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SERIES REPRESENTATION OF DAUBECHIES' WAVELETS*

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Abstract

This paper gives a kind of series representation of the scaling functions ϕ_{N} and the associated wavelets ψ_{N} constructed by Daubechies. Based on Poission summation formula, the functions $\phi_{N}(x+N-1),\ \phi_{N}(x+N),\cdots,\phi_{N}(x+2N-2)(0\le x\le 1)$ are linearly represented by $\phi_N(x),\phi_N(x+1),\cdots,\phi_N(x+2N-2)$ and some polynomials of order less than N, and $\mathbf{\Phi}_0(x):=(\phi_{N}(x),\phi_{N}(x+1),\cdots,\phi_{N}(x+N-2))^t$ is translated into a solution of a nonhomogeneous vector–valued functional equation

$$\mathbf{f}(x)=\mathbf{A}_d\mathbf{f}(2x-d)+\mathbf{P}_d(x),\ x\in[\frac{d}{2},\frac{d+1}{2}],\ d=0,1,$$

where $\mathbf{A}_0,\mathbf{A}_1$ are $(N-1)\times(N-1)$–dimensional matrices, the components of $\mathbf{P}_0(x),\mathbf{P}_1(x)$ are polynomials of order less than N. By iteration, $\mathbf{\Phi}_0(x)$ is eventualy represented as an $(N-1)$–dimensional vector series $\sum_{k=0}^{\infty}\mathbf{u}_k(x)$ with vector norm $\|\mathbf{u}_k(x)\|\le C\beta^k$, where $\beta=\beta_{N}<1$ and $\beta_{N}\searrow 0$ as $N\to\infty$.

1. Introduction.

In this paper we study the representation of Daubechies' wavelets. Daubechies[1] constructed a family of compactly supported regular scaling functions $\phi_{N}(x)$ and the associated regular wavelets $\psi_{N}(x)(N\ge 2)$:

$$\psi_{N}(x):=\sum_{n=-1}^{2N-2}(-1)^nC_{N}(n+1)\phi_{N}(2x+n),\qquad x\in\mathbf{R},\tag{1.1}$$

$$\phi_{N}(x):=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\hat{\phi}_{N}(\xi)e^{-\mathrm{i}\xi x}d\xi,\qquad x\in\mathbf{R},\ \mathrm{i}=\sqrt{-1},$$

where $\hat{\phi}_{N}\in L^1(\mathbf{R})$ defined by

$$\hat{\phi}_{N}(\xi):=\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty}m_{N}(2^{-j}\xi),\ \hat{\phi}_{N}(0)=\frac{1}{\sqrt{2\pi}},$$

$$m_{N}(\xi):=\frac{1}{2}\sum_{n=0}^{2N-1}C_{N}(n)e^{\mathrm{i}n\xi}=\Big[\frac{1}{2}(1+e^{\mathrm{i}\xi})\Big]^N\sum_{k=0}^{N-1}q_{N}(k)e^{\mathrm{i}k\xi},\tag{1.2}$$

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the polynomial $\sum_{k=0}^{N-1} q_{_N}(k)z^k$ satisfies

$$\left|\sum_{k=0}^{N-1} q_{_N}(k)e^{\mathrm{i}k\xi}\right|^2 = \sum_{k=0}^{N-1}\binom{k+N-1}{k}\sin^{2k}(\frac{\xi}{2}), \quad \xi\in\mathbf{R}, \tag{1.3}$$

with $\sum_{k=0}^{N-1} q_{_N}(k)=1, q_{_N}(k)\in\mathbf{R}, k=0,1,\cdots,N-1$. It is known that[1] for each $N\geq 2$, $\operatorname{supp}\phi_{_N}=[0,2N-1]$, $\operatorname{supp}\psi_{_N}=[-(N-1),N]$ and the wavelet $\psi_{_N}$ generates by its dilations and translations an orthornormal basis $\{\sqrt{2^j}\psi_{_N}(2^jx-k)\}_{j,k\in\mathbf{Z}}$ of $L^2(\mathbf{R})$. The functions $\phi_{_N}$ and $\psi_{_N}$ have been proved to be very useful in numerical analysis[2,3]. On the aspect of representation, however, comparing to some nonorthogonal wavelets, the wavelets ψ_N and (any) other orthogonal regular wavelets seem to be hardly written in very explicit forms. This is not strange because for any wavelet ψ, its regularity, orthogonality (i.e. orthogonality of $\{\sqrt{2^j}\psi(2^jx-k)\}_{j,k\in\mathbf{Z}}$ in $L^2(\mathbf{R})$), symmetry, support compactness and representation (in the sense of computing) can not be satisfied simultaneously. So far there are two methods for approximating or representing the scaling functions ϕ_N, both of them are based on the two–scale difference equation[1,4,5]

$$\phi_N(x)=\sum_{n=0}^{2N-1} C_N(n)\phi_N(2x-n),\ x\in\mathbf{R}, \tag{1.4}$$

and homogeneous iterative approximation. One method is the iterative approximation scheme $f_n=Vf_{n-1}$, where V is a linear operator

$$Vf(x):=\sum_{k=0}^{2N-1} C_{_N}(k)f(2x-k)$$

acting on a function space. The $\phi_{_N}$ is therefore a fixed point of V, $V\phi_{_N}=\phi_{_N}$, computed by $\lim_{n\to\infty}V^nf_0(x)=\phi_{_N}(x)$ with a suitable initial function f_0, e.g., interpolating spline. The convergence is uniform or pointwise depending on the choice of f_0[1,4]. Another method[5] is similar to that scheme but with vector (matrix) forms: Let $\mathbf{\Phi}(x)=(\phi_{_N}(x),\phi_{_N}(x+1),\cdots,\phi_{_N}(x+2N-2))^t, \mathbf{T}_0,\mathbf{T}_1\in\mathbf{R}^{(2N-1)\times(2N-1)}, (\mathbf{T}_d)_{ij}=C_{_N}(2i-j-1+d), d=0,1$ ($C_{_N}(n)=0\,\text{for}\,n<0\,\text{or}\,n>2N-1$). Then (1.4) is written $\mathbf{\Phi}(x)=\mathbf{T}_{d_1(x)}\mathbf{\Phi}(\tau(x))$, $x\in[0,1]$ since $\operatorname{supp}\phi_{_N}=[0,2N-1]$. Iteratively,

$$\mathbf{\Phi}(x)=\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\cdots\mathbf{T}_{d_n(x)}\mathbf{\Phi}(\tau^n(x)), x\in[0,1],$$

where the index $d_j(x)$ is the jth digit in the binary expansion for $x\in[0,1], \tau(x)$ is the shift operator: $\tau(x)=0.d_2(x)d_3(x)\cdots$, (see section 2). All the infinite products $\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\mathbf{T}_{d_3(x)}\cdots$ of the matrices $\mathbf{T}_0,\mathbf{T}_1$ are convergent in matrix norm and for a suitable initial function $\mathbf{v}_0(x)\in\mathbf{R}^{2N-1}$,

$$\mathbf{\Phi}(x)=\lim_{n\to\infty}\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\cdots\mathbf{T}_{d_n(x)}\mathbf{v}_0(\tau^n(x)),\ x\in[0,1]. \tag{1.5}$$

Both the schemes can achieve approximation degree as $O(2^{-\alpha n})(n\to\infty), \alpha>0$. In this paper we give a different method to represent (approximate) the scaling functions $\phi_{_N}$

and therefore to the wavelets ψ_N via (1.1). Dividing $\phi_N(x), \phi_N(x+1), \cdots, \phi_N(x+2N-2)$ into two parts, for instance, $\phi_N(x), \phi_N(x+1), \cdots, \phi_N(x+N-2)$, and $\phi_N(x+N-1), \phi_N(x+N), \cdots, \phi_N(x+2N-2) (0 \le x \le 1)$, we prove that the second part can be linearly determined by the first part and some polynomials of order $\le N-1$. Then we expand, through a nonhomogeneous iterative sheme (see section 3 (3.2)), the first part as a vector–valued series $\sum_{k=0}^{\infty} \mathbf{u}_k(x)$ in which each term $\mathbf{u}_k(x)$ is an $(N-1)$–dimensional vector with vector norm $\|\mathbf{u}_k(x)\| \le C\beta^k$, where $\beta = \beta_N < 1$ and $\beta_N \searrow 0$ as $N \to \infty$. As a result, we reduce the dimension $2N-1$ in (1.5) to $N-1$ in the series. The main tools we used are (1) decay estimates for derivatives of analytic functions, such decay estimates have many applications in dealing with convergence problems; (2) some results of [1], [4], [5]; (3) some further properties of the polynomial $\sum_{k=0}^{N-1} q_N(k) z^k$.

