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MUSCL TYPE SCHEMES AND DISCRETE ENTROPY CONDITIONS*1)

N. Zhao
(*Department of Aerodynamics, Nanjing University of Aeronautics and Astronautics, Nanjing, China*)
H.M. Wu
(*State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Chinese Academy of Sciences, Beijing, China*)

Abstract

In this paper, the semi-discrete entropy conditions with so called the proper discrete entropy flux of a class of high resolution MUSCL type schemes are discussed for genuinely nonlinear scalar conservation laws. It is shown that the high resolution schemes satisfying such semi-discrete entropy conditions cannot preserve second order accuracy in the rarefaction region.

1. Introduction

Consider the hyperbolic conservation laws:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$
$$u(x,0)=u_0(x). \tag{1.1}$$

The research of numerical methods for equations (1.1) has been developed rapidly in this decade. Since the appearance of the concept of TVD(total variation diminishing) schemes, various high resolution schemes have been proposed[1,2,3,4] and successfully applied to computational fluid dynamics. It is well known that the convergence of the numerical methods for hyperbolic conservation laws depends on the entropy condition of the numerical solutions[5]. Previously the construction of difference schemes was always based on some kinds of total variation stability (TVD, TVB, and ENO etc.). In order to satisfy the entropy condition the constructed schemes are modified in such a way by introducing some quantities depending on the grid width[6]. Generally, the difference schemes for homogenious problems like (1.1) only depend on the grid ratio but independ on the grid width itself explicitly. So, it is meaningful to construct schemes satisfying the entropy condition without introducing quantities depending explicitly on the grid width. Merriam[7] and Sonar[8] put out the concept of the proper discrete entropy flux,

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that is, discretizating the entropy flux in such a proper way that the entropy condition can be satisfied and simultaneously the difference solution satisfies some kind of total variation stability. In [9], Zhao and Wu discussed the relationship between entropy conditions and high resolution schemes for 1-D scalar linear conservation laws, and obtained second order accurate TVD schemes using limiters. In [10], Zhao and Tang discuss the relationship between the discrete entropy conditions. The MmB property for linear scalar hyperbolic conservation laws in two dimensions are presented in [11].

In this paper, we discuss the accuracy of high resolution MUSCL type schemes and their semi-discrete entropy condition for 1-D genuinely nonlinear conservation laws. In section 2, we define the concept of the proper discrete entropy flux and discuss the entropy condition of three point monotone schemes. In section 3, the results of section 2 are generalized to five point MUSCL type schemes. The relationship between proper discrete entropy conditions and the TVD property of high resolution MUSCL type schemes is analyzed. Our main result is that the high resolution schemes of MUSCL type satisfying semi-discrete entropy conditions with proper discrete entropy flux cannot preserve second order accuracy in the region of rarefaction.

2. Three Point Monotone Schemes and Entropy

To begin with, let us consider the three point monotone semi-discrete schemes in conservative form

$$\frac{\partial}{\partial t}u_j + \frac{1}{\Delta x}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) = 0 \tag{2.1}$$

$$h_{j+\frac{1}{2}} = h(u_{j+1}, u_j) \quad , \quad h(u,u) = f(u) \tag{2.2}$$

where Δx is the variable meshsize in x-direction. The schemes (2.1), (2.2) is monotone if

$$\frac{\partial h(v,w)}{\partial v} \leq 0 \quad \text{and} \quad \frac{\partial h(v,w)}{\partial w} \geq 0\ . \tag{2.3}$$

As is well known that the weak solution of (1.1) is not unique. Let $U(u)$ be any convex function, the so-called entropy function, and corresponding function $F(u)$, the entropy flux satisfying $F'(u) = U'(u)f'(u)$. (U, F) is called an entropy pair. If the weak solution u of (1.1) satisfies the inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0 \tag{2.4}$$

in the sense of distribution for every entropy pair (U, F), then u is the unique physical solution of (1.1). The inequality (2.4) is called the entropy inequality (or the entropy condition).

