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SUPERCONVERGENCE PHENOMENA ON THREE-DIMENSIONAL MESHES

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Abstract. We give an overview of superconvergence phenomena in the finite element method for solving three-dimensional problems, in particular, for elliptic boundary value problems of second order over uniform meshes. Some difficulties with superconvergence on tetrahedral meshes are presented as well. For a given positive integer m we prove that there is no tetrahedralization of R^3 whose all edges are m-valent.

Key Words. Linear and quadratic tetrahedral elements, acute partitions, Poisson equation, postprocessing, supercloseness, averaging and smoothing operators, regular polytopes, combinatorial topology.

1. Introduction

In 1966 Babuška, Práger, and Vitásek (see [2, Sect. 4.3]) developed a special finite difference method for the equation

$$-(pu')' + qu = f \text{ in } (0,1)$$

with mixed boundary conditions. Using the Marchuk identities and sophisticated numerical quadrature rules, they obtained a numerical scheme yielding the accuracy $\mathcal{O}(h^6)$ at nodal points. The associated system of linear algebraic equations has only a tridiagonal matrix (like for linear finite elements). In 1972 Douglas and Dupont (see [16]) called (for the first time) a similar high order accuracy phenomenon in the finite element method *superconvergence*. Some very early finite element superconvergence results from the period 1966–1969 are mentioned in [64, p. vi]. Surveys on other superconvergence phenomena can be found, e.g., in [10], [12], [13], [14], [27], [30], [31], [32], [42], [64], [71], [72]. Superconvergence is a useful tool in a posteriori error estimations, mesh refinement and adaptivity. At present, the total number of papers on superconvergence is about 1000.

A key assumption in proving many superconvergence phenomena is a high regularity of the exact solution and also some regular structure of the partitions used (uniform, piecewise uniform, locally quasiuniform, locally periodic, locally point-symmetric, self-similar etc.). Throughout this paper we shall use standard face-to-face partitions into elements in R^d, $d \in \{1, 2, 3, \dots\}$. Also we shall only consider

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regular families of partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$, i.e., there exists a constant $C > 0$ such that for all elements $T \in \mathcal{T}_h$ and all partitions $\mathcal{T}_h \in \mathcal{F}$ we have

$$\text{meas}_d T \geq C h_T^d,$$

where h_T is the diameter of T and meas_d stands for the d-dimensional Lebesgue measure.

In the next section we deal with superconvergence of linear elements on tetrahedral meshes for solving the Poisson equation with Dirichlet boundary conditions. The main idea is based on the fact that the gradient of the Ritz-Galerkin solution is superclose to the gradient of the Lagrange interpolation when uniform meshes are employed.

In the third section we recall some superconvergence phenomena for standard quadratic tetrahedral elements, which are frequently used in applications. These phenomena are based on some special properties of an important subclass of basis functions, namely piecewise quadratic bubble functions.

In the fourth section we give an overview of some other superconvergence phenomena for the finite element method in three-dimensional space. In particular, we present superconvergence results obtained for the solution and its gradient of second order boundary value problems of elliptic type when using rectangular trilinear and triquadratic elements, Serendipity elements, etc.

Finally, in the last section we present an unusual difficulty with superconvergence on three-dimensional meshes, which does not arise in solving two-dimensional problems. Namely, one can prove $\mathcal{O}(h^4)$-superconvergence at nodal points when solving the Poisson equation by linear elements over triangulations consisting solely of equilateral triangles. Such a result cannot be generalized to three-dimensional space, since the regular tetrahedron is not a space-filler.

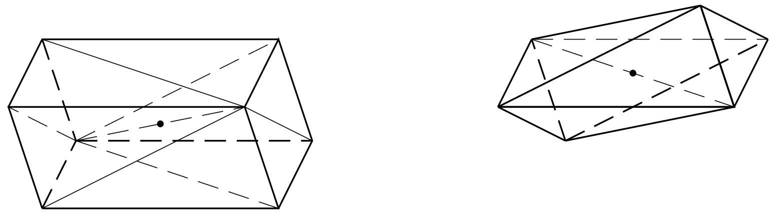
2. Superconvergence of linear elements on tetrahedral meshes

Let $\Omega \subset R^d$ be a bounded polytopic (polyhedral for $d = 3$) domain with Lipschitz boundary. We shall use the standard Sobolev space notation of norms and seminorms. For simplicity, let us consider only the Poisson equation

$$-\Delta u = f \quad \text{ in } \Omega \tag{1}$$

with homogeneous Dirichlet boundary conditions.

