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A CLASS OF C^1 DISCRETE INTERPOLANTS OVER TETRAHEDRA*

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Abstract

Smooth interpolants defined over tetrahedra are currently being developed for they have many applications in geography, solid modeling, finite element analysis, etc. In this paper, we will characterize a certain class of C^1 discrete tetrahedral interpolants with only C^1 data required. As special cases of the class characterized, we give two C^1 discrete tetrahedral interpolants which have concise expressions.

1. Introduction

The purpose of the paper is to characterize a certain class of C^1 discrete interpolants defined over tetrahedra with only C^1 data required. We assume that a polyhedral domain in three-space or a set of 3D scattered data have been tessellated into tetrahedra with any two of which share only one face. As for this preprocessing stage, one may refer to [2] and [3] and here we omit it. In the paper, we only describe the characterization of an interpolant over a single tetrahedron for the interpolants have the same form. Now we begin our paper with some conceptions and notations.

A discrete interpolant $\mathcal{P}$ is said to interpolate a linear functional $\mathcal{L}$ if $\mathcal{LP}f = f$ for any function f. For simplicity, we sometimes use $\mathcal{P}$ to denote $\mathcal{P}f$ for any function f being interpolated. Denote a general tetrahedron by V with vertices V_i, $i = 1, \cdots, 4$. Denote its faces by F_i, $i = 1, \cdots, 4$, with F_i opposite to vertex V_i, and edges by E_i^j, $j \neq i$, with E_i^j opposite to vertices V_j and V_i, i.e.,

$$E_i^j(t) = (1-t)V_k + tV_l, \quad k, l \neq i, j, \quad k \neq l.$$

Denote edge vectors by $e_{ij} = V_j - V_i$. Furthermore, denote(see Fig.1)

$$\boldsymbol{n}_i = e_{li} - \frac{\boldsymbol{n}_l^i \cdot e_{li}}{\boldsymbol{n}_l^i \cdot \boldsymbol{n}_l^i}\boldsymbol{n}_l^i, \quad \boldsymbol{n}_l^i = e_{kl} - \frac{e_{jk} \cdot e_{kl}}{e_{jk} \cdot e_{jk}} e_{jk} \tag{1.1}$$

where $(i, j, k, l) \in \Lambda := \{(1, 2, 3, 4), (2, 3, 4, 1), (3, 4, 1, 2), (4, 1, 2, 3)\}$, then the directional derivative is computed to be

$$\frac{\partial b_i}{\partial \boldsymbol{n}_i} = 1, \quad \frac{\partial b_j}{\partial \boldsymbol{n}_i} = -\frac{\boldsymbol{n}_l^i \cdot e_{li}}{\boldsymbol{n}_l^i \cdot \boldsymbol{n}_l^i} \cdot \frac{e_{jk} \cdot e_{kl}}{e_{jk} \cdot e_{jk}} \tag{1.2}$$

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$$\frac{\partial b_k}{\partial \boldsymbol{n}_i} = -\frac{\boldsymbol{n}_l^i \cdot e_{li}}{\boldsymbol{n}_l^i \cdot \boldsymbol{n}_l^i} \cdot \frac{e_{jk} \cdot e_{lj}}{e_{jk} \cdot e_{jk}}, \quad \frac{\partial b_l}{\partial \boldsymbol{n}_i} = -1 - \frac{\boldsymbol{n}_l^i \cdot e_{li}}{\boldsymbol{n}_l^i \cdot \boldsymbol{n}_l^i}, \tag{1.3}$$

where (b_1, b_2, b_3, b_4) is the barycentric coordinate of a point P on tetrahedron V:

$$P = b_1V_1 + b_2V_2 + b_3V_3 + b_4V_4, \quad b_1 + b_2 + b_3 + b_4 = 1.$$

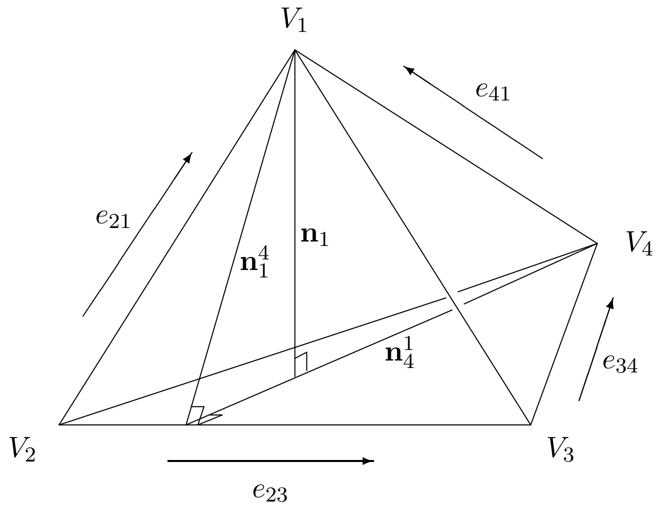

Fig. 1 Notational conventions on a tetrahedron

2. Interpolation Requirements

Our goal is to characterize the set of discrete tetrahedral interpolants $\mathcal{P}$ which satisfies the following requirements:

(2.1). $\mathcal{P}$ forms C^1 joins with adjacent interpolants.