Corresponding to the conservative scheme (2.1), the semi-discrete entropy inequality is defined as

$$\frac{\partial}{\partial t}U(u_j) + \frac{1}{\Delta x}(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \leq 0 \tag{2.5}$$

where the discrete entropy flux

$$H_{j+\frac{1}{2}} = H(u_{j+1}, u_j) \quad , \quad H(u,u) = F(u) \ . \tag{2.6}$$

From the scheme (2.1), the semi-discrete entropy inequality (2.5) can be rewritten as

$$-U'(u_j)(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) + (H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \leq 0. \tag{2.7}$$

The numerical entropy flux $H(.,.)$ for $F(.)$ is not unique. So, the entropy flux should be discretized properly in such a way that not only the discrete entropy flux satisfies the consistent condition (2.6), but also some kinds of nonlinear stability such as TVD or TVB take place. This kind of discrete form of the entropy flux is called the proper discrete entropy flux. For the conservative scheme (2.1), we define the proper entropy flux as follows.

Definition 2.1. *The discrete entropy flux $H(v,w)$ of the conservative scheme* (2.1) *is called proper, if*

$$\frac{\partial H(v,w)}{\partial v} = U'(v)\frac{\partial h(v,w)}{\partial v} \tag{2.8a}$$

and

$$\frac{\partial H(v,w)}{\partial w} = U'(w)\frac{\partial h(v,w)}{\partial w}. \tag{2.8b}$$

Here are some examples of proper discrete entropy fluxes. For the Lax-Friedrichs type scheme

$$h^{LF}(v,w) = \frac{1}{2}(f(v) + f(w)) - \frac{\alpha}{2\lambda}(v - w), \quad 0 < \alpha \leq 1 \tag{2.9}$$

we can use

$$H^{LF}(v,w) = \frac{1}{2}(F(v) + F(w)) - \frac{\alpha}{2\lambda}(U(v) - U(w)). \tag{2.10}$$

It is easy to see that $H^{LF}(v,w)$ is proper.

Another example is the Engquist-Osher scheme

$$h^{E-O}(v,w) = f(v) + \int_v^w X(s)f'(s)ds \tag{2.11}$$

where

$$X(s) = \begin{cases} 1 & \text{if } f'(s) > 0 \\ 0 & \text{if } f'(s) \leq 0. \end{cases}$$

In this case we define

$$H^{E-O}(v,w) = F(v) + \int_v^w X(s)F'(s)ds. \tag{2.12}$$

Then $H^{E-O}(v,w)$ is also proper.

Now, let us discuss the conditon (2.7) for the three point monotone conservative scheme (2.1),(2.2),(2.3) with the proper discrete entropy flux.

Omitting the superscript rewrite the condition (2.7) as

$$-U'(u_j)(h(u_{j+1},u_j)-h(u_j,u_{j-1}))+(H(u_{j+1},u_j)-H(u_j,u_{j-1}))\leq 0\ . \tag{2.13}$$

The left hand side of (2.13) can be reformed as a curve integral

$$LHS = \int_{(u_j,u_{j-1})}^{(u_{j+1},u_j)} \Big(\frac{\partial H(v,w)}{\partial v}-U'(u_j)\frac{\partial h(v,w)}{\partial v}\Big)\,dv+\Big(\frac{\partial H(v,w)}{\partial w}-U'(u_j)\frac{\partial h(v,w)}{\partial w}\Big)\,dw. \tag{2.14}$$

From the definition of the proper entropy flux, we have

$$\begin{aligned} LHS &= \int_{(u_j,u_{j-1})}^{(u_{j+1},u_j)} (U'(v)-U'(u_j))\frac{\partial h(v,w)}{\partial v}dv+(U'(w)-U'(u_j))\frac{\partial h(v,w)}{\partial w}dw \\ &= \int_{u_j}^{u_{j+1}} (U'(v)-U'(u_j))\frac{\partial h(v,u_j)}{\partial v}dv+\int_{u_{j-1}}^{u_j} (U'(w)-U'(u_j))\frac{\partial h(u_j,w)}{\partial w}dw. \end{aligned} \tag{2.15}$$

From the monotone assumption of the scheme (2.1),(2,2),(2.3) and the convexity of $U(u)$, it can be found that the inequality (2.13) holds. In other words, the three point monotone scheme (2.1),(2,2),(2.3) satisfies the semi-discrete entropy condition (2.7) for the proper discrete entropy flux.