2. Notation and Lemmas

(1). We make an appointment throughout this paper.

In the binary expansion of $x \in [0,1]$,

$$x = 0.d_1 d_2 d_3 \cdots = \sum_{j=1}^{\infty} 2^{-j} d_j,\ d_j \in \{0,1\}, \tag{$*$}$$

we restrict that d_j *vanishes for some infinite* j *which depend on* $x \in [0,1[$, *but for* $x = 1$ *we always write* $1 = 0.111\cdots$.

This appointment insures the uniqueness of the expansion (*) and yields a family of two–valued functions $d_j(x)$ well defined on [0,1] by $d_j(x) = d_j$ according to (*). Let $\tau : [0,1] \to [0,1]$ be the shift operator

$$\tau(x) := \sum_{j=1}^{\infty} 2^{-j} d_{j+1}(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ 2x-1, & \frac{1}{2} \le x \le 1. \end{cases} \tag{2.1}$$

By the uniqueness of expansion (*), it is easy to check that the following relation between $d_j(x)$ and $\tau^k(x)$ hold :

$$d_1(x) = \chi_{[\frac{1}{2},1]}(x),\ d_{k+1}(x) = d_k(\tau(x)) = d_1(\tau^k(x)),\ x \in [0,1], \tag{2.2}$$

$$\tau^0(x) = x, \tau^k(x) = 2^k(x - 0.d_1(x)\cdots d_k(x)) = 0.d_{k+1}(x) d_{k+2}(x) \cdots. \tag{2.3}$$

(2). For $k \notin [0, N-1]$ and $n \notin [0, 2N-1]$ we define $q_N(k) = C_N(n) = 0$. Let $\mathbf{T}_0, \mathbf{T}_1$ and $\mathbf{B} \in \mathbf{R}^{(2N-1)\times(2N-1)}$ be $(2N-1)\times(2N-1)$–dimensional matrices defined in [5]:

$$(\mathbf{T}_d)_{i,j} = C_N(2i-j-1+d),\ 1 \le i,j \le 2N-1,\ d = 0,1, \tag{2.4}$$

$$(\mathbf{B})_{i,j} = \begin{cases} (i-1)!\binom{j-1}{i-1}, & 1 \le i \le N, \\ (N-1)!\binom{j-i+N-1}{N-1}, & N+1 \le i \le 2N-1. \end{cases} \tag{2.5}$$

$\mathbf{B}$ is an up–triangular matrix, the inverse $\mathbf{B}^{-1}$ is given by

$$(\mathbf{B}^{-1})_{i,j} = \begin{cases} (-1)^{i+j}\binom{j-1}{i-1}[(j-1)!]^{-1}, & 1 \le j \le N, \\ (-1)^{i+j}\binom{N}{i-j+N}[(N-1)!]^{-1}, & N+1 \le j \le 2N-1. \end{cases} \tag{2.6}$$

Here we use the standard convention that binomial coefficient $\binom{n}{m}$ vanishes if $m < 0$ or $m > n$. In the proofs of our main results we will use a result from [5] that

$$\mathbf{B}\mathbf{T}_d\mathbf{B}^{-1} = \begin{bmatrix} \mathbf{D}_d & \mathbf{0} \\ \mathbf{C}_d & \mathbf{Q}_d \end{bmatrix} \in \begin{bmatrix} \mathbf{R}^{N\times N} & \mathbf{R}^{N\times(N-1)} \\ \mathbf{R}^{(N-1)\times N} & \mathbf{R}^{(N-1)\times(N-1)} \end{bmatrix}, \qquad d = 0, 1. \tag{2.7}$$

Define $\mathbf{U} \in \mathbf{R}^{N\times(2N-1)}$ and its submatrices $\mathbf{U}_0, \mathbf{U}_1$ by

$$\begin{cases} (\mathbf{U})_{i,j} = j^{i-1}, & 1 \le i \le N, 1 \le j \le 2N-1, \\ \Big[\mathbf{U}_0, \mathbf{U}_1\Big] = \mathbf{U}, & \mathbf{U}_0 \in \mathbf{R}^{N\times(N-1)}, \mathbf{U}_1 \in \mathbf{R}^{N\times N}. \end{cases} \tag{2.8}$$

Instead of the $(2N-1)\times(2N-1)$–dimensional matrices $\mathbf{T}_0, \mathbf{T}_1$, we consider in this paper the $(N-1)\times(N-1)$–dimensional matrices $\mathbf{A}_0, \mathbf{A}_1$ defined by

$$\mathbf{A}_d = \mathbf{T}_{11,d} - \mathbf{T}_{12,d}\mathbf{U}_1^{-1}\mathbf{U}_0, \qquad d = 0, 1, \tag{2.9}$$

where $\mathbf{T}_{11.d}, \mathbf{T}_{12,d}$ are submatrices of $\mathbf{T}_d$ given by

$$\mathbf{T}_d = \begin{bmatrix} \mathbf{T}_{11,d} & \mathbf{T}_{12,d} \\ \mathbf{T}_{21,d} & \mathbf{T}_{22,d} \end{bmatrix} \in \begin{bmatrix} \mathbf{R}^{(N-1)\times(N-1)} & \mathbf{R}^{(N-1)\times N} \\ \mathbf{R}^{N\times(N-1)} & \mathbf{R}^{N\times N} \end{bmatrix}, d = 0, 1. \tag{2.10}$$

Define the submatrices $\mathbf{B}_{11}, \mathbf{B}_{12}, \mathbf{B}_{22}$ of $\mathbf{B}$ by

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix} \in \begin{bmatrix} \mathbf{R}^{N\times(N-1)} & \mathbf{R}^{N\times N} \\ \mathbf{R}^{(N-1)\times(N-1)} & \mathbf{R}^{(N-1)\times N} \end{bmatrix}. \tag{2.11}$$

(3).

$$\mathbf{P}(x) := \mathbf{U}_1^{-1}\Big(1, \sum_{j=0}^{1} b_j\binom{1}{j}(1-x)^{1-j}, \cdots, \sum_{j=0}^{N-1} b_j\binom{N-1}{j}(1-x)^{N-1-j}\Big)^t \tag{2.12}$$

$$\mathbf{P}_0(x) := \mathbf{T}_{12,0}\mathbf{P}(2x), \qquad \mathbf{P}_1(x) := \mathbf{T}_{12,1}\mathbf{P}(2x-1), \tag{2.13}$$

where $b_j = \sqrt{2\pi}(-\mathrm{i})^j\hat{\phi}_N^{(j)}(0)$ are real numbers determined by the following recursion (because of $\hat{\phi}_{_N}(2\xi) = m_{_N}(\xi)\hat{\phi}_{_N}(\xi)$):

$$\begin{cases} b_0 = 1 \\ b_s = (2^s-1)^{-1}\sum_{j=0}^{s-1} b_j\binom{s}{j}(-\mathrm{i})^{s-j}m_{_N}^{(s-j)}(0), & s = 1, 2, 3, \cdots, \mathrm{i} = \sqrt{-1}. \end{cases} \tag{2.14}$$

(4). Denote by $\mathbf{A}(k;x)$ the right product of the matrices $\mathbf{A}_{d_1}(x), \mathbf{A}_{d_2}(x), \cdots, \mathbf{A}_{d_k}(x)$, i.e.

$$\mathbf{A}(k;x) := \mathbf{A}_{d_1(x)}\mathbf{A}_{d_2(x)}\cdots\mathbf{A}_{d_k(x)},\ \mathbf{A}(0;x) := \mathbf{I} \quad \text{(identitymatrix)}. \tag{2.15}$$

For any $\mathbf{x} = (x_1, x_2, \cdots, x_n)^t \in \mathbf{R}^n$, $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n\times n}$ we use in this paper the following vector norm $\|\mathbf{x}\|$ and the corresponding matrix norm $\|\mathbf{A}\|$:

$$\|\mathbf{x}\| := |x_1| + |x_2| + \cdots + |x_n|, \qquad \|\mathbf{A}\| := \sup\Big\{\|\mathbf{A}\mathbf{x}\| \,|\mathbf{x} \in \mathbf{R}^n,\ \|\mathbf{x}\| = 1\Big\},$$

and denote by $\mid \mathbf{x} \mid, \mid \mathbf{A} \mid$ the nonnegative vector and nonnegative matrix respectively, i.e.,

$$\mid \mathbf{x} \mid := (\mid x_1 \mid, \mid x_2 \mid, \cdots, \mid x_n \mid)^t, \qquad \mid \mathbf{A} \mid := (\mid a_{ij} \mid).$$