(2.2). $\mathcal{P}$ is locally defined, i.e., evaluating the interpolant at a point within a tetrahedron domain requires only data defined on it. This ensures that local changes in the data will only have local effects.

Usually, the construction of a C^1 tetrahedral interpolant requires the positions and the first derivatives at vertices of the tetrahedron. But the above information is not sufficient to insure C^1 joins with adjacent interpolants. We restrict $\mathcal{P}$ to satisfy

(2.3). $\mathcal{P}$ interpolates some cross boundary derivatives in addition to the positions and the first derivatives at corners. In order to maintain the interpolation precision and shape fidelity, we insist that

(2.4). $\mathcal{P}$ has cubic precision, i.e., $\mathcal{P}$ reproduces any trivariate polynomial up to cubic degree.

Because a great deal of flexibility can be obtained by using rational functions, we also insist that

(2.5). $\mathcal{P}$ is a rational polynomial in form.

Finally, due to aesthetic reasons we insist that the form of the interpolant $\mathcal{P}$ is not affected by the ordering of the tetrahedron vertices, that is:

(2.6). $\mathcal{P}$ is of tetrahedral symmetric form.

In terms of linear functionals, the interpolation properties (2.3) of $\mathcal{P}$ can be expressed as follows:

$$\begin{cases} \mathcal{F}_i f = f(V_i), & i = 1,2,3,4. \\ \mathcal{T}_{ij} f = \dfrac{\partial f}{\partial (V_j - V_i)}\Big|_{V_i}, & i,j = 1,2,3,4, \quad j \neq i, \\ \mathcal{N}_{ip}^{\alpha} f = \dfrac{\partial f}{\partial D_{ip}^{\alpha}}\Big|_{P_{ip}^{\alpha}}, & \alpha = 1,2,3,4 \quad i \neq \alpha, \quad p = 1,2,\ldots,s, \end{cases} \tag{2.7}$$

here D_{ip}^{α} are some vectors on face F_α and $P_{ip}^{\alpha} = P_{\alpha p}^{i}$ some points on edge E_i^{α} where cross boundary derivative evaluations take place. From the cubic precision (2.4) and fact(derived from (2.3)) that $\mathcal{P}$ restricted on edge E_i^{α} interpolates $\mathcal{F}_j$, $\mathcal{F}_k$, $\mathcal{T}_{jk}$ and $\mathcal{T}_{kj}$, we conclude that $\mathcal{P}f$ is a cubic on each edge of the domain tetrahedron. Without loss of generality(for details, see [1]), we suppose points P_{ip}^{α} and P_{iq}^{α} are not equal for $p \neq q$ and vectors D_{ip}^{α} are perpendicular to E_i^{α}, and their arrangement is symmetrical about the edge midpoint.

3. Cardinal Functions

Denote

$$\begin{cases} \omega_i(V) = b_i^2(3 - 2b_i), & i = 1,2,3,4, \\ \omega_{ij}(V) = b_i^2 b_j, & i,j = 1,2,3,4, \quad i \neq j, \\ \omega_{ijk}(V) = b_i b_j b_k, & (i,j,k) \in \{(1,2,3),(2,3,4),(3,4,1),(4,1,2)\}, \end{cases} \tag{3.1}$$

then we have

Theorem 1. *The first sixteen functions of* (3.1), ω_i, ω_{ij}, *are cardinal with respect to the first sixteen linear functionals of* , $\mathcal{F}_i$, $\mathcal{T}_{ij}$, *respectively, where* $i,j = 1,2,3,4$, $i \neq j$. *This is,*

$$\begin{cases} \mathcal{F}_j \omega_i = \omega_i(V_j) = \delta_{ij}, \\ \mathcal{T}_{jk} \omega_i = \dfrac{\partial \omega_i}{\partial (V_k - V_j)}\Big|_{V_i} = 0, \quad k \neq j, \end{cases}$$

and

$$\begin{cases} \mathcal{F}_k \omega_{ij} = \omega_{ij}(V_k) = 0, \\ \mathcal{T}_{kl} \omega_{ij} = \dfrac{\partial \omega_{ij}}{\partial (V_l - V_k)}\Big|_{V_k} = \delta_{ik}\delta_{jl}, \quad l \neq k. \end{cases}$$

Corollary 1. *The twenty functions of* (3.1) *form a basis for the trivariate cubic polynomials.*

Corollary 2. *The interpolant* $\mathcal{C}$ *formed by the linear combination of the first sixteen linear functionals of* (2.7),

$$\begin{aligned} \mathcal{C} &= \sum_{i=1}^{4} \omega_i \mathcal{F}_i + \sum_{j \neq i} \omega_{ij} \mathcal{T}_{ij} \\ &= \omega_1 \mathcal{F}_1 + \omega_2 \mathcal{F}_2 + \omega_3 \mathcal{F}_3 + \omega_4 \mathcal{F}_4 + \omega_{12} \mathcal{T}_{12} + \omega_{13} \mathcal{T}_{13} + \omega_{14} \mathcal{T}_{14} + \omega_{21} \mathcal{T}_{21} \\ &\quad + \omega_{23} \mathcal{T}_{23} + \omega_{24} \mathcal{T}_{24} + \omega_{31} \mathcal{T}_{31} + \omega_{32} \mathcal{T}_{32} + \omega_{34} \mathcal{T}_{34} + \omega_{41} \mathcal{T}_{41} + \omega_{42} \mathcal{T}_{42} + \omega_{43} \mathcal{T}_{43} \end{aligned}$$

interpolates these linear functionals.