3. High Resolution MUSCL Type Schemes and Entropy

In this section we derive a general entropy inequality for high resolution MUSCL type schemes. We obtain the restrictions on the flux limiter function Φ which are necessory for the proper discrete entropy condition to be satisfied, and then prove a conclusion that the high resolution schemes with the above properties can not preserve second order accuracy in the regions of rarefaction.

As is well known that monotone schemes have only first order accuracy. To convert the three point monotone scheme (2.1) into a second order accurate scheme, we should modify the variables of the numerical flux $h_{j+\frac{1}{2}}$, for instance, in the following way

$$\frac{\partial}{\partial t}u_j+\lambda(h(u_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}})-h(u_{j-\frac{1}{2}},\tilde{u}_{j-\frac{1}{2}}))=0 \tag{3.1}$$

where

$$u_{j+\frac{1}{2}}=u_{j+1}-\frac{1}{2}p_{j+1}\ \ ,\ \ \tilde{u}_{j+\frac{1}{2}}=u_j+\frac{1}{2}q_j \tag{3.2}$$

with

$$p_{j+1}=\Phi(r_{j+1})\Delta_+u_{j+1}\ \ ,\ \ q_j=\Phi(s_j)\Delta_+u_j$$

$$\Delta_+u_j=u_{j+1}-u_j\ ,\ r_j=\frac{\Delta_+u_{j-1}}{\Delta_+u_j}\ ,\ s_j=\frac{\Delta_+u_j}{\Delta_+u_{j-1}}\quad .$$

and $\Phi(x)$ is the limiter function. It is well known that the condition $\Phi(1) = 1$ is necessary for scheme (3.1) to be second order accurate in space in regions where the solution is smooth.

Now let us discuss the semi-discrete entropy condition (2.7) of the scheme (3.1). Using the modifications (3.1),(3.2), we convert (2.15) into

$$\int_{(u_{j-\frac{1}{2}},\tilde{u}_{j-\frac{1}{2}})}^{(u_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}})} \Big(\frac{\partial H(v,w)}{\partial v} - U'(u_j)\frac{\partial h(v,w)}{\partial v}\Big)dv + \Big(\frac{\partial H(v,w)}{\partial w} - U'(u_j)\frac{\partial h(v,w)}{\partial w}\Big)dw. \tag{3.3}$$

Being the proper discrete entropy flux, $H(v,w)$ satisfies (2.9). It leads (3.3) to the following integral

$$\int_{(u_{j-\frac{1}{2}},\tilde{u}_{j-\frac{1}{2}})}^{(u_{j+\frac{1}{2}},\tilde{u}_{j+\frac{1}{2}})} (U'(v) - U'(u_j))\frac{\partial h(v,w)}{\partial v}dv + (U'(w) - U'(u_j))\frac{\partial h(v,w)}{\partial w}dw$$

$$= \int_{u_{j-\frac{1}{2}}}^{u_{j+\frac{1}{2}}} (U'(v) - U'(u_j))\frac{\partial h(v,\tilde{u}_{j-\frac{1}{2}})}{\partial v}dv + \int_{\tilde{u}_{j-\frac{1}{2}}}^{\tilde{u}_{j+\frac{1}{2}}} (U'(w) - U'(u_j))\frac{\partial h(u_{j+\frac{1}{2}},w)}{\partial w}dw. \tag{3.4}$$

If the entropy inequality for (3.1),(3.2) is satisfied, at least one of the two integrals of the RHS of (3.4) must be nonpositive.

In this paper we discuss the Lax-Friedrichs type scheme with $\alpha = 1$. The general case can be considered in a similar manner. The numerical flux and entropy flux of the Lax-Friedrichs scheme are

$$h^{LF}(v,w) = \frac{1}{2}(f(v) + f(w)) - \frac{1}{2\lambda}(v - w) \tag{3.5}$$

and

$$H^{LF}(v,w) = \frac{1}{2}(F(v) + F(w)) - \frac{1}{2\lambda}(U(v) - U(w)) \tag{3.6}$$

where the CFL condition $\lambda|f'(v)| < 1$ is required. Let the flux function $f(u)$ be convex, $f''(u) > 0$. Consider the special entropy function $U(u)$, namely the square entropy $U(u) = \frac{u^2}{2}$, $U(u) = \frac{u^2}{2}$.