By a *tetrahedralization* (finite or infinite) we shall mean any face-to-face partition of a polyhedron or R^3 into closed tetrahedra. In this and the next section, we shall only consider so-called *uniform tetrahedralizations of* $\overline{\Omega}$, i.e., for each internal edge $\ell \not\subset \partial\Omega$ the patch of tetrahedra sharing ℓ is a point-symmetric set with respect to the midpoint M of ℓ (see Figure 1).



We shall look for $u_h \in V_h$ such that $(\nabla u_h, \nabla v_h)_0 = (f, v_h)_0$ for all $v_h \in V_h$, where $V_h \subset H_0^1(\Omega)$ is the space of continuous piecewise linear functions over a given tetrahedralization $\mathcal{T}_h$. In 1969 Oganesjan and Ruhovec (see [49]) proved for linear triangular elements over uniform partitions (i.e., when any two adjacent triangles form a parallelogram) a remarkable approximation theoretical phenomenon, namely that

$$|u_h - L_h u|_1 \le Ch^2 |u|_3, \tag{2}$$

where L_h is the standard linear Lagrange interpolation operator. Later, this phenomenon was called *supercloseness* (cf. [64]), since

$$|u - L_h u|_1 \le Ch|u|_2, \quad |u - u_h|_1 \le Ch|u|_2$$

are the optimal error estimates, which cannot be improved, in general. Note that the seminorm $|\cdot|_1$ is equivalent with the norm $\|\cdot\|_1$ due to the Friedrichs' inequality. Estimate (2) is at the basis of many superconvergence phenomena due to the following triangle inequality

$$|\nabla u - K_h \nabla u_h|_0 \le |\nabla u - K_h \nabla L_h u|_0 + |K_h \nabla (L_h u - u_h)|_0, \tag{3}$$

where K_h is a suitable postprocessing operator that makes both terms on the right-hand side of order $\mathcal{O}(h^2)$.

The first superconvergence result for linear tetrahedral elements was introduced by Chen [11] (see also [12, p. 90]). His averaging of gradients of linear tetrahedral elements at midpoints M of those edges (see the left part of Figure 1), which are surrounded by 6 tetrahedra $T_1, \dots, T_6$, gives

$$\frac{1}{6} \sum_{i=1}^{6} \nabla L_h u|_{T_i} = \nabla u(M) + \mathcal{O}(h^2).$$

The same formula holds when $L_h u$ is replaced by u_h (see [14, p. 278]).

Later, a similar result was derived independently by Kantchev and Lazarov [23] who proved that the tangential component of ∇u_h along edges is a superconvergent approximation to the tangential component of ∇u at midpoints of edges when solving (1) on a parallelepiped. Moreover, they compute the average of all constant vectors $\nabla u_h|_T$, where $T \in \mathcal{T}_h$ are tetrahedra incident with a given nodal point N. Note that each interior nodal point in a uniform tetrahedralization is surrounded by 24 tetrahedra. Setting

$$K_h \nabla u_h(N) = \frac{1}{24} \sum_{N \cap T \neq \emptyset} \nabla u_h|_T \tag{4}$$

for every interior nodal point N, Kantchev and Lazarov proved that the averaged gradient $K_h \nabla u_h$ of the finite element solution at nodal points has second order accuracy in the discrete L^2-norm (generalizing the result of [28] for $d = 2$ to $d = 3$). Using (4), we can for sufficiently small h uniquely define a continuous piecewise linear vector field $K_h \nabla u_h$ over a fixed domain $\Omega_0 \subset\subset \Omega$. Then we have

$$|\nabla u - K_h \nabla u_h|_{0,\Omega_0} \le Ch^2 |u|_{3,\Omega}. \tag{5}$$

This estimate was generalized by Goodsell [19] to the L^∞-norm and variable coefficients.