(5). Given a vector–valued function $\mathbf{f} : [a,b] \to \mathbf{R}^n,\ \mathbf{f}(x) = (f_1(x), f_2(x), \cdots, f_n(x))^t$, we define, as usual,

$$\mathbf{f}'(x) = \frac{d\mathbf{f}(x)}{dx} := \Big(f_1'(x), f_2'(x), \cdots, f_n'(x)\Big)^t,$$
$$\int_a^b \mathbf{f}(t)dt := \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \cdots, \int_a^b f_n(t)dt\right)^t,$$

provided every component f_j is differentiable at x or Lebesgue integrable on $[a,b]$ respectively. Let $1 \le p \le \infty$. Define $\mathbf{f} \in L^p([a,b], \mathbf{R}^n) \iff \forall j, f_j \in L^p[a,b]; \mathbf{f}$ is absolutely continuous on $[a,b] \iff \forall j,\ f_j$ is absolutely continuous on $[a,b]$. Obviously, $\mathbf{f}$ is absolutely continuous on $[a,b] \iff \exists \mathbf{g} \in L^1([a,b], \mathbf{R}^n)$ such that $\mathbf{f}(x) = \mathbf{f}(a) + \int_a^x \mathbf{g}(t)dt, x \in [a,b] \iff \mathbf{f}$ is differentiable almost everywhere in $[a,b], \mathbf{f}' \in L^1([a,b], \mathbf{R}^n)$ and $\mathbf{f}(x) = \mathbf{f}(a) + \int_a^x \mathbf{f}'(t)dt,\ x \in [a,b]$.

Lemma 1. *The polynomial* $\sum_{k=0}^{N-1} q_N(k)z^k$ *(see* (1.2)*) satisfies*

$$\operatorname{sign}(q_N(k)) = (-1)^k\sigma,\ k = 0, 1, \cdots, N-1, (\sigma = 1 \text{ or } -1) \tag{2.16}$$

$$\sum_{k=0}^{N-1} \mid q_N(k) \mid = 2^{N-1}\Big[2 \cdot \frac{(2N-1)!!}{(2N)!!}\Big]^{1/2}. \tag{2.17}$$

Proof. Let $q(z) = \sum_{k=0}^{N-1} q_N(k)(-1)^k z^k$. Since $q_N(k)$ are real numbers, we have by (1.3) for all $z \in \{e^{\mathrm{i}\xi} | \xi \in \mathbf{R}\}$

$$q(z)q(z^{-1}) = \sum_{k=0}^{N-1} \binom{k+N-1}{k} 2^{-k}(\frac{1}{2}z^2 + z + \frac{1}{2})^k z^{-k},$$

and so for all $z \in \mathbf{C}$

$$\begin{aligned} q(z)q(z^{-1})z^{N-1} &= q(z) \sum_{k=0}^{N-1} q_N(k)(-1)^k z^{N-1-k} \\ &= \sum_{k=0}^{N-1} \binom{k+N-1}{k} 2^{-k}(\frac{1}{2}z^2 + z + \frac{1}{2})^k z^{N-1-k}. \end{aligned}$$

Letting $z = 0$ we obtain $q_N(0)q_N(N-1)(-1)^{N-1} = 2^{-2(N-1)}\binom{2N-2}{N-1} \neq 0$. Let $p(z) = \sum_{k=0}^{N-1} \binom{k+N-1}{k} 2^{-k} z^k$. We observe that the coefficients $a_k = \binom{k+N-1}{k} 2^{-k}$ satisfy $a_k \ge a_{k-1} > 0, k = 1, 2, \cdots, N-1$, which imply $p(z) \neq 0$ for all $|z| > 1$. In fact, if $|z| > 1$, then $|(z-1)p(z)| \geq a_{N-1}|z|^N - \sum_{k=1}^{N-1}(a_k - a_{k-1})|z|^N - a_0 = a_0(|z|^N - 1) > 0$. Take any $z \in \mathbf{C}$ with Re$z \geq 0$. If $|z| = 1$ or $z = 0$, then $q(z) \neq 0$ because of (1.3) and $q_N(0) \neq 0$. If $|z| \neq 1$ and $z \neq 0$, then $w := 1 + \frac{1}{2}(z + z^{-1})$ satisfies $|w| > 1$ and so $q(z)q(z^{-1}) = p(w) \neq 0$. This means that all zeros of $q(z)$ are in the open left-half plane $Re(z) < 0$. Thus the polynomial $q(z)/q_N(N-1)(-1)^{N-1}$ is a product of linear factors $z + a$ and quadratic factors $z^2 + bz + c$, each with positive coefficients. Therefore all coefficients of $q(z)/q_N(N-1)(-1)^{N-1}$ are positive, i.e. (2.16) holds. (2.16) together with (1.3) yield (2.17):

$$\Big[\sum_{k=0}^{N-1} | q_N(k) |\Big]^2 = \Big|\sum_{k=0}^{N-1} q_N(k) e^{\mathrm{i}k\pi}\Big|^2 = \sum_{k=0}^{N-1} \binom{k+N-1}{k} = 4^{N-1} \cdot 2 \cdot \frac{(2N-1)!!}{(2N)!!}.$$

Lemma 2. *Let* $\mathbf{f} : [0,1] \to \mathbf{R}^{N-1}, k \in \mathbf{N}$. *Then*

$$\begin{aligned} &\mathbf{A}(k;x)\mathbf{f}(\tau^k(x)) \\ &= \sum_{m=0}^{2^k-1} \mathbf{A}(k; 2^{-k}m)\chi_{[2^{-k}m, 2^{-k}(m+1)[}(x)\mathbf{f}(2^k x - m),\ x \in [0,1[, \end{aligned} \tag{2.18}$$

$$\sum_{m=0}^{2^k-1} \mathbf{A}(k; 2^{-k}m) = (\mathbf{A}_0 + \mathbf{A}_1)^k. \tag{2.19}$$

Proof. $\forall x \in [0,1[$, choose $s = \sum_{j=1}^{k} 2^{k-j} d_j(x)$. Then $x \in [2^{-k}s, 2^{-k}(s+1)[$, and by definitions and properties of $d_j(x)$ and $\tau(x)$ (see (2.1)—(2.3)) we have $d_j(x) = d_j(2^{-k}s)$, $j = 1, 2, \cdots, k$; $\tau^k(x) = 2^k x - s$. Therefore by (2.15),

$$\mathbf{A}(k;x)\mathbf{f}(\tau^k(x)) = \mathbf{A}(k; 2^{-k}s)\mathbf{f}(2^k x - s) = \text{the right} - \text{hand side of (2.18)}.$$

Now let $\mathbf{f} \in L^1([0,1], \mathbf{R}^{N-1})$ be arbitrary. Then (2.18), (2.15), (2.2) and (2.1) yield

$$\begin{aligned} &2^{-k}\sum_{m=0}^{2^k-1} \mathbf{A}(k; 2^{-k}m) \int_0^1 \mathbf{f}(t)dt = \int_0^1 \mathbf{A}(k;t)\mathbf{f}(\tau^k(t))dt \\ &= \int_0^{\frac{1}{2}} \boldsymbol{A}_0\mathbf{A}(k-1; 2t)\mathbf{f}(\tau^{k-1}(2t))dt + \int_{\frac{1}{2}}^1 \mathbf{A}_1\mathbf{A}(k-1; 2t-1)\mathbf{f}(\tau^{k-1}(2t-1))dt \\ &= \frac{1}{2}(\mathbf{A}_0 + \mathbf{A}_1)\int_0^1 \mathbf{A}(k-1;t)\mathbf{f}(\tau^{k-1}(t))dt = 2^{-k}(\mathbf{A}_0 + \mathbf{A}_1)^k \int_0^1 \mathbf{f}(t)dt. \end{aligned}$$

This implies (2.19).