The proofs of the above three results are similar to those of [1] and [4] and thus are omitted.

4. Characterization of the Interpolants

The first characterization of the set of discrete tetrahedral interpolants $\mathcal{P}$ satisfying (2.1) through (2.6) is

Theorem 2. *If $\mathcal{P}$ is an interpolant which satisfies* (2.1) *through* (2.6)*, then $\mathcal{P}$ has the following form*

$$\begin{aligned}\mathcal{P} &= \mathcal{C} + b_2b_3b_4\sum_{i\neq 1}\sum_{p=1}^{s}\bar{x}_{ip}^{1}\overline{\mathcal{N}}_{ip}^{1} + b_3b_4b_1\sum_{i\neq 2}\sum_{p=1}^{s}\bar{x}_{ip}^{2}\overline{\mathcal{N}}_{ip}^{2} \\ &\quad + b_4b_1b_2\sum_{i\neq 3}\sum_{p=1}^{s}\bar{x}_{ip}^{3}\overline{\mathcal{N}}_{ip}^{3} + b_1b_2b_3\sum_{i\neq 4}\sum_{p=1}^{s}\bar{x}_{ip}^{4}\overline{\mathcal{N}}_{ip}^{4} \\ &= \mathcal{C} + \sum_{i,j,k,l} b_ib_jb_k\sum_{q\neq l}\sum_{p=1}^{s}\bar{x}_{ip}^{l}\overline{\mathcal{N}}_{ip}^{l},\end{aligned}$$

where $\overline{\mathcal{N}}_{ip}^{l} := \mathcal{N}_{ip}^{l}(\mathcal{I}-\mathcal{C})$, and $\bar{x}_{ip}^{l}$ is some unspecified trivariate rational function. ($\mathcal{I}$ is the identical operator)

Proof. $\mathcal{P}$ is of cubic precision, and $\mathcal{C}$ is a cubic interpolant, thus $\mathcal{P}\mathcal{C} = \mathcal{C}$ and

$$\mathcal{P} = \mathcal{P} + \mathcal{C} - \mathcal{P}\mathcal{C} = \mathcal{C} + \mathcal{P}(\mathcal{I}-\mathcal{C}).$$

Since $\mathcal{P}$ is the linear combination of $\mathcal{F}_i, \mathcal{T}_{ij}$ and $\mathcal{N}_{ip}^{\alpha}$ with rational polynomials as coefficients, the form of $\mathcal{P}(\mathcal{I}-\mathcal{C})$ is simpler and is the combination of $\mathcal{N}_{ip}^{\alpha}(\mathcal{I}-\mathcal{C})$. This is due to the fact that $\mathcal{F}_i(\mathcal{I}-\mathcal{C}) = \mathcal{F}_i - \mathcal{F}_i\mathcal{C} = 0$ and $\mathcal{T}_{ij}(\mathcal{I}-\mathcal{C}) = \mathcal{T}_{ij} - \mathcal{T}_{ij}\mathcal{C} = 0$. Thus, $\mathcal{P}(\mathcal{I}-\mathcal{C})$ only contains linear functionals $\overline{\mathcal{N}}_{ip}^{\alpha} \equiv \mathcal{N}_{ip}^{\alpha}(\mathcal{I}-\mathcal{C})$.

Now restrict our consideration to each face of the tetrahedron, for example, $b_4 = 0$. First, $\mathcal{P}\Big|_{\substack{b_4=0\\ b_1=0}}$ is a univariate cubic polynomial from cubic precision. Since a univariate cubic polynomial is uniquely determined by the position values and first derivatives at the two ends, we have $\mathcal{P} \equiv \mathcal{C}$ on each edge, i.e., $\mathcal{P}(\mathcal{I}-\mathcal{C}) \equiv 0$ on each edge. Note that $\mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0}$ is a rational polynomial, so $\mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0}$ contains b_1, b_2 and b_3 as factors. Hence, we can write

$$\mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0} = b_1b_2b_3\sum_{\alpha=1}^{4}\sum_{i\neq\alpha}\sum_{p=1}^{s}\bar{x}_{ip}^{\alpha}\overline{\mathcal{N}}_{ip}^{\alpha}$$

From the local property, $\mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0}$ should be independent of the $\overline{\mathcal{N}}_{ip}^{\alpha}$, for $\alpha \neq 4$. Thus,

$$\mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0} = b_1b_2b_3\sum_{i\neq 4}\sum_{p=1}^{s}\bar{x}_{ip}^{4}\overline{\mathcal{N}}_{ip}^{4}$$