By the Sobolev imbedding theorem, if $s > d/2$ then $u \in H^s(\Omega)$ has a representation as a continuous function, and hence the nodal linear Lagrange interpolation $L_h u$ of u is well defined. According to [8, p. 503], for $s = 3$ if $d \le 5$ and $s > d/2$ if $d \ge 6$ we get

$$|u_h - L_h u|_1 \le Ch^2 |u|_3 \tag{6}$$

for uniform simplectic partitions (for their definition in the case $d \geq 4$ see also [8]). Using (3) and (6), estimate (5) was generalized to arbitrary dimension $d \in \{1, 2, 3, \dots\}$, i.e., we have (see [8])

$$|\nabla u - K_h \nabla u_h|_0 \leq Ch^2 |u|_3.$$

We meet the case $d \geq 4$ in problems of financial mathematics, particle and statistical physics, general relativity, etc.

Kantchev [22] generalized the above techniques for the gradient of the solution of a quasilinear boundary value problem over a three-dimensional bounded domain that can be partitioned into cubes. Each cube is then decomposed into 6 tetrahedra that share one spatial diagonal of the cube (cf. also [52]). Lazarov in [34] investigates superconvergence of the derivatives when solving a linear elasticity problem by linear tetrahedral elements. A survey of averaging schemes for the gradient recovery of linear tetrahedral elements is given by Goodsell and Whiteman [68].

Results from [55], [64] for locally point-symmetric meshes cover the above superconvergence phenomena for the gradient of linear tetrahedral elements (since any uniform tetrahedralization is locally point-symmetric with respect to midpoints of interior edges). For nonlinear problems see [64, Chapt. 9].

In [26] a weighted averaged gradient of linear simplicial elements was proposed in R^d. Its interpolation order in the L^q-norm is $\mathcal{O}(h^2)$ for $d < 2q$ and for general irregular simplicial partitions (see [21]). However, to get superconvergence of finite element solutions, quasiuniform triangulations are required [20].

Next we discuss a smoothing technique suggested by Vladimir A. Steklov. Let $v \in L^1(\Omega)$, let Ω be a rectangular bounded domain in R^d, and let $\Omega_0 \subset\subset \Omega$. The d-dimensional Steklov smoothing integral operator is given by the formula

$$(S_h v)(x) = \frac{1}{(2h)^d} \int_{D_h} v(x+y)\,\mathrm{d}y,$$

where $D_h = (-h, h) \times (-h, h) \times \cdots \times (-h, h)$ (d-times). For linear triangular elements Oganesjan and Ruhovec [50, p. 94 and 189] proved that

$$\|u - S_h u_h\|_{1,\Omega_0} \leq Ch^{3/2} \|u\|_{3,\Omega},$$

where $S_h u_h$ can be easily evaluated analytically, since u_h is a piecewise linear function over $\mathcal{T}_h$ consisting of right isosceles triangles. However, numerical tests showed that this estimate was not optimal and recently Kolman [25] derived that

$$\|u - S_h u_h\|_{1,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega}$$

for any dimension $d \in \{1, 2, 3, \dots\}$. For the numerical integration the 11 and 24 point formulae from [24] were employed.

3. Superconvergence of quadratic elements on tetrahedral meshes

In engineering society quadratic elements are even more popular than linear elements. Throughout this section u_h denotes the standard finite element solution using quadratic elements.

By Schatz [54, p. 246] at each vertex N of a given uniform tetrahedralization we have (see also [55] for locally point-symmetric meshes)

$$|(u - u_h)(N)| \leq C(u)h^4. \tag{7}$$

By [55] superconvergence of the same order holds also at midpoints of edges, since the uniform tetrahedralization is locally point symmetric with respect to the midpoints of edges (for $d = 2$ see [29, p. 97]).

In 1981 Zhu [70] (see also [72, p. 191]) proved supeconvergence of the gradient of quadratic triangular elements at the two Gaussian points of each edge. Later his result was extended into tetrahedral elements using estimate (7) and special properties of quadratic bubble functions. In particular, the following supercloseness result for derivatives holds (see [9])

$$|u_h - Q_h u|_1 \leq C(u)h^3, \tag{8}$$

where Q_h is the standard Lagrange quadratic interpolation operator. This theoretical bound has been numerically confirmed in [9], too, and extended for a second order elliptic equation with smooth variable coefficients. We also introduced a suitable postprocessing operator K_h which yields the following superconvergence

$$|\nabla u - K_h \nabla u_h|_0 \leq C(u)h^3.$$

In particular, sampling at the two Gaussian points of each edge leads to the superconvergence of the tangential component of the gradient along edges. A short note [51] reflects also the quadratic case, but unfortunately without any (reference to a) proof.