Note that if $N = 2, i.e., \mathbf{A}_0 = a, \mathbf{A}_1 = b$ are (real) numbers, then (2.19) becomes

$$(a+b)^k = \sum_{m=0}^{2^k-1} a^{k-\sigma_k(m)} b^{\sigma_k(m)} \qquad \text{(see section 4 for N = 2).}$$

Lemma 3.[5] $(\mathbf{Q}_d)_{i,j} = 2^{-N+1} q_N(2i - j - 1 + d), 1 \leq i, j \leq N-1, d = 0, 1,$ *where* $\mathbf{Q}_d$ *is defined in* (2.7).

Remark. This expression of $\mathbf{Q}_d$ was mentioned in [5, p.1061] without proof. For the sake of insurance of our main results, we give the lemma a proof in Appendix.

Lemma 4. *Let* $N \geq 2$, $\beta = \beta_{_N} := \left[\frac{1}{2}\frac{(2N-1)!!}{(2N)!!}\right]^{1/2} + 2^{-N}$, $\lambda = \lambda_{_N} := \left[2 \cdot \frac{(2N-1)!!}{(2N)!!}\right]^{1/2}$. *We have*

$$\mathbf{A}_d = \mathbf{S}^{-1}\mathbf{Q}_d\mathbf{S}, \qquad d = 0,1, \qquad where\, \mathbf{S} := \mathbf{B}_{22}\mathbf{B}_{12}^{-1}\mathbf{B}_{11}. \tag{2.20}$$

$$\begin{aligned} \| \mathbf{Q}_d \| = \||\mathbf{Q}_d\|| &= \beta, \qquad N \geq 3, \qquad d = 0,1, \\ &\leq \beta, \qquad N = 2, \qquad d = 0,1. \end{aligned} \tag{2.21}$$

$$\||\mathbf{Q}_0| + |\mathbf{Q}_1|\|| = \rho(|\mathbf{Q}_0| + |\mathbf{Q}_1|) = \lambda. \tag{2.22}$$

$$\max_{x\in[0,1]} \| \mathbf{A}(k;x) \| \leq C\beta^k, \; k \in \mathbf{N}. \tag{2.23}$$

$$2^k \int_0^1 \| \mathbf{A}(k;t)\mathbf{f}(\tau^k(t)) \| \, dt \leq C\lambda^k \int_0^1 \| \mathbf{f}(t) \| \, dt, \; k \in \mathbf{N}, \tag{2.24}$$

$\mathbf{f} \in L^1([0,1], \mathbf{R}^{N-1})$, *where* $C = \| \mathbf{S}^{-1} \| \cdot \| \mathbf{S} \|$, $\rho(\mathbf{A})$ *denotes the usual spectral radius.*

Proof.

(a). In polynomials $(x-1)(x-2)\cdots(x-i+1) = \sum_{k=1}^{i} g_{ik}x^{k-1}, (2 \leq i \leq N)$ taking $x = j = 1, 2, \cdots, 2N-1$ yield $(i-1)!\binom{j-1}{i-1} = \sum_{k=1}^{i} g_{ik}j^{k-1}, \quad 1 \leq i \leq N (g_{11} = 1)$, or equivalently in matrix form with an invertible matrix $\mathbf{G}$, $[\mathbf{B}_{11}, \mathbf{B}_{12}] = \mathbf{GU} = [\mathbf{GU}_0, \mathbf{GU}_1]$ which gives $\mathbf{B}_{12}^{-1}\mathbf{B}_{11} = \mathbf{U}_1^{-1}\mathbf{U}_0$ since det $\mathbf{B}_{12} = \det\mathbf{G} \cdot \det\mathbf{U}_1 \neq 0$. This equality together with (2.7), (2.10), (2.11) and (2.9) deduce

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11,d} & \mathbf{T}_{12,d} \\ \mathbf{T}_{21,d} & \mathbf{T}_{22,d} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\mathbf{U}_1^{-1}\mathbf{U}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_d & \mathbf{0} \\ \mathbf{C}_d & \mathbf{Q}_d \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\mathbf{B}_{12}^{-1}\mathbf{B}_{11} \end{bmatrix},$$

and so

$$\begin{aligned} &\mathbf{B}_{11}\mathbf{A}_d + \mathbf{B}_{12}(\mathbf{T}_{21,d} - \mathbf{T}_{22,d}\mathbf{U}_1^{-1}\mathbf{U}_0) = \mathbf{0}, \\ &\mathbf{B}_{22}(\mathbf{T}_{21,d} - \mathbf{T}_{22,d}\mathbf{U}_1^{-1}\mathbf{U}_0) = -\mathbf{Q}_d\mathbf{B}_{22}\mathbf{B}_{12}^{-1}\mathbf{B}_{11}. \end{aligned}$$

Thus $\mathbf{SA}_d = \mathbf{Q}_d\mathbf{S}$. Since $\mathbf{B}$ is inverseble, by (2.11) $\mathbf{S}$ is inverseble also.

(b). Let $\mathbf{x} = (x_1, x_2, \cdots, x_{_{N-1}})^t \in \mathbf{R}^{N-1}$. By definition of $\| \mathbf{x} \|$, Lemma 3 and (2.17) we have

$$\| \mathbf{Q}_d\mathbf{x} \| = 2^{-N+1}\sum_{i=1}^{N-1}\left|\sum_{j=1}^{N-1} q_{_N}(2i-j-1+d)x_j\right|, \tag{2.25}$$

$$\||\mathbf{Q}_d|\mathbf{x}\| \leq 2^{-N+1}\max\Big\{\sum_k |q_{_N}(2k)|, \sum_k |q_{_N}(2k-1)|\Big\} \| \mathbf{x} \|, \tag{2.26}$$

$$\| (|\mathbf{Q}_0| + |\mathbf{Q}_1|)\mathbf{x} \| \leq 2^{-N+1}(\sum_{k=0}^{N-1} |q_{_N}(k)|) \| \mathbf{x} \| = \lambda \| \mathbf{x} \| . \tag{2.27}$$

Observe that (2.16), (2.17) and $\sum\limits_{k=0}^{N-1} q_{_N}(k) = 1$ imply

$$\begin{aligned} &\sum_k |q_{_N}(2k)| + \sum_k |q_{_N}(2k-1)| = 2^{N-1}\lambda, \\ &\sum_k |q_{_N}(2k)| - \sum_k |q_{_N}(2k-1)| = \pm 1. \end{aligned}$$

Since $\| \mathbf{x} \|=\| \mid \mathbf{x} \mid \|$ and $\| \mathbf{Q}_d\mathbf{x} \|\leq\| \mid \mathbf{Q}_d \mid \mid \mathbf{x} \mid\|$, we obtain by (2.26)

$$\| \mathbf{Q}_d \|\leq\| \mid \mathbf{Q}_d \mid \|\leq 2^{-N+1}(2^{N-1}\lambda+1)/2=\beta.$$

For $N\geq 3, d\in\{0,1\}$, choose $\mathbf{y}=(y_1,y_2,\cdots,y_{N-1})^t$ such that $\forall j, y_{2j-1}=1, y_{2j}=0$, or $\forall j,\ y_{2j-1}=0,\ y_{2j}=1$. Then (2.25), (2.16) yield $\| \mathbf{Q}_d\mathbf{y} \|=\beta \| \mathbf{y} \|\neq 0$. Thus $\| \mathbf{Q}_d \|=\| \mid \mathbf{Q}_d \mid \|=\beta$, and (2.21) holds. Choose an $(N-1)$–dimensional row vector $\mathbf{v}=(1,1,\cdots,1)$. Then $\mathbf{v}(\mid \mathbf{Q}_0 \mid+\mid \mathbf{Q}_1 \mid)=\lambda\mathbf{v}$, which together with (2.27) lead to (2.22).