$\mathcal{P}(\mathcal{I}-\mathcal{C})$ is rational polynomial, so $\mathcal{P}(\mathcal{I}-\mathcal{C})$ can be rewritten as

$$\mathcal{P}(\mathcal{I}-\mathcal{C}) = \mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_4=0} + b_4^{\ell}\mathcal{Q}_4$$

with $\ell > 0$ and either $\mathcal{Q}_4 \equiv 0$ or $\mathcal{Q}_4|_{b_4=0} \neq 0$. Again from the symmetry, it follows that

$$\begin{aligned}\mathcal{P}(\mathcal{I}-\mathcal{C}) &= \sum_{i=1}^{4} \mathcal{P}(\mathcal{I}-\mathcal{C})|_{b_i=0} + (b_1b_2b_3b_4)^\ell \mathcal{Q} \\ &= \sum_{(i,j,k,l)\in\Lambda} b_ib_jb_k \sum_{p=1}^{s} \bar{x}_{ip}^l \overline{\mathcal{N}}_{ip}^l + (b_1b_2b_3b_4)^\ell \mathcal{Q}\end{aligned}$$

with $\ell > 0$ and either $\mathcal{Q} \equiv 0$ or $\mathcal{Q}|_{b_i=0} \neq 0$, $i = 1, 2, 3, 4$. From $\mathcal{P}(b_1b_2b_3) = b_1b_2b_3$, $\mathcal{C}(b_1b_2b_3) = 0$ and $\overline{\mathcal{N}}_{ip}^\alpha(b_1b_2b_3) = 0$, for $\alpha \neq 4$, we have

$$b_1b_2b_3 = b_1b_2b_3 \sum_{p=1}^{s} \bar{x}_{ip}^4 \overline{\mathcal{N}}_{ip}^4(b_1b_2b_3) + (b_1b_2b_3b_4)^\ell \mathcal{Q}(b_1b_2b_3)$$

i.e.,

$$1 = \sum_{p=1}^{s} \bar{x}_{ip}^4 \overline{\mathcal{N}}_{ip}^4(b_1b_2b_3) + (b_1b_2b_3)^{\ell-1} b_4^\ell \mathcal{Q}(b_1b_2b_3)$$

Since $\sum_{p=1}^{s} \bar{x}_{ip}^4 \overline{\mathcal{N}}_{ip}^4(b_1b_2b_3)$ and $\mathcal{Q}(b_1b_2b_3)$ do not contain b_4 as a factor, thus $\ell = 0$. But this is impossible, therefore, $\mathcal{Q}(b_1b_2b_3) = 0$. Similarly, we have $\mathcal{Q}(b_2b_3b_4) = 0$, $\mathcal{Q}(b_3b_4b_1) = 0$ and $\mathcal{Q}(b_4b_1b_2) = 0$. Besides, it is not difficult to show $\mathcal{QC} = 0$. Combining these equalities, we get $\mathcal{Q}f = 0$ for any cubic f. This means $\mathcal{Q} \equiv 0$ for $\mathcal{Q}$ is the combination of $\mathcal{F}_i$, $\mathcal{T}_{ij}$ and $\mathcal{N}_{ip}^\alpha$. Thus the proof is complete.

The final characterization of the interpolant is as follows:

Theorem 3. *If $\mathcal{P}$ is an interpolant which satisfies* (2.1) *through* (2.6)*, then $\mathcal{P}$ has the following form*

$$\mathcal{P} = \mathcal{C} + \sum_{(i,j,k,l)\in\Lambda} b_ib_jb_k \frac{\sum_{p=1}^{s} (b_ib_jf_{kp}^l\overline{\mathcal{N}}_{kp}^l + b_jb_kf_{ip}^l\overline{\mathcal{N}}_{ip}^l + b_kb_if_{jp}^l\overline{\mathcal{N}}_{jp}^l)}{\sum_{p=1}^{s} (b_ib_jf_{kp}^lc_{kp}^l + b_jb_kf_{ip}^lc_{ip}^l + b_kb_if_{jp}^lc_{jp}^l)},$$

where $c_{i'p}^l = \overline{\mathcal{N}}_{i'p}^l(b_ib_jb_k)$. The weight functions $\{f_{ip}^l\}$ satisfy the symmetry of tetrahedra and zero conditions: $f_{i'p}^l = f_{i'p}^l(b_i, b_j, b_k)$ are zeros on all $P_{i'q}^l$ other than $P_{i'p}^l$, $(i, j, k, l) \in \Lambda$, $i' \in \{i, j, k\}$.

Proof. From Theorem 2, we have

$$\begin{aligned}\mathcal{P} = \mathcal{C} &+ b_1b_2b_3 \sum_{i\neq 4}\sum_{p=1}^{s} \bar{x}_{ip}^4\overline{\mathcal{N}}_{ip}^4 + b_2b_3b_4 \sum_{i\neq 1}\sum_{p=1}^{s} \bar{x}_{ip}^1\overline{\mathcal{N}}_{ip}^1 \\ &+ b_3b_4b_1 \sum_{i\neq 2}\sum_{p=1}^{s} \bar{x}_{ip}^2\overline{\mathcal{N}}_{ip}^2 + b_4b_1b_2 \sum_{i\neq 3}\sum_{p=1}^{s} \bar{x}_{ip}^3\overline{\mathcal{N}}_{ip}^3.\end{aligned}$$