Suppose the first term of the RHS of (3.4) is nonpositive. This is equivalent to have

$$\int_{u_{j-\frac{1}{2}}}^{u_{j+\frac{1}{2}}} (v - u_j)(\lambda f'(v) - 1)dv \le 0. \tag{3.7}$$

Case 1. $u_j < u_{j+1}$, and $u_j < u_{j-1}$ (or $u_j > u_{j+1}$, and $u_j > u_{j-1}$).

This is the case of local extreme point. Take $\Phi(r_j) = 0$, then $u_{j-\frac{1}{2}} = u_j$. From the CFL condition, it can be found easily that the semi-discrete entropy condition (2.7) is always satisfied.

Case 2. $u_{j+1} \leq u_j \leq u_{j-1}$.

This is the case of compression (shock region). The inequality (3.7) converts into

$$(1-\lambda f'(\xi_1))\left(u_{j+\frac{1}{2}}-u_j\right)^2 \geq (1-\lambda f'(\xi_2))\left(u_j-u_{j-\frac{1}{2}}\right)^2 \tag{3.8}$$

where $\xi_1 \in (u_{j+\frac{1}{2}}, u_j)$ and $\xi_2 \in (u_j, u_{j-\frac{1}{2}})$. Note that

$$\begin{aligned} &u_{j+1} \leq u_{j+\frac{1}{2}} \leq u_j \\ &u_j \leq u_{j-\frac{1}{2}} \leq u_{j-1} \end{aligned} \tag{3.9}$$

provided the following condition

$$0 \leq \left\{\frac{\Phi(r)}{r}, \Phi(r)\right\} \leq 2 \tag{3.10}$$

is satisfied.

Let

$$\begin{aligned} &1-\lambda \max_u |f'(u)| = \alpha > 0\ ,\ M_1 = \max_u |f'(u)| \\ &\max_j |u_j| \leq M_0\ ,\ \max_u |f''(u)| \leq M_2. \end{aligned} \tag{3.11}$$

From the convexity of the flux function $f(u)$, (3.9) holds if

$$(u_{j+\frac{1}{2}}-u_j)^2 \geq (u_j-u_{j-\frac{1}{2}})^2. \tag{3.12}$$

From (3.2), the above inequality is identical with

$$\left(1-\frac{1}{2}\frac{\Phi(r_{j+1})}{r_{j+1}}\right)^2 (u_{j+1}-u_j)^2 \geq \left(\frac{\Phi(r_j)}{2}\right)^2 (u_{j+1}-u_j)^2.$$

Under the condition (3.10), it can be shown that the above inequality holds if the limiters $\Phi(r)$ satisfy

$$0 \leq \left\{\frac{\Phi(r)}{r}, \Phi(r)\right\} \leq 1\ . \tag{3.13}$$

Finally consider the last case.

Case 3. $u_{j+1} \geq u_j \geq u_{j-1}$.

This is the case of rarefaction.

In a similar way we can get the inequality (3.8), but with $\xi_1 \in (u_j, u_{j+\frac{1}{2}})$, $\xi_2 \in (u_{j-\frac{1}{2}}, u_j)$, and

$$u_{j-1} \leq u_{j-\frac{1}{2}} \leq u_j \leq u_{j+\frac{1}{2}} \leq u_{j+1} \tag{3.14}$$

under the assumption (3.10).

So, the inequality (3.9) is identical with

$$\begin{aligned} &(1-\lambda f'(\xi_1))(u_{j+\frac{1}{2}}-u_j)^2-(1-\lambda f'(\xi_2))(u_{j+\frac{1}{2}}-u_j)^2 \\ &\geq (1-\lambda f'(\xi_2))\left[(u_j-u_{j-\frac{1}{2}})^2-(u_{j+\frac{1}{2}}-u_j)^2\right]. \end{aligned}$$

That is

$$\lambda f''(\eta)(\xi_1-\xi_2)(u_{j+\frac{1}{2}}-u_i)^2 \le (1-\lambda f'(\xi_2))(u_{j+\frac{1}{2}}-u_{j-\frac{1}{2}})(u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}-2u_j)$$

where $\eta\in(\xi_2,\xi_1)$.