(c). (2.23) follows from (2.20), (2.21). Note that according to our choice for vector norm $\| \cdot \|$,

$$\int_0^1 \| \mathbf{g}(t) \| \, dt=\| \int_0^1 \mid \mathbf{g}(t) \mid dt \|, \qquad \mathbf{g}\in L^1([0,1],\mathbf{R}^{N-1}).$$

By (2.20), (2.18), (2.19) and (2.22) we obtain (2.24):

$$\begin{aligned}
&2^k\int_0^1 \| \mathbf{A}(k;t)\mathbf{f}(\tau^k(t)) \| \, dt\\
&\leq 2^k \| \mathbf{S}^{-1} \|\| \int_0^1 |\mathbf{Q}_{d_1(t)}|\cdots|\mathbf{Q}_{d_k(t)}||\mathbf{S}\mathbf{f}(\tau^k(t))|dt \|\\
&=\| \mathbf{S}^{-1} \|\| (\mid \mathbf{Q}_0 \mid+\mid \mathbf{Q}_1 \mid)^k\int_0^1 \mid \mathbf{S}\mathbf{f}(t) \mid dt \|\leq\| \mathbf{S}^{-1} \|\| \mathbf{S} \| \lambda^k\int_0^1 \| \mathbf{f}(t) \| \, dt.
\end{aligned}$$

Remark. The Wallis' inequality $(2N-1)!!/(2N)!!<(\pi N)^{-1/2}$ gives explicit estimates for β_N and λ_N:

$$\beta_N<(4\pi N)^{-1/4}+2^{-N}, \quad \lambda_N<(\frac{4}{\pi N})^{1/4}.$$

The following Lemma 5 gives decay estimates for derivatives of analytic functions, which have many applications in dealing with some kinds of convergence problems.

Lemma 5.(decay estimates for derivatives) *Let* $f:\mathbf{R}\to\mathbf{C}, f\in C^\infty(\mathbf{R})$ *satisfy*

$$\mid f(x) \mid\leq C(1+\mid x \mid)^{-\alpha}, x\in\mathbf{R},\ \textit{and}\ \sup_{x\in\mathbf{R}}|f^{(n)}(x)|\leq B^n, n\in\mathbf{N}$$

with some positive constants α, C *and* B. *Then there exists a constant* $M>0$ *which depends only on* α , C *and* B *such that*

$$\mid f^{(s)}(x) \mid\leq M^s(1+\mid x \mid)^{-\alpha},\ x\in\mathbf{R}, \qquad s=1,2,\cdots. \tag{2.28}$$

Proof. Let $s\in\mathbf{N}$ be given. Define

$$m:=\max\{n\in\mathbf{N}\,\Big|\,e^{\frac{n-1}{\alpha}}-1\leq\frac{2n}{e^2B}\}, \qquad \eta_s:=\max\{e^{\frac{s}{\alpha}}-1, e^{\frac{m}{\alpha}}-1\}.$$

For any $\mid x \mid>\eta_s$, choose an integer n such that $\alpha log(1+|x|)<n\leq 1+\alpha log(1+\mid x \mid)$. Then $n>\max\{s,m\}$ and $|x|>2n/e^2B$. Now we consider the polynomial p_n in variable $t\in\mathbf{R}$:

$$p_n(t):=\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}\rho^k t^k, \tag{2.29}$$

where $\rho = n/e^2B$. By Taylor formula we have

$$p_n(t) = f(x+\rho t) - \frac{\rho^n t^n}{(n-1)!}\int_0^1 f^{(n)}(x+\theta\rho t)(1-\theta)^{n-1}d\theta,$$

which gives, by assumption and $|x| > 2\rho$, for all $t \in [-1,1]$

$$\begin{aligned} | p_n(t) | &\leq | f(x+\rho t) | + \frac{\rho^n B^n}{n!} \leq C(1+|x+\rho t|)^{-\alpha} + \frac{n^n}{n!}e^{-2n} \\ &\leq C(1+\frac{1}{2}|x|)^{-\alpha} + e^{-n} \leq C(1+\frac{1}{2}|x|)^{-\alpha} + (1+|x|)^{-\alpha}. \end{aligned} \tag{2.30}$$

On the other hand, for the elliptic curve

$$\Gamma = \{z \in \mathbf{C} | \frac{(Rez)^2}{a^2} + \frac{(Imz)^2}{b^2} = 1\} \quad \text{with } a = \frac{1}{2}(r+r^{-1}),\ b = \frac{1}{2}(r-r^{-1}),$$

where $r = (\frac{n+s}{n-s})^{\frac{1}{2}}$, using Bernstein inequality ([6])

$$\max_{z\in\Gamma} | p_n(z) | \leq r^n \max_{-1\leq t\leq 1} | p_n(t) |$$

and the inequality $b^{-s}r^n \leq s^{-s}e^s(n+s)^s < s^{-s}(2en)^s$ we obtain

$$\begin{aligned} | p_n^{(s)}(0) | &\leq s!b^{-s}\max_{|z|=b} | p_n(z) | \leq s!b^{-s}\max_{z\in\Gamma} | p_n(z) | \\ &\leq (2en)^s \max_{-1\leq t\leq 1} | p_n(t) |, \end{aligned}$$

where we have used principle of the maximum. Combining this with (2.29) and (2.30) lead to

$$| f^{(s)}(x) | = \rho^{-s} | p_n^{(s)}(0) | \leq (2e^3B)^s(2^\alpha C+1)(1+ | x |)^{-\alpha},\ (|x| > \eta_s)$$

which implies (2.28) with $M = \max\{2e^3B(2^\alpha C+1), e^mB\}$ since $\sup_{x\in\mathbf{R}} | f^{(s)}(x) | \leq B^s$ and $(1+\eta_s)^\alpha \leq e^{ms}$.

As an application of Lemma 5 we can extend the decay estimate of $\hat{\phi}_N$ obtained in [1] to its all derivatives $\hat{\phi}_N^{(s)}$.

Lemma 6. (i) *There exists a positive constant M depending only on $N(\geq 2)$ such that*

$$| \hat{\phi}_N^{(s)}(\xi) | \leq M^{s+1}(1+ | \xi |)^{-1-\delta N}, \qquad \xi \in \mathbf{R},\ s = 0,1,2,\cdots, \tag{2.31}$$

where $\delta > 0$ is an absolute constant.

(ii)

$$\sum_{n=1}^{2N-1} n^s\phi_N(x+n-1) = \sum_{j=0}^{s}\binom{s}{j} r_j(x)(1-x)^{s-j}, \qquad x \in [0,1], s = 0,1,2,\cdots, \tag{2.32}$$

where

$$r_j(x) = \begin{cases} b_j = (-\mathrm{i})^j\sqrt{2\pi}\hat{\phi}_N^{(j)}(0), & 0 \leq j \leq N-1, \\ (-\mathrm{i})^j\sqrt{2\pi}\sum_{n\in\mathbf{Z}}\hat{\phi}_N^{(j)}(2n\pi)e^{-\mathrm{i}2n\pi x}, & j \geq N. \end{cases} \quad \mathrm{i} = \sqrt{-1}$$

Proof. From [1] we know that there is a constant $C > 0$ depending only on N such that

$$|\hat{\phi}_N(\xi)| \leq C(1+|\xi|)^{-1-\delta N}, \qquad \xi \in \mathbf{R},$$

and by $|m_N(\xi)| \leq 1$ and $m_N(0) = 1$([1]), it is easy to prove that $\hat{\phi}_N \in Lip1$ on $\mathbf{R}$. Thus

$$\hat{\phi}_N(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{2N-1} \phi_N(x) e^{\mathrm{i}\xi x} dx, \qquad \xi \in \mathbf{R},$$

and so $\hat{\phi}_N \in C^\infty(\mathbf{R})$,

$$\sup_{\xi \in \mathbf{R}} |\hat{\phi}_N^{(n)}(\xi)| \leq \Big(\frac{1}{\sqrt{2\pi}} \int_0^{2N-1} |\phi_N(x)|\, dx\Big)(2N-1)^n, \qquad n \in \mathbf{N}.$$

Therefore (2.31) follows from Lemma 5. Then we are allowed to use Poisson summation formula to the compactly supported functions $x^s\phi_N(x)$ (see, e.g., [7, pp.250–253]) and obtain

$$\sum_{n\in\mathbf{Z}} (x+n)^s \phi_N(x+n) = (-\mathrm{i})^s \sqrt{2\pi} \sum_{n\in\mathbf{Z}} \hat{\phi}_N^{(s)}(2n\pi) e^{-\mathrm{i}2n\pi x}, x \in \mathbf{R}, s = 0, 1, 2, \cdots. \quad (2.33)$$

Moreover, by definition of $m_N(\xi)$ and the relation $\hat{\phi}_N(2\xi) = m_N(\xi)\hat{\phi}_N(\xi)$, one finds that $\hat{\phi}_N^{(j)}(2n\pi) = 0$ for all $n \in \mathbf{Z} \setminus \{0\}$ and all $0 \leq j \leq N-1$. Combining (2.33) with the identities $n^s = \sum_{j=0}^{s} \binom{s}{j} (n+x-1)^j (1-x)^{s-j}$ and $\mathrm{supp}\phi_N = [0, 2N-1]$ yield (2.32).