Applying $\mathcal{P}$ to function $b_1b_2b_3$, we can get, from $\mathcal{P}(b_1b_2b_3) = b_1b_2b_3$, $\mathcal{C}(b_1b_2b_3) = 0$ and $\overline{\mathcal{N}}_{ip}^\alpha(b_1b_2b_3) = 0$,

$$1 = \sum_{i=1}^{3}\sum_{p=1}^{s} \bar{x}_{ip}^4\overline{\mathcal{N}}_{ip}^4(b_1b_2b_3)$$

Let $\bar{x}_{ip}^4 = x_{ip}^4/g$, for all $\bar{x}_{ip}^4$, where x_{ip}^4 and g are polynomials about b_1, b_2, b_3. Then

$$g = \sum_{i=1}^{3}\sum_{p=1}^{s} x_{ip}^4 c_{ip}^4,$$

here we denote $c_{ip}^4 = \overline{\mathcal{N}}_{ip}^4(b_1b_2b_3)$. Now let us consider the cross boundary derivative $\dfrac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}$:

$$\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} = \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} + \frac{\partial}{\partial \boldsymbol{n}_4}\left(b_1b_2b_3 \frac{\sum_{i=1}^{3}\sum_{p=1}^{s} x_{ip}^4 \overline{\mathcal{N}}_{ip}^4}{\sum_{i=1}^{3}\sum_{p=1}^{s} x_{ip}^4 c_{ip}^4}\right)\Bigg|_{b_4=0}$$
$$+ \left(b_2b_3 \frac{\sum_{i\neq 1}\sum_{p=1}^{s} x_{ip}^1 \overline{\mathcal{N}}_{ip}^1}{\sum_{i\neq 1}\sum_{p=1}^{s} x_{ip}^1 c_{ip}^1} + b_3b_1 \frac{\sum_{i\neq 2}\sum_{p=1}^{s} x_{ip}^2 \overline{\mathcal{N}}_{ip}^2}{\sum_{i\neq 2}\sum_{p=1}^{s} x_{ip}^2 c_{ip}^2} + b_1b_2 \frac{\sum_{i\neq 3}\sum_{p=1}^{s} x_{ip}^3 \overline{\mathcal{N}}_{ip}^3}{\sum_{i\neq 3}\sum_{p=1}^{s} x_{ip}^3 c_{ip}^3}\right)\Bigg|_{b_4=0}$$

In order to satisfy the C^1 and local properties, x_{ip}^α, for $\alpha \neq 4, i \neq \alpha, 4$, should contain b_4 as a factor. Thus,

$$\begin{aligned}\mathcal{P} = \mathcal{C} &+ b_1b_2b_3 \frac{\sum_{p=1}^{s}(b_2b_3 f_{1p}^4 \overline{\mathcal{N}}_{1p}^4 + b_3b_1 f_{2p}^4 \overline{\mathcal{N}}_{2p}^4 + b_1b_2 f_{3p}^4 \overline{\mathcal{N}}_{3p}^4)}{\sum_{p=1}^{s}(b_2b_3 f_{1p}^4 c_{1p}^4 + b_3b_1 f_{2p}^4 c_{2p}^4 + b_1b_2 f_{3p}^4 c_{3p}^4)}\\
&+ b_2b_3b_4 \frac{\sum_{p=1}^{s}(b_3b_4 f_{2p}^1 \overline{\mathcal{N}}_{2p}^1 + b_4b_2 f_{3p}^1 \overline{\mathcal{N}}_{3p}^1 + b_2b_3 f_{4p}^1 \overline{\mathcal{N}}_{4p}^1)}{\sum_{p=1}^{s}(b_3b_4 f_{2p}^1 c_{2p}^1 + b_4b_2 f_{3p}^1 c_{3p}^1 + b_2b_3 f_{4p}^1 c_{4p}^1)}\\
&+ b_3b_4b_1 \frac{\sum_{p=1}^{s}(b_4b_1 f_{3p}^2 \overline{\mathcal{N}}_{3p}^2 + b_1b_3 f_{4p}^2 \overline{\mathcal{N}}_{4p}^2 + b_3b_4 f_{1p}^2 \overline{\mathcal{N}}_{1p}^2)}{\sum_{p=1}^{s}(b_4b_1 f_{3p}^2 c_{3p}^2 + b_1b_3 f_{4p}^2 c_{4p}^2 + b_3b_4 f_{1p}^2 c_{1p}^2)}\\
&+ b_4b_1b_2 \frac{\sum_{p=1}^{s}(b_1b_2 f_{4p}^3 \overline{\mathcal{N}}_{4p}^3 + b_2b_4 f_{1p}^3 \overline{\mathcal{N}}_{1p}^3 + b_4b_1 f_{2p}^3 \overline{\mathcal{N}}_{1p}^3}{\sum_{p=1}^{s}(b_1b_2 f_{4p}^3 c_{4p}^3 + b_2b_4 f_{1p}^3 c_{1p}^3 + b_4b_1 f_{2p}^3 c_{2p}^3)}.\end{aligned}$$