4. Other superconvergence results in the three-dimensional space

Superconvergence results mentioned in this section require some regular structure of the meshes used (like in Sections 2 and 3). This is specified in the references cited in the text.

The first superconvergence result for $d = 3$ goes back to 1978 when Zlámal in [73] proved higher order convergence of the gradient at Gaussian points of a three-dimensional isoparametric quadratic element of the Serendipity family having 20 degrees of freedom defined on incomplete quartic polynomials. The degrees of freedom correspond to vertices and midpoints of edges. Superconvergence at the Gaussian points of isoparametric block elements was numerically observed already in 1975 by Xie [69] for a linear elasticity problem. Similar results for problem (1) solved by trilinear, triquadratic, tricubic,... elements were later derived by Chen in [12] (see also Chen and Huang [14, p. 278]) by means of the orthogonal expansion method. In particular, sampling at centroids for trilinear elements leads to $\mathcal{O}(h^2)$-superconvergence of the gradient.

For superconvergence of u_h to u at the Lobatto points of block elements we refer to [14, p. 278]. This result is based on Bakker's classical result [4] for $d = 1$. Superconvergence of finite element solution at Lobatto points for d-dimensional rectangular Q_k-type elements for any $k \geq 1$ and any $d \geq 1$ was proved by Bo Li in [35]. Supercloseness when solving a nonlinear second order elliptic problem of nonmonotone type by rectangular elements was demonstrated by Liu et al. in [46]. This result can be directly extended to block elements. Superconvergence for triangular prismatic elements is mentioned in [12, p. 97] and [14, p. 278].

The Galerkin solution can be post-processed by a special convolution with a kernel proposed by Bramble and Schatz [6]. This technique yields interior $\mathcal{O}(h^{2k})$-superconvergence of function values. It was extended into several directions, namely to superconvergence of derivatives, to negative norm estimates and to parabolic problems (see [56], [59], [60], [62]).

According to [55] and [64], locally point-symmetric meshes yield superconvergence at nodal points for even degree polynomials and superconvergence of the gradient for odd degree polynomials (see also [63], [65]). Zlotnik in [75] examines superconvergence of the gradient when the right-hand side of (1) is discontinuous (see also [74]).

Lin and Yan in [42, p. 251] approximate problem (1) by rectangular block elements (possibly smoothly deformed). Using special integral identities, they prove supercloseness which gives superconvergence by means of an appropriate postprocessing. For an open problem concerning superconvergence of Raviart-Thomas mixed finite elements for a second order diffusion equation in three-dimensional space we refer to [17]. Schmid in [57] proves superconvergence of mixed and nonconforming hexahedral finite elements. Lin, Tobiska, and Zhou in [40] examine superconvergence of nonconforming low order finite elements applied to the Poisson equation (1).

Li and Chang in [36] derived superconvergence estimates for computation blending surfaces in three dimensions (see also [37], [38]). Lin and Yan in [43] deal with global superconvergence of mixed rectangular finite elements for Maxwell's equations in R^3 (see also [42]). For the superconvergence analysis a special technique based on an integral identity is developed (cf. also [39], [41]).

Superconvergence of a finite element method for three-dimensional stationary Stokes and Navier-Stokes problems is studied in [48]. Some superconvergence phenomena for the boundary element method applied to the three-dimensional stationary Stokes problem are mentioned in [66].

Thomée and Westergren [61] show in R^d that if a finite difference solution converges to u at certain rate as the mesh-size h tends to zero, then an appropriate difference quotient of the finite difference solution converges to the corresponding derivative of u at the same rate. See also [67] for some interior superconvergence-type estimates in R^d for the finite difference method and [62] for the finite element method.