3. Main Result and Proof

First of all, we note that since the wavelet ψ_N is defined by ϕ_N via (1.1), every series representation of ϕ_N yields a series representation of ψ_N.

Theorem . *Let* $N \geq 2$, $\mathbf{\Phi}_0(x) := (\phi_N(x), \phi_N(x+1), \cdots, \phi_N(x+N-2))^t$, $\mathbf{\Phi}_1(x) := (\phi_N(x+N-1), \phi_N(x+N), \cdots, \phi_N(x+2N-2))^t$. *Then*

(i)

$$\mathbf{\Phi}_1(x) = -\mathbf{U}_1^{-1}\mathbf{U}_0\mathbf{\Phi}_0(x) + \mathbf{P}(x), \qquad x \in [0,1], \quad (3.1)$$

$$\mathbf{\Phi}_0(x) = \mathbf{A}_{d_1(x)}\mathbf{\Phi}_0(\tau(x)) + \mathbf{P}_{d_1(x)}(x), \qquad x \in [0,1], \quad (3.2)$$

$$\mathbf{\Phi}_0(0) = (\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{P}_0(0), \mathbf{\Phi}_0(1) = (\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{P}_1(1). \quad (3.3)$$

(ii) $\mathbf{\Phi}_0$ *(*and so $\mathbf{\Phi}_1$) *is absolutely continuous on* $[0,1]$ *and the following three types of series representation of* $\mathbf{\Phi}_0$ *hold with absolute convergence:*

$$\mathbf{\Phi}_0(x) = \sum_{k=0}^{\infty} \mathbf{A}(k;x)\mathbf{P}_{d_{k+1}(x)}(\tau^k(x)), \qquad x \in [0,1], \quad (3.4)$$

$$\mathbf{\Phi}_0(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{2^k-1} \mathbf{A}(k; 2^{-k}m)\Big[\chi_{[2^{-k}m, 2^{-k}(m+1/2)[}(x)\mathbf{P}_0(2^k x - m) \quad (3.5)$$

$$+ \chi_{[2^{-k}(m+1/2), 2^{-k}(m+1)[}(x)\mathbf{P}_1(2^k x - m)\Big], \ x \in [0,1],$$

$$\mathbf{\Phi}_0(x) = \mathbf{\Phi}_0(0) + \sum_{k=0}^{\infty} 2^k \int_0^x \mathbf{A}(k;t)\mathbf{D}_{d_{k+1}(t)}(\tau^k(t))dt \tag{3.6}$$
$$= \mathbf{\Phi}_0(0) + \sum_{k=0}^{\infty} 2^k \sum_{m=0}^{2^k-1} \mathbf{A}(k;2^{-k}m)\Big[\int_0^x \chi_{[2^{-k}m, 2^{-k}(m+1/2)[}(t)\mathbf{D}_0(2^k t - m)dt$$
$$+ \int_0^x \chi_{[2^{-k}(m+1/2), 2^{-k}(m+1)[}(t)\mathbf{D}_1(2^k t - m)dt\Big],\ x \in [0,1],$$

where $\mathbf{D}_0(x) := \mathbf{P}_0'(x)$, $\mathbf{D}_1(x) := \mathbf{P}_1'(x)$.

Proof. (i). By Lemma 6 (ii) we have

$$\sum_{n=1}^{2N-1} n^s \phi_N(x+n-1) = \sum_{j=0}^{s} b_j \binom{s}{j}(1-x)^{s-j}, \qquad x \in [0,1], s = 0,1,\cdots,N-1. \tag{3.7}$$

Let $\mathbf{\Phi}(x) = (\phi_N(x), \phi_N(x+1), \cdots, \phi_N(x+2N-2))^t$. Then (3.7) is written in vector form by (2.8):

$$\mathbf{U}_0\mathbf{\Phi}_0(x) + \mathbf{U}_1\mathbf{\Phi}_1(x) = \mathbf{U}\mathbf{\Phi}(x)$$
$$= \Big(1, \sum_{j=0}^{1} b_j \binom{1}{j}(1-x)^{1-j}, \cdots, \sum_{j=0}^{N-1} b_j \binom{N-1}{j}(1-x)^{N-1-j}\Big)^t,\ x \in [0,1].$$

This yields (3.1) via (2.12). On the other hand, (1.4) and supp$\phi_N = [0, 2N-1]$ imply $\mathbf{\Phi}(x) = \mathbf{T}_{d_1(x)}\mathbf{\Phi}(\tau(x)), x \in [0,1]$. Combining this with (2.10), (2.9), (3.1) and (2.13) lead to (3.2). (3.3) is obvious because, by (2.20) and (2.21), the matrices $\mathbf{I}-\mathbf{A}_0, \mathbf{I}-\mathbf{A}_1$ are inverseble.

(ii). Let us now equip the linear space $L^1([0,1], \mathbf{R}^{N-1})$ and its linear subspace $L^\infty([0,1], \mathbf{R}^{N-1})$ with the norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$ respectively, given by

$$\| \mathbf{f} \|_1 := \int_0^1 \| \mathbf{f}(t) \| \, dt = \| \int_0^1 | \mathbf{f}(t) | \, dt \|, \ \mathbf{f} \in L^1([0,1], \mathbf{R}^{N-1}),$$
$$\| \mathbf{f} \|_\infty := \text{ess sup}_{x\in[0,1]} \| \mathbf{f}(x) \|, \qquad \mathbf{f} \in L^\infty([0,1], \mathbf{R}^{N-1}).$$

Then both the spaces are real Banach space. Define a linear operator $\mathbf{T} : L^p([0,1], \mathbf{R}^{N-1}) \to L^p([0,1], \mathbf{R}^{N-1})$,

$$\mathbf{T}\mathbf{f}(x) = \mathbf{A}_{d_1(x)}\mathbf{f}(\tau(x)), \qquad x \in [0,1],$$

with the norm

$$\| \mathbf{T} \|_{L^p} := \sup\Big\{\| \mathbf{T}\mathbf{f} \|_p \ \Big| \mathbf{f} \in L^p([0,1], \mathbf{R}^{N-1}), \| \mathbf{f} \|_p = 1\Big\},\ p = 1 \text{ or } \infty.$$

By (2.2), (2.15) and (2.23) we have

$$\mathbf{T}^k\mathbf{f}(x) = \mathbf{A}(k;x)\mathbf{f}(\tau^k(x)), \qquad x \in [0,1], \tag{3.8}$$

$$\| \mathbf{T}^k \|_{L^\infty} = \max_{x\in[0,1]} \| \mathbf{A}(k;x) \| \le C\beta^k, \qquad k \in \mathbf{N}. \tag{3.9}$$

(3.9) insures the existance of the inverse operator $(\mathbf{I}-\mathbf{T})^{-1}=\sum_{k=0}^{\infty}\mathbf{T}^k$ which is convergent in the norm $\|\cdot\|_{L^\infty}$ since $\beta=\beta_N<1$. Thus for any $\mathbf{g}\in L^\infty([0,1],\mathbf{R}^{N-1})$, the equation $\mathbf{f}=\mathbf{Tf}+\mathbf{g}$ has a unique solution: $\mathbf{f}=(\mathbf{I}-\mathbf{T})^{-1}\mathbf{g}=\sum_{k=0}^{\infty}\mathbf{T}^k\mathbf{g}$. Specifically, for $\mathbf{g}(x)=\mathbf{P}_{d_1(x)}(x)$, the function $\mathbf{\Phi}_0(x)$ is the corresponding solution because of (3.2). Therefore (3.4) follows from (3.8) and (2.2). (3.4), (2.18) and (2.2) then imply(3.5). To prove absolute continuity of $\mathbf{\Phi}_0$ and the representation (3.6), we define

$$K(\mathbf{f})(x):=\mathbf{Tf}(x)+\mathbf{P}_{d_1(x)}(x),\qquad \mathbf{f}\in L^1([0,1],\mathbf{R}^{N-1}),$$
$$W:=\Big\{\mathbf{f}\in L^1([0,1],\mathbf{R}^{N-1})\mid \mathbf{f}\text{ is absolutely continuous on }[0,1]\text{and}$$
$$\mathbf{f}(0)=\mathbf{\Phi}_0(0),\qquad \mathbf{f}(1)=\mathbf{\Phi}_0(1)\Big\}.$$