The above inequality holds if

$$\lambda M_2(u_{j+\frac{1}{2}}-u_j)^2 \le \alpha(u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}-2u_j).$$

From (3.2)

$$\lambda M_2\left(1-\frac{1}{2}\frac{\Phi(r_{j+1})}{r_{j+1}}\right)^2(u_{j+1}-u_j)^2 \le \alpha\left(1-\frac{1}{2}(\frac{\Phi(r_{j+1})}{r_{j+1}}+\Phi(r_j))\right)(u_{j+1}-u_j).$$

Denote $\frac{\Phi(r)}{r}=X$, and $0\le\Phi(r)\le 1$ and $0\le\frac{\Phi(r)}{r}\le X_0$, then we have

$$\lambda M_2(2-X)^2(u_{j+1}-u_j)\le 2\alpha(1-X). \tag{3.15}$$

If $u_{j+1}=u_j$, take $X_0=1$, otherwise,

$$\lambda M_2(u_{j+1}-u_j)X^2-(4\lambda M_2(u_{j+1}-u_j)-2\alpha)X+4\lambda M2(u_{j+1}-u_j)-2\alpha\le 0$$

i.e.

$$1^\circ \quad \alpha\ge 2\lambda M_2(u_{j+1}-u_j)$$

$$2^\circ \quad X\le\frac{2\lambda M_2(u_{j+1}-u_j)-\alpha+\sqrt{\alpha(\alpha-2\lambda M_2(u_{j+1}-u_j))}}{\lambda M_2(u_{j+1}-u_j)}. \tag{3.16}$$

So, (3.8) holds if

$$\lambda\le\frac{1}{M_1+\beta}\ ,\ 0\le\Phi(r)\le 1\ ,\ 0\le\frac{\Phi(r)}{r}\le X_0=\frac{2\sqrt{\delta-1}}{\sqrt{\delta}+\sqrt{\delta+1}} \tag{3.17}$$

where $\beta=4M_2M_0$, $\delta=\alpha/(\lambda\beta)\ge 1$, and $\alpha=1-\lambda M_1$.

Similarily, the second term of (3.4) is nonpositive if

$$\begin{cases} \lambda\le\frac{1}{M_1+\beta}\ ,\ 0\le\Phi(s)\le 1\ ,\ 0\le\frac{\Phi(s)}{s}\le\frac{2\delta-1}{2\delta}, & \text{if}\quad u_{j+1}>u_j>u_{j-1}\\ \lambda\le\frac{1}{M_1}\ ,\ \Phi(s_j)=\Phi(s_{j-1})=0 & \text{if}\quad u_j<u_{j+1},\quad u_j<u_{j-1}\\ & \ \text{or}\ u_j>u_{j+1},\quad u_j>u_{j-1}\\ \lambda\le\frac{1}{M_1}\ ,\ 0\le\left\{\Phi(s),\frac{\Phi(s)}{s}\right\}\le 1 & \text{if}\quad u_{j+1}\le u_j\le u_{j-1}. \end{cases} \tag{3.18}$$

The estimates (3.10),(3.13),(3.16),(3.17), and (3.18) give us the following theorem.

Theorem 3.1. *Let the flux function f(u) be convex, then, the scheme* (3.1) *with the Lax-Friedrichs type numerical flux satisfies the semi-discrete entropy condition with the proper discrete entropy flux, if*

i)

$$\lambda\le\frac{1}{M_1+\beta}$$

ii)