5. Unusual difficulties with superconvergence for tetrahedral elements

In the previous sections we saw that many superconvergence phenomena which were proved in R^2 can be extended to R^3 or even to R^d for integers $d \geq 4$. Still, there may appear various difficulties. First of all note that Bo Li in [35] for simplicial P_k-type elements with $k > d > 1$ proved that the standard Lagrange interpolant and the finite element solution are not superclose in either the H^1- or the L^2-norm (cf. (6) and (8)). Other difficulty is the loss of orthogonality for $d = 3$, which is described in detail in [7].

The study of superconvergence by a computer-based approach developed by Babuška et al. [3] requires to examine harmonic polynomials in the plane. Note that dimension of harmonic polynomials of degree $k \in \{1, 2, \dots\}$ in two variables is only 2, whereas the dimension of harmonic polynomials in three variables is $2k+1$. This makes superconvergence analysis for $d = 3$ much more difficult (see [45]) than for $d = 2$. The likelihood of $2k + 1$ polynomial graphs passing through a common point is therefore, much smaller than the probability of two intersecting polynomial graphs.

Now we concentrate on a superconvergence phenomenon that holds in R^2, but cannot be generalized to R^3. Consider triangulations of a polygonal domain in R^2 consisting only of equilateral triangles. Then each interior node is surrounded by 6 congruent triangles. Linear triangular elements on such highly regular triangulations exhibit superconvergence (ultraconvergence) of order h^4 at nodal points when solving the Poisson equation (see [5], [44]).

Unfortunately, this strong and really beautiful result cannot be generalized into three-dimensional space, since R^3 cannot be decomposed into regular tetrahedra. Aristotle in his book *On the Heaven* (350 BC, Vol. 3, Chapt. 8) asserted that this is possible (cf. [1]) and that each edge is surrounded by 5 regular tetrahedra (which would require that the dihedral angle between two faces of the regular tetrahedron is 72°). Since Aristotle was such a respectable personage, this mistake remained unnoticed until the 16th century, when it was found that this angle is $\arccos\frac{1}{3}$, which is about 71°.

In this section we shall examine general structure of tetrahedralizations in detail. In Proposition 5.7 below we prove a surprising statement, namely, that there is no tetrahedralization of R^3 such that each edge is surrounded by exactly 5 tetrahedra (each of a different shape, in general). In Theorem 5.1 we state a more general result by means of combinatorial topology. Its proof will follow from Propositions 5.2, 5.5, 5.6, 5.7, and the fact that no edge can be surrounded by less than three tetrahedra.

Theorem 5.1. *Let m be an arbitrary fixed positive integer. Then there is no tetrahedralization of R^3 such that each edge is surrounded by m tetrahedra (i.e., all edges are m-valent).*

Using purely combinatorial arguments, we first prove a more simple statement:

Proposition 5.2. *There is no tetrahedralization of R^3 such that each edge is surrounded by at least 6 tetrahedra.*

Proof. Suppose, to the contrary, that such a tetrahedralization $\mathcal{T}$ exists and take an arbitrary nodal point N. Consider the polyhedron

$$P = \bigcup_{\substack{N\cap T\neq\emptyset \\ T\in\mathcal{T}}} T, \tag{9}$$

which is composed of all tetrahedra from $\mathcal{T}$ that share the vertex N. Triangular faces opposite to N form a triangulation of the surface ∂P of P. For this triangulation the well-known Euler formula holds:

$$v + t = e + 2, \tag{10}$$

where t is the number of triangles, e is the number of edges, and v is the number of vertices on ∂P. Since the surface is covered only by triangles, we have

$$2e = 3t \tag{11}$$

(which implies that the number of triangles on ∂P is always even). By (11) the Euler formula (10) reduces to the form

$$t = 2v - 4. \tag{12}$$

Let $v_i \geq 0$ be the number of vertices that are shared by exactly i triangles on the surface ∂P. Hence,

$$v = v_6 + v_7 + v_8 + \cdots + v_n, \tag{13}$$

where n is the maximum number of triangles around one vertex. Moreover, since each triangle on ∂P has three vertices, we have

$$3t = 6v_6 + 7v_7 + \cdots + nv_n.$$

From this, (12) and (13), we get

$$6(v_6 + v_7 + \cdots + v_n) - 12 = 6v_6 + 7v_7 + \cdots + nv_n, \tag{14}$$

which is a contradiction, since the expression on left-hand side is smaller than the one on the right-hand side. □