Clearly, W is nonempty; the function $\mathbf{\Phi}_0(0)+x[\mathbf{\Phi}_0(1)-\mathbf{\Phi}_0(0)]$ is a member of W. We now prove that $K(W)\subset W$ and

$$K(\mathbf{f})(x)=\mathbf{\Phi}_0(0)+\int_0^x[\mathbf{D}(t)+2\mathbf{Tf}'(t)]dt,\qquad x\in[0,1],\mathbf{f}\in W,\tag{3.10}$$

where $\mathbf{D}(x):=\mathbf{D}_{d_1(x)}(x)$. The end conditions $K(\mathbf{f})(0)=\mathbf{\Phi}_0(0),K(\mathbf{f})(1)=\mathbf{\Phi}_0(1)$ are satisfied for $\mathbf{f}\in W$ because of (3.2) or (3.3). Let $\hat K(\mathbf{f})(x)$ denote the right–hand side of (3.10). Then for $f\in W$ and $0\le x<\frac12$,

$$\begin{aligned}\hat K(\mathbf{f})(x)&=\mathbf{\Phi}_0(0)+\int_0^x\mathbf{P}_0'(t)dt+2\mathbf{A}_0\int_0^x\mathbf{f}'(2t)dt\\&=\mathbf{P}_0(x)+\mathbf{A}_0\mathbf{f}(2x)+\mathbf{\Phi}_0(0)-\mathbf{P}_0(0)-\mathbf{A}_0\mathbf{f}(0)=K(\mathbf{f})(x).\end{aligned}$$

Note that continuity of $\mathbf{\Phi}_0$ and $\mathbf{f}\in W$ imply

$$\hat K(\mathbf{f})(\frac12)=\mathbf{P}_0(\frac12)+\mathbf{A}_0\mathbf{\Phi}_0(1)=\lim_{x\nearrow\frac12}\mathbf{\Phi}_0(x)=\mathbf{\Phi}_0(\frac12)=\mathbf{P}_1(\frac12)+\mathbf{A}_1\mathbf{\Phi}_0(0).$$

Then we have for $\frac12\le x\le 1$,

$$\begin{aligned}\hat K(\mathbf{f})(x)&=\hat K(\mathbf{f})(\frac12)+\int_{1/2}^x\mathbf{P}_1'(t)dt+2\mathbf{A}_1\int_{1/2}^x\mathbf{f}'(2t-1)dt\\&=\mathbf{P}_1(x)+\mathbf{A}_1\mathbf{f}(2x-1)=K(\mathbf{f})(x).\end{aligned}$$

Hence $K(\mathbf{f})\in W$ and (3.10) holds. Iterating (3.10) leads to

$$K^n(\mathbf{f})(x)=\mathbf{\Phi}_0(0)+\sum_{k=0}^{n-1}2^k\int_0^x\mathbf{T}^k\mathbf{D}(t)dt+2^n\int_0^x\mathbf{T}^n\mathbf{f}'(t)dt,\ n\in\mathbf{N}.\tag{3.11}$$

On the other hand, (3.8) and (2.24) yield $2^k\parallel\mathbf{T}^k\parallel_{L^1}\le C\lambda^k,k\in\mathbf{N}$, and so $\mathbf{I}-2\mathbf{T}$ has a bounded inverse on $L^1([0,1],\mathbf{R}^{N-1})$; $(\mathbf{I}-2\mathbf{T})^{-1}=\sum_{k=0}^{\infty}2^k\mathbf{T}^k$ converges in the norm $\|\cdot\|_{L^1}$ since $\lambda=\lambda_N<1$ (see Lemma 4). Take $\mathbf{h}(x)=(\mathbf{I}-2\mathbf{T})^{-1}\mathbf{D}(x)$. Then $\mathbf{h}\in L^1([0,1],\mathbf{R}^{N-1})$ and

$$\int_0^x\mathbf{h}(t)dt=\sum_{k=0}^{\infty}2^k\int_0^x\mathbf{T}^k\mathbf{D}(t)dt=\sum_{k=0}^{\infty}2^k\int_0^x\mathbf{A}(k;t)\mathbf{D}(\tau^k(t))dt.\tag{3.12}$$

Since $\mathbf{\Phi}_0 = K(\mathbf{\Phi}_0)$, by (3.9) we have for $\mathbf{f} \in W$,

$$\begin{aligned}\| K^n(\mathbf{f}) - \mathbf{\Phi}_0 \|_\infty =& \| K^n(\mathbf{f}) - K^n(\mathbf{\Phi}_0) \|_\infty \\ &=\| \mathbf{T}^n(\mathbf{f} - \mathbf{\Phi}_0) \|_\infty \le C\beta^n \| \mathbf{f} - \mathbf{\Phi}_0 \|_\infty,\ n \in \mathbf{N}.\end{aligned}$$

This estimate together with (3.11), (3.12) yield

$$\mathbf{\Phi}_0(x) = \lim_{n\to\infty} K^n(\mathbf{f})(x) = \mathbf{\Phi}_0(0) + \int_0^x \mathbf{h}(t)dt, \qquad x \in [0,1]. \tag{3.13}$$

Hence $\mathbf{\Phi}_0$ is absolutely continuous on [0,1] and $\mathbf{\Phi}_0'(x) = \mathbf{h}(x)$ a.e. in [0,1]. (3.6) then follows from (3.13), (3.12) (with $\mathbf{D}(t) = \mathbf{D}_{d_1(t)}(t)$), (2.18) and (2.2).

Remark. It is known that for $N \ge 3, \phi_N$ belongs to $\mathrm{C}^k(\mathbf{R})$ with $k = k_N \ge 1$ ([1], [5]). For N=2, Daubechies and Lagarias in [5] proved that ϕ_2 is differentiable a.e. in $\mathbf{R}$. Our Theorem gives its further regularity, i.e., ϕ_2 is even absolutely continuous on $\mathbf{R}$ (see also below for $N = 2$).

4. Representation of ϕ_2 and ϕ_3

As special cases of the Theorem for N=2,3, we give here the representation of ϕ_2 with explicit numerical series and ϕ_3 with vector forms. The following numerical values of $q_N(k), C_N(n) (N = 2,3)$ are taken from [1], [5]. (From [1], [5] we know that the values of the coefficients $q_N(k)$ and therefore $C_N(n)$ for $N \ge 4$ can not be written in explicit forms.)

(1) N=2. $\mathbf{A}_0 = \frac{1}{2}q_2(0) = \frac{1+\sqrt{3}}{4}$, $\mathbf{A}_1 = \frac{1}{2}q_2(1) = \frac{1-\sqrt{3}}{4}$,

$$C_2(0) = \frac{1+\sqrt{3}}{4}, C_2(1) = \frac{3+\sqrt{3}}{4},\ C_2(2) = \frac{3-\sqrt{3}}{4}, C_2(3) = \frac{1-\sqrt{3}}{4};$$

$$\begin{bmatrix} \phi_2(x+1) \\ \phi_2(x+2) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \phi_2(x) + \begin{bmatrix} x + \frac{1+\sqrt{3}}{2} \\ -x + \frac{1-\sqrt{3}}{2} \end{bmatrix}, \qquad x \in [0,1],$$

$$\begin{aligned}\phi_2(x) &= \frac{1}{4}\Big[1 + (1 - 2d_1(x))\sqrt{3}\Big]\phi_2(\tau(x)) + d_1(x)\frac{1+\sqrt{3}}{2}(x + \frac{\sqrt{3}-1}{4}) \\ &= \frac{1+\sqrt{3}}{2}\sum_{k=0}^{\infty}(\frac{1-\sqrt{3}}{4})^{s_k(x)}(\frac{1+\sqrt{3}}{4})^{k-s_k(x)}d_{k+1}(x)[\tau^k(x) + \frac{\sqrt{3}-1}{4}], \\ &\quad x \in [0,1];\end{aligned}$$

$$\begin{aligned}\phi_2(x) = \frac{1+\sqrt{3}}{2}&\sum_{k=0}^{\infty}\sum_{m=0}^{2^k-1}(\frac{1-\sqrt{3}}{4})^{\sigma_k(m)} \\ &\cdot (\frac{1+\sqrt{3}}{4})^{k-\sigma_k(m)}\chi_{[2^{-k}(m+1/2),2^{-k}(m+1)[}(x)(2^kx - m + \frac{\sqrt{3}-1}{4}),\end{aligned}$$

