since the generalized solution $u(x,t)$ of the boundary problem (3) and (4) for the parabolic system (1) is unique, the limiting m-dimensional vector function is always the same one m-dimensional vector function $u(x,t) \in W_2^{(2,1)}(Q_T)$.

Now suppose that there is a sequence $\{h_{(i)}, \tau_{(i)}\}$ which converges to zero as $i \to \infty$, such that the corresponding sequence $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ is not convergent. Then there exist two subsequences $\{h_{(i)}, \tau_{(i)}\}$ and $\{h_{(\bar{i})}, \tau_{(\bar{i})}\}$ with the properties that as $i \to \infty$, $\bar{i} \to \infty$ and then $h_{(i)}, h_{(\bar{i})}, \tau_{(i)}, \tau_{(\bar{i})} \to 0$, such that the corresponding sequences $\{h_{(i)}, \tau_{(i)}\}$ and $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ and $\{v_{h_{(\bar{i})}}^{\tau_{(\bar{i})}}(x,t)\}$ satisfy the relation

$$\|v_{h_{(i)}}^{\tau_{(i)}} - v_{h_{(\bar{i})}}^{\tau_{(\bar{i})}}\|_{L_\infty(Q_T)} > \varepsilon,$$

where $\varepsilon > 0$ is a certain constant. Let subsequences $\{v_{\tilde{h}_{(i)}}^{\tilde{\tau}_{(i)}}(x,t)\}$ of the sequence $\{v_{h_{(i)}}^{\tau_{(i)}}(x,t)\}$ and the subsequences $\{v_{\bar{\tilde{h}}_{(i)}}^{\bar{\tilde{\tau}}_{(i)}}(x,t)\}$ of the sequence $\{v_{h_{(\bar{i})}}^{\tau_{(\bar{i})}}(x,t)\}$ be uniformly convergent to $u_1(x,t)$ and $u_2(x,t)$ respectively in Q_T, then we have

$$\|u_1 - u_2\|_{L_\infty(Q_T)} \geq \varepsilon.$$

This contradiction shows that for any sequence $\{h_{(i)}, \tau_{(i)}\}$ convergent to zero, the corresponding sequences $\{v_{h_{(i)}}^{\tau_{(i)}}\}$ converges always to the unique generalized solution $u(x,t)$ of the boundary problem (3) and (4) for the system (1).

Hence theorems are proved.

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