Below, we shall investigate tetrahedralizations, where all edges are surrounded by 3, or 4, or 5 tetrahedra. Then polyhedron (9) will be a tetrahedron, octahedron,

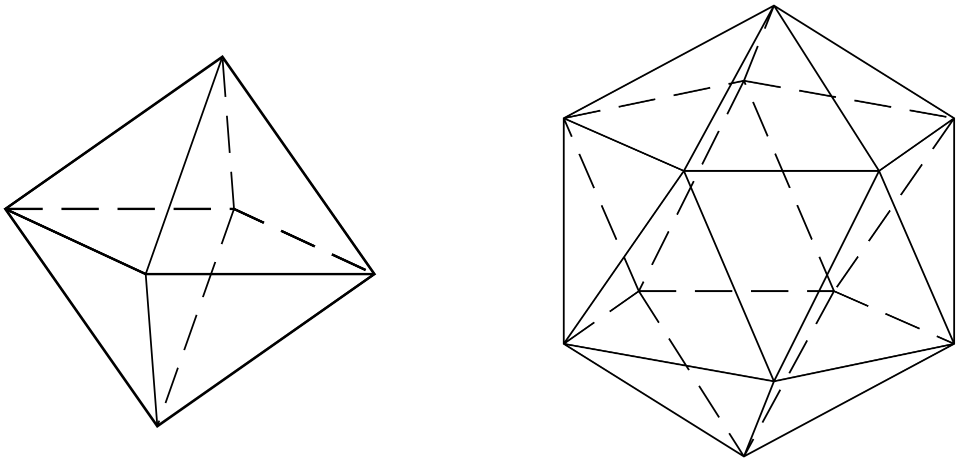


Fig. 2

or icosahedron, respectively (cf. Figure 2). In what follows, we keep the same notation v_i for the number of vertices that are shared by $i \in \{3, 4, \dots, n\}$ triangles, as in the proof of Proposition 5.2.

Lemma 5.3. *Let P be a simply connected polyhedron (i.e., with no handles), whose surface ∂P is connected (i.e., with no cavities in the interior of P). If ∂P is triangulated, then*

$$3v_3 + 2v_4 + v_5 = v_7 + 2v_8 + 3v_9 + \cdots + (n-6)v_n + 12. \tag{15}$$

Proof. Under the assumptions on P the Euler formula still holds, since the genus of ∂P is 0. Therefore, analogously to (14) we have

$$6(v_3 + v_4 + v_5 + \cdots + v_n) - 12 = 3v_3 + 4v_4 + 5v_5 + \cdots + nv_n,$$

and thus (15) follows. □

Corollary 5.4. *If $v_i = 0$ for all $i \notin \{5, 6\}$ then $v_5 = 12$.*

The proof follows directly from (15). A special case of Corollary 5.4 is illustrated on the right part of Figure 2.

Proposition 5.5. *There is no tetrahedralization of R^3 such that each edge is surrounded by* 3 *tetrahedra (i.e., with all edges* 3*-valent).*

Proof. Suppose, to the contrary, that such a tetrahedralization $\mathcal{T}$ exists and consider again the polyhedron P defined by (9). Then $v_i = 0$ for all integers $i > 3$ and by (15) we find that $v_3 = 4$. By (12), P is a tetrahedron subdivided into 4 tetrahedra from $\mathcal{T}$ that share the common vertex N. The interior node (which was chosen arbitrarily) is, therefore, 4-valent. However, the other vertices on the surface are 4-valent as well (each vertex of P shares three edges lying on the surface and one edge coming to N). Consequently, the tetrahedralization $\mathcal{T}$ is finite (with five 4-valent vertices only and ten edges, see the left part of Figure 3). Note the associated graph corresponds to a simplex in R^4 (the so-called 4-simplex). Thus, tetrahedra from $\mathcal{T}$ do not fill the whole space R^3, which is a contradiction. □

Proposition 5.6. *There is no tetrahedralization of R^3 such that each edge is surrounded by* 4 *tetrahedra (i.e., with all edges* 4*-valent).*

Proof. Suppose, to the contrary, that such a tetrahedralization $\mathcal{T}$ exists and consider again the polyhedron P defined by (9). Then $v_i = 0$ for all integers $i \neq 4$ and by (15) we obtain $v_4 = 6$. By (12) we find that P is an octahedron (see the left part of Figure 2). It is subdivided into 8 tetrahedra from $\mathcal{T}$ that share the common vertex N. The interior node N is, therefore, 6-valent and the valence of all the other nodes has to be the same, since N was chosen arbitrarily.