Without loss of generality, let us calculate the cross boundary derivative of $\mathcal{P}(\mathcal{I}-\mathcal{C})$ on edge E_1^4:

$$\frac{\partial \mathcal{P}(\mathcal{I}-\mathcal{C})}{\partial \boldsymbol{n}_4}\Bigg|_{\substack{b_4=0\\ b_1=0}} = b_2b_3 \frac{\sum_{p=1}^{s} f_{1p}^4 \overline{\mathcal{N}}_{1p}^4}{\sum_{p=1}^{s} f_{1p}^4 c_{1p}^4}\frac{\partial b_1}{\partial \boldsymbol{n}_4} + b_2b_3 \frac{\sum_{p=1}^{s} f_{4p}^1 \overline{\mathcal{N}}_{4p}^1}{\sum_{p=1}^{s} f_{4p}^1 c_{4p}^1}.$$

From the interpolation properties of $\mathcal{P}$, $\mathcal{N}_{1p}^4$ and $\mathcal{N}_{1q}^4$, $\mathcal{N}_{4p}^1$ and $\mathcal{N}_{4q}^1$, $p \neq q$, are independent of each other. These will happen only if f_{1p}^4 is zero on $\{P_{1q}^4\}$ other than P_{1p}^4 and f_{4p}^1 is zero on $\{P_{4q}^1\}$ other than $P_{4p}^1 = P_{1p}^4$. Finally, the symmetry of f_{1p}^4 and f_{4p}^1 are derived from (2.6).

5. Application

In this section, we use Theorem 3 to give two discrete C^1 interpolants over tetrahedra, one with cubic precision and the other quadratic precision.

(1) A discrete C^1 tetrahedral interpolant with cubic precision

We are interested in creating a discrete C^1 tetrahedral interpolant which interpolates the positions and gradients at the vertices of a general tetrahedron, in addition to the perpendicular cross boundary derivatives at midpoints of its edges(see Fig. 2). For simplicity, we omit the second subscripts of $\mathcal{N}_{ip}^{\alpha}$, $\overline{\mathcal{N}}_{ip}^{\alpha}$ and c_{ip}^{α} since only one cross boundary derivative is required at each edge. Thus we have

$$\mathcal{N}_i^{\alpha} = \frac{\partial}{\partial \boldsymbol{n}_i^{\alpha}}\Big|_{E_i^{\alpha}(1/2)},$$

and $c_i^{\alpha} = \dfrac{1}{4}$, for all possible i and α. The choice of weight functions f_i^{α} should be $f_i^{\alpha} \equiv 1$ for no zero conditions are needed in this case. Hence, the C^1 tetrahedral interpolant with cubic precision can be expressed as

$$\begin{aligned}\mathcal{P} = \mathcal{C} &+ 4b_1b_2b_3\frac{b_2b_3\overline{\mathcal{N}}_1^4 + b_3b_1\overline{\mathcal{N}}_2^4 + b_1b_2\overline{\mathcal{N}}_3^4}{b_2b_3 + b_3b_1 + b_1b_2} + 4b_2b_3b_4\frac{b_3b_4\overline{\mathcal{N}}_2^1 + b_4b_2\overline{\mathcal{N}}_3^1 + b_2b_3\overline{\mathcal{N}}_4^1}{b_3b_4 + b_4b_2 + b_2b_3} \\ &+ 4b_3b_4b_1\frac{b_4b_1\overline{\mathcal{N}}_3^2 + b_1b_3\overline{\mathcal{N}}_4^2 + b_3b_4\overline{\mathcal{N}}_1^2}{b_4b_1 + b_1b_3 + b_3b_4} + 4b_4b_1b_2\frac{b_1b_2\overline{\mathcal{N}}_4^3 + b_2b_4\overline{\mathcal{N}}_1^3 + b_4b_1\overline{\mathcal{N}}_2^3}{b_1b_2 + b_2b_4 + b_4b_1}.\end{aligned} \tag{5.1}$$

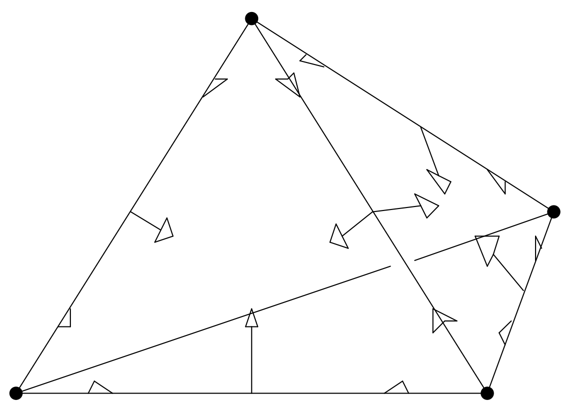

Fig. 2 The data required for the interpolant with cubic precision: filled circles denote position values, arrows denote given directional derivatives.