$$\begin{cases} \Phi(r_{j+1}) = \Phi(r_j) = \Phi(s_{j+1}) = \Phi(s_j) = 0, & \textit{if} \quad u_j < u_{j+1}, \quad u_j < u_{j-1} \\ & \textit{or}\ u_j > u_{j+1}, \quad u_j > u_{j-1} \\ 0 \le \left\{ \Phi(r), \frac{\Phi(r)}{r}, \Phi(s), \frac{\Phi(s)}{s}, \right\} \le 1, & \textit{if} \quad u_{j+1} \le u_j \le u_{j-1} \\ 0 \le \{\Phi(r), \Phi(s)\} \le 1 \ , \ 0 \le \begin{cases} \frac{\Phi(r)}{r} \le \frac{2\sqrt{\delta-1}}{\sqrt{\delta}+\sqrt{\delta-1}} \\ \frac{\Phi(s)}{s} \le \frac{2\delta-1}{2\delta} \end{cases} & \textit{if} \quad u_{j+1} > u_j > u_{j-1} \end{cases} \tag{3.19}$$

where $\beta = 4M_2M_0$, $\delta = \alpha/(\lambda\beta) \ge 1$, *and* $\alpha = 1 - \lambda M_1$.

The restrictions on the limiter functions Φ in Theorem 3.1 show that the following theorem is valid.

Theorem 3.2. *The scheme* (3.1) *with Lax-Friedrichs type numerical flux is TVD, if it satisfies the conditions in Theorem* 3.1*.*

The restrictions on $\frac{\Phi(r)}{r}$ and $\frac{\Phi(s)}{s}$ in (3.17), (3.18) or (3.19) show $\Phi(1)$ can not be equal to 1 in the case of rarefaction. Hence, we have the following main result.

Theorem 3.3. *Under the conditions in Theorem* 3.1*, the scheme* (3.1) *can not preserve the second order accuracy near the rarefaction region no matter whether it is TVD or not.*

Remarks: For the other kind of numerical flux, such as $h^{E-O}(v,w)$, we can obtain results similar to Theorem 3.1,Theorem 3.2, and Theorem 3.3.

4. Discussion

We have discussed the semi-discrete entropy conditions of the high resolution MUSCL type schemes obtained by using the theory of proper discrete entropy flux for genuinely nonlinear hyperbolic conservation laws. The way of discretizing the entropy condition seems quite natural, but unfortunately, our analysis shows that the high resolution MUSCL type schemes with Lax-Friedrichs type or Engquist-Osher numerical fluxes, which satisfy the entropy condition with the proper discrete entropy flux, can not preserve the second order accuracy. We emphasize that the above conclusion is also valid for fully-discrete high resolution schemes. To see this let us consider the fully discrete scheme

$$u_j^{n+1} = u_j^n - \lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \tag{4.1}$$

where $\lambda = \Delta t/\Delta x$ is the mesh ratio and Δt the time meshsize.

Corresponding to the scheme (2.8), the fully discrete entropy inequality has the form:

$$U(u_j^{n+1}) - U(u_j^n) + \lambda(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \le 0. \tag{4.2}$$

Using (4.1), (4.2) we have

$$\begin{aligned} U(u_j^{n+1}) - U(u_j^n) &= U'(u_j^n)(u_j^{n+1} - u_j^n) + \frac{U''(\theta)}{2}(u_j^{n+1} - u_j^n)^2 \\ &= -\lambda U'(u_j^n)(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) + \frac{\lambda^2 U''(\theta)}{2}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}})^2 \end{aligned}$$

where θ is located in between u_j^{n+1} and u_j^n. So, the inequality (4.1) can be converted into

$$-U'(u_j^n)(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) + H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}} + \frac{\lambda U''(\theta)}{2}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}})^2 \leq 0. \tag{4.3}$$

Thanks to the convexity of $U(u)$, we have

Theorem 4.1. *If the difference solution* $\{u_j^n\}$ *of the fully discrete scheme* (4.1) *satisfies the condition* (4.2), *then the semi-discrete entropy condition* (2.7) *must be satisfied.*

This theorem shows us the correctness of our statement.

It should be studied in future how to discretize the entropy condition and to design the high resolution schemes for genuinely nonlinear hyperbolic conservation laws in such a way that not only the discrete entropy condition can be satisfied by the numerical solution but also the nonlinear stability (such as TVD, MmB and ENO etc.) and the second order accuracy of the difference schemes remain.

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