$x \in [0,1]$, where $s_k(x) = d_1(x) + d_2(x) + \cdots + d_k(x), s_0(x) = 0, \sigma_k(m) = s_k(2^{-k}m)$. Since ϕ_2 is continuous on $\mathbf{R}$ and $\mathrm{supp}\phi_2 = [0,3]$, by the Theorem we see that ϕ_2 is also absolutely continuous on $\mathbf{R}$ and

$$\phi_2(x) = \int_0^x \phi_2'(t)dt$$
$$= \frac{1+\sqrt{3}}{2}\sum_{k=0}^{\infty} 2^k \sum_{m=0}^{2^k-1} (\frac{1-\sqrt{3}}{4})^{\sigma_k(m)} (\frac{1+\sqrt{3}}{4})^{k-\sigma_k(m)} \int_0^x \chi_{[2^{-k}(m+1/2), 2^{-k}(m+1)[}(t)dt,$$
$$x \in [0,1].$$

(2). N=3. $q_3(0) = \frac{1}{4}(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})$, $q_3(1) = \frac{1}{2}(1-\sqrt{10})$, $q_2(2) = \frac{1}{4}(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})$,

$$C_3(0) = \frac{1}{16}(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}),$$
$$C_3(1) = \frac{1}{16}(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}),$$
$$C_3(2) = \frac{1}{16}(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}),$$
$$C_3(3) = \frac{1}{16}(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}),$$
$$C_3(4) = \frac{1}{16}(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})$$
$$C_3(5) = \frac{1}{16}(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}),$$
$$b_1 = \frac{1}{2}(5-\sqrt{5+2\sqrt{10}}),\ b_2 = \frac{1}{2}(15+\sqrt{10}-5\sqrt{5+2\sqrt{10}}),$$

$$\mathbf{A}_0 = \begin{bmatrix} c_0 & 0 \\ \\ c_2-6c_0 & c_1-3c_0 \end{bmatrix},\ \mathbf{A}_1 = \begin{bmatrix} c_1 & c_0 \\ \\ c_3-6c_1+8c_0 & c_2-3c_1+3c_0 \end{bmatrix}$$

where $c_k = C_3(k)$.

$$\begin{bmatrix} \phi_3(x+2) \\ \phi_3(x+3) \\ \phi_3(x+4) \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 8 & 3 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} \phi_3(x) \\ \\ \phi_3(x+1) \end{bmatrix} + \begin{bmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \end{bmatrix}, \qquad x \in [0,1],$$

$$p_1(x) = \frac{1}{2}x^2 + (\frac{7}{2}-b_1)x - \frac{7}{2}b_1 + \frac{1}{2}b_2 + 6,$$
$$p_2(x) = -x^2 + (2b_1-6)x + 6b_1 - b_2 - 8,$$
$$p_3(x) = \frac{1}{2}x^2 + (\frac{5}{2}-b_1)x - \frac{5}{2}b_1 + \frac{1}{2}b_2 + 3,$$

$$\begin{bmatrix} \phi_3(x) \\ \\ \phi_3(x+1) \end{bmatrix} = \mathbf{A}_{d_1(x)} \begin{bmatrix} \phi_3(\tau(x)) \\ \\ \phi_3(\tau(x)+1) \end{bmatrix} + \mathbf{P}_{d_1(x)}(x), \qquad x \in [0,1],$$

$$\mathbf{P}_0(x) = \begin{bmatrix} 0 \\ \\ c_0 p_1(2x) \end{bmatrix}, \qquad \mathbf{P}_1(x) = \begin{bmatrix} 0 \\ \\ c_1 p_1(2x-1) + c_0 p_2(2x-1) \end{bmatrix}.$$

Final Remark. As we have mentioned in §1, a common criticism on wavelet orthonormal bases is that one could not give their explicit representations except for Haar basis (see also [8]). Our representation for ϕ_2 and therefore for wavelet ψ_2 ($= -\frac{1+\sqrt{3}}{4}\phi_2(2x-1) + \frac{3+\sqrt{3}}{4}\phi_2(2x) - \frac{3-\sqrt{3}}{4}\phi_2(2x+1) + \frac{1-\sqrt{3}}{4}\phi_2(2x+2)$ by (1.1)) is then so far the first example among the non–Haar orthogonal wavelets which can be represented at least in explicit numerical series forms. The main methods used in this paper can be in fact also used to study more general scaling functions or refinable functions.

5. Appendix : Proof of Lemma 3

The proof given here is based on the following combinational identities:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{p-k}\binom{n}{k} = (-1)^{\lfloor \frac{1}{2}p \rfloor}\chi(p)\binom{n}{\lfloor \frac{1}{2}p \rfloor}, \tag{5.1}$$

$$\sum_{k=0}^{p}(-1)^k\binom{k+m}{k}\binom{n}{p-k} = \binom{n-m-1}{p}, \; n \ge m+1, \; p \ge 0, \tag{5.2}$$

where $n, m, p \in \mathbf{Z}, n, m \ge 0$, and $\chi(p) = 1$ for $0 \le \frac{1}{2}p \in \mathbf{Z}; \chi(p) = 0$, otherwise; $\lfloor x \rfloor$ denotes the largest integer not exceeding x . (5.1), (5.2) can be easily derived by comparing the coefficients of t^p in both sides of the following power series in $\mid t \mid <$ 1(using $(1+t)^{-m-1} = \sum_{k=0}^{\infty}\binom{k+m}{k}(-1)^k t^k$):

$$\sum_{s=0}^{\infty}\left[\sum_{k=0}^{s}(-1)^k\binom{n}{s-k}\binom{n}{k}\right]t^s = (1-t)^n(1+t)^n = \sum_{s=0}^{n}\binom{n}{s}(-1)^s t^{2s},$$

$$\sum_{s=0}^{\infty}\left[\sum_{k=0}^{s}(-1)^k\binom{k+m}{k}\binom{n}{s-k}\right]t^s = (1+t)^n(1+t)^{-m-1} = \sum_{s=0}^{n-m-1}\binom{n-m-1}{s}t^s.$$

Here and below we use again that $\binom{n}{m} = 0$ if $m < 0$ or $m > n$, and $q_N(k) = 0$ for all $k \notin [0, N-1], C_N(n) = 0$ for all $n \notin [0, 2N-1]$. From identity (1.2) we have

$$C_N(n) = 2^{-N+1}\sum_{k\in\mathbf{Z}} q_N(k)\binom{N}{n-k}, \qquad n \in \mathbf{Z}. \tag{5.3}$$

Now let $1 \le i, j \le N-1, d \in \{0,1\}$ and write $r = 2i - j - 1 + d$. Then by (2.4)–(2.7), (5.3), (5.1) and (5.2) we obtain

$$(\mathbf{Q}_d)_{i,j} = (\mathbf{B}\mathbf{T}_d\mathbf{B}^{-1})_{N+i,N+j}$$

$$
\begin{aligned}
&= \sum_{k=N+i}^{2N-1} \sum_{s=1}^{N+j} \binom{k-(N+i)+N-1}{N-1} C_N(2k-s-1+d)(-1)^{N+j-s} \binom{N}{N+j-s} \\
&= \sum_{k=0}^{N-1-i} \sum_{s=0}^{N} \binom{k+N-1}{N-1} C_N(N+2k+s+r)(-1)^s \binom{N}{s} \\
&= 2^{-N+1} \sum_{m\in\mathbf{Z}} q_N(m) \sum_{k=0}^{N-1-i} \binom{k+N-1}{k} \sum_{s=0}^{N} \binom{N}{m-2k-s-r} (-1)^s \binom{N}{s} \\
&= 2^{-N+1} \sum_{m\in\mathbf{Z}} q_N(m) \cdot \sum_{k=0}^{N-1-i} \binom{k+N-1}{k} (-1)^{\lfloor \frac{m-r}{2} \rfloor - k} \binom{N}{\lfloor \frac{m-r}{2} \rfloor - k} \chi(m-r-2k) \\
&= 2^{-N+1} \sum_{0\le \frac{m-r}{2} \in\mathbf{Z}} q_N(m) \sum_{k=0}^{N-1-i} \binom{k+N-1}{k} (-1)^{\frac{m-r}{2}-k} \binom{N}{\frac{m-r}{2}-k} \\
&= 2^{-N+1} \sum_{0\le \frac{m-r}{2} \in\mathbf{Z}} q_N(m)(-1)^{\frac{m-r}{2}} \binom{0}{\frac{m-r}{2}} = 2^{-N+1} q_N(r).
\end{aligned}
$$

This proves the lemma.

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