Now we show (5.1) is a C^1 interpolant. To this end, we calculate the cross boundary derivative of $\mathcal{P}$ on the face F_4. It is not difficult to obtain

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} =& \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} + 4b_2b_3\overline{\mathcal{N}}_4^1 + 4b_3b_1\overline{\mathcal{N}}_4^2 + 4b_1b_2\overline{\mathcal{N}}_4^3 \\ &+ \frac{\partial}{\partial \boldsymbol{n}_4}\Big(4b_1b_2b_3\frac{b_2b_3\overline{\mathcal{N}}_1^4 + b_3b_1\overline{\mathcal{N}}_2^4 + b_1b_2\overline{\mathcal{N}}_3^4}{b_2b_3 + b_3b_1 + b_1b_2}\Big)\end{aligned}$$

and

$$\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} = \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{\substack{b_4=0\\ b_1=0}} + \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{\substack{b_4=0\\ b_2=0}} + \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\ Big|_{\substack{b_4=0\\ b_3=0}} - b_1^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_1} - b_2^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_2} - b_3^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_3}$$

$$- 6b_1(1-b_1)\frac{\partial b_1}{\partial \boldsymbol{n}_4}\mathcal{F}_1 - 6b_2(1-b_2)\frac{\partial b_2}{\partial \boldsymbol{n}_4}\mathcal{F}_2 - 6b_3(1-b_3)\frac{\partial b_3}{\partial \boldsymbol{n}_4}\mathcal{F}_3.$$

Combining these two equalities, we have

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{b_4=0} =&\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{\substack{b_4=0\\ b_1=0}} + \frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{\substack{b_4=0\\ b_2=0}} + \frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{\substack{b_4=0\\ b_3=0}} - b_1^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_1} - b_2^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_2} - b_3^2\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{V_3}\\ &- 6b_1(1-b_1)\frac{\partial b_1}{\partial \boldsymbol{n}_4}\mathcal{F}_1 - 6b_2(1-b_2)\frac{\partial b_2}{\partial \boldsymbol{n}_4}\mathcal{F}_2 - 6b_3(1-b_3)\frac{\partial b_3}{\partial \boldsymbol{n}_4}\mathcal{F}_3\\ &- 4b_2b_3\frac{\partial b_1}{\partial \boldsymbol{n}_4}\overline{\mathcal{N}}_1^4 - 4b_3b_1\frac{\partial b_2}{\partial \boldsymbol{n}_4}\overline{\mathcal{N}}_2^4 - 4b_1b_2\frac{\partial b_3}{\partial \boldsymbol{n}_4}\overline{\mathcal{N}}_3^4\\ &+ \frac{\partial}{\partial \boldsymbol{n}_4}\Big(4b_1b_2b_3\frac{b_2b_3\overline{\mathcal{N}}_1^4 + b_3b_1\overline{\mathcal{N}}_2^4 + b_1b_2\overline{\mathcal{N}}_3^4}{b_2b_3+b_3b_1+b_1b_2}\Big).\end{aligned}$$

Thus we can conclude from the above equality that (5.1) is a C^1 interpolant.

This scheme requires a user to supply not only the values of positions and gradients but also cross boundary derivatives on each edge of the tetrahedron. However, sometimes it is difficult to supply cross boundary derivatives. So we will construct a condensed version of the above interpolant which can relieve the user's need to supply any cross boundary derivatives but at the sacrifice of precision to only polynomials of degree up to two.

(2) A discrete C^1 tetrahedral interpolant with quadratic precision

For this interpolant, the supplied data are only values of positions and gradients at the vertices of the tetrahedron. We need to construct the approximations of the cross boundary derivatives at edge midpoints. In fact, we only need to give the approximations of $\overline{\mathcal{N}}_i^\alpha$. Without loss of generality, we construct $\overline{\mathcal{N}}_1^4$ and $\overline{\mathcal{N}}_4^1$.

Notice that the cross boundary derivatives along each edge are univariate quadratic polynomials, we can determine $\overline{\mathcal{N}}_1^4$ and $\overline{\mathcal{N}}_4^1$ by enforcing $\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_i}\Big|_{E_i^j(t)}$ to be linear. This can be realized by setting

$$\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(1/2)} = \frac{1}{2}\Big(\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(0)} + \frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(1)}\Big), \tag{5.2}$$

$$\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{E_1^4(1/2)} = \frac{1}{2}\Big(\frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{E_1^4(0)} + \frac{\partial \mathcal{P}}{\partial \boldsymbol{n}_4}\Big|_{E_1^4(1)}\Big). \tag{5.3}$$

Thus, from (5.2) we have

$$\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(1/2)} + \overline{\mathcal{N}}_1^4 + \frac{\partial b_4}{\partial \boldsymbol{n}_1}\overline{\mathcal{N}}_4^1 = \frac{1}{2}\Big(\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(0)} + \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(1)}\Big),$$

i.e.,

$$\overline{\mathcal{N}}_1^4 + \frac{\partial b_4}{\partial \boldsymbol{n}_1}\overline{\mathcal{N}}_4^1 = \frac{1}{2}\Big(\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{V_2} + \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{V_3}\Big) - \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{(V_2+V_3)/2}. \tag{5.4}$$

In the same way, from (5.3) we have

$$\frac{\partial b_1}{\partial \boldsymbol{n}_4}\overline{\mathcal{N}}_4^1 + \overline{\mathcal{N}}_1^4 = \frac{1}{2}\Big(\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{V_2} + \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{V_3}\Big) - \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{(V_2+V_3)/2}. \tag{5.5}$$

Since the cross boundary derivatives at any point $E_4^1(t)$ is computed to be

$$\begin{aligned}\frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_1}\Big|_{E_4^1(t)} =&6t(1-t)\frac{\partial b_2}{\partial \boldsymbol{n}_1}F_2+6t(1-t)\frac{\partial b_3}{\partial \boldsymbol{n}_1}F_3+(1-t)^2T_{21}\\&+\Big(2t(1-t)\frac{\partial b_2}{\partial \boldsymbol{n}_1}+(1-t)^2\frac{\partial b_3}{\partial \boldsymbol{n}_1}\Big)T_{23}+(1-t)^2\frac{\partial b_4}{\partial \boldsymbol{n}_1}T_{24}\\&+t^2T_{31}+\Big(t^2\frac{\partial b_2}{\partial \boldsymbol{n}_1}+2t(1-t)\frac{\partial b_3}{\partial \boldsymbol{n}_1}\Big)T_{32}+t^2\frac{\partial b_4}{\partial \boldsymbol{n}_1}T_{34},\\ \frac{\partial \mathcal{C}}{\partial \boldsymbol{n}_4}\Big|_{E_4^1(t)} =&6t(1-t)\frac{\partial b_2}{\partial \boldsymbol{n}_4}F_2+6t(1-t)\frac{\partial b_3}{\partial \boldsymbol{n}_4}F_3+\frac{\partial b_1}{\partial \boldsymbol{n}_4}(1-t)^2T_{21}\\&+\Big(2t(1-t)\frac{\partial b_2}{\partial \boldsymbol{n}_4}+(1-t)^2\frac{\partial b_3}{\partial \boldsymbol{n}_4}\Big)T_{23}+(1-t)^2T_{24}\\&+\frac{\partial b_3}{\partial \boldsymbol{n}_4}t^2T_{31}+\Big(t^2\frac{\partial b_2}{\partial \boldsymbol{n}_4}+2t(1-t)\frac{\partial b_3}{\partial \boldsymbol{n}_4}\Big)T_{32}+t^2T_{34},\end{aligned}$$

it follows from (5.4) and (5.5) that

$$\begin{cases}\overline{\mathcal{N}}_1^4=\Big(c_1-\dfrac{\partial b_4}{\partial \boldsymbol{n}_1}c_2\Big)\cdot\Big(1-\dfrac{\partial b_4}{\partial \boldsymbol{n}_1}\dfrac{\partial b_1}{\partial \boldsymbol{n}_4}\Big)^{-1},\\ \overline{\mathcal{N}}_4^1=\Big(c_2-\dfrac{\partial b_1}{\partial \boldsymbol{n}_4}c_1\Big)\cdot\Big(1-\dfrac{\partial b_4}{\partial \boldsymbol{n}_1}\dfrac{\partial b_1}{\partial \boldsymbol{n}_4}\Big)^{-1},\end{cases} \tag{5.6}$$

where

$$\begin{aligned}c_1=&-\frac{3}{2}\frac{\partial b_2}{\partial \boldsymbol{n}_1}F_2+\frac{\partial b_3}{\partial \boldsymbol{n}_1}F_3+\frac{1}{4}T_{21}+\frac{1}{4}\Big(-2\frac{\partial b_2}{\partial \boldsymbol{n}_1}+\frac{\partial b_3}{\partial \boldsymbol{n}_1}\Big)T_{23}\\&+\frac{1}{4}\frac{\partial b_4}{\partial \boldsymbol{n}_1}T_{24}+\frac{1}{4}T_{31}+\frac{1}{4}\Big(\frac{\partial b_2}{\partial \boldsymbol{n}_1}-2\frac{\partial b_3}{\partial \boldsymbol{n}_1}\Big)T_{32}+\frac{1}{4}\frac{\partial b_4}{\partial \boldsymbol{n}_1}T_{34},\\ c_2=&\frac{3}{2}\frac{\partial b_2}{\partial \boldsymbol{n}_4}F_2+\frac{\partial b_3}{\partial \boldsymbol{n}_4}F_3+\frac{1}{4}\frac{\partial b_1}{\partial \boldsymbol{n}_4}T_{21}+\frac{1}{4}\Big(-2\frac{\partial b_2}{\partial \boldsymbol{n}_4}+\frac{\partial b_3}{\partial \boldsymbol{n}_4}\Big)T_{23}\\&+\frac{1}{4}T_{24}+\frac{1}{4}\frac{\partial b_1}{\partial \boldsymbol{n}_4}T_{31}+\frac{1}{4}\Big(\frac{\partial b_2}{\partial \boldsymbol{n}_4}-2\frac{\partial b_3}{\partial \boldsymbol{n}_4}\Big)T_{32}+\frac{1}{4}T_{34}.\end{aligned}$$

Thus, combining (5.1) and (5.6), we get a condensed scheme with quadratic precision.

6. Conclusions

In the paper, we have characterized a class of discrete C^1 tetrahedral interpolants. Furthermore, two C^1 discrete tetrahedral interpolants, one with cubic precision and the other quadratic precision, are created.

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