Now take an arbitrary tetrahedron T' from the tetrahedralization $\mathcal{T}$ and consider another polyhedron (see the right part of Figure 3).

$$P' = \bigcup_{\substack{T' \cap T \neq \emptyset \\ T \in \mathcal{T}}} T. \tag{16}$$

It is composed of all tetrahedra from $\mathcal{T}$ that have a point in common with T' (4 tetrahedra have just a common face with T', 6 tetrahedra have only a common edge with T', and 4 tetrahedra have only a common vertex with T'). Hence, this tetrahedralization is finite (with eight 6-valent vertices and 24 edges, see the right part of Figure 3). Note the associated graph corresponds to a 4-orthoplex (see [58]). The tetrahedralization $\mathcal{T}$ is again finite and thus, it does not fill the whole space R^3, which is a contradiction. □

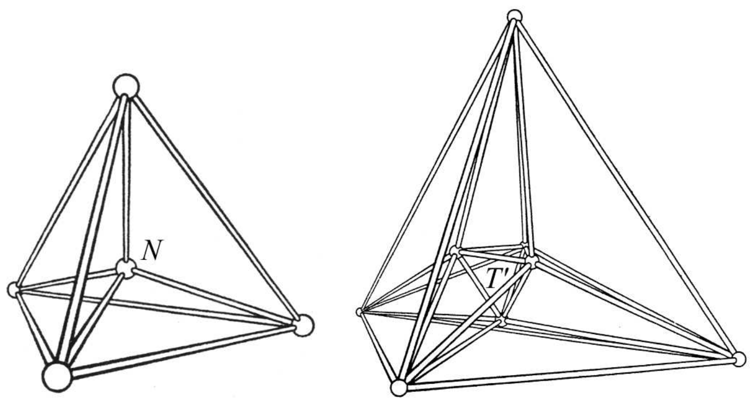


Fig. 3

The graphs of Figure 3 correspond to the regular 4-simplex and the regular 4-orthoplex (natural generalizations of the classical Platonic bodies into R^4). These are two of the three regular polytopes in R^4 whose three-dimensional faces (cells) are regular tetrahedra. The third body will be used in the next proposition.

Proposition 5.7. *There is no tetrahedralization of R^3 such that each edge is surrounded by* 5 *tetrahedra (i.e., with all edges* 5*-valent).*

Proof. Suppose, to the contrary, that such a tetrahedralization $\mathcal{T}$ exists and consider again the polyhedron P defined by (9). From Corollary 5.4 we deduce that P has exactly $v_5 = 12$ vertices, since $v_6 = 0$. Consequently, the valence of each node in the tetrahedralization is 12. Moreover, P has 20 triangular faces by (12) and thus, P is an icosahedron consisting of 20 tetrahedra that share a common point N.

According to [15], [58], there exists a special regular polytope in R^4, namely, the regular 600-cell. Before we establish the number of its edges, we first recall the general Euler-Poincaré formula (see, e.g., [47], [53]) for the three-dimensional surface of a convex polytope in R^4,

$$V + F = E + S, \tag{17}$$

where V, E, F, and S is the number of vertices, edges, faces, and polyhedra, respectively. In the case of the regular 600-cell, all its two-dimensional faces are equilateral triangles. All cells (polyhedra) from its three-dimensional surface are regular tetrahedra (i.e., regular 3-simplexes) and we denote this set by $\mathcal{C}$. The regular 600-cell has $V = 120$ vertices (as its dual is regular 120-cell, see [15] or [58] for associated graphs). Clearly,

$$2F = 4S \tag{18}$$

and thus, we have $F = 1200$. Now by (17) and (18) we get $E = V + S = 120 + 600 = 720$. This and the fact that any edge contains two vertices imply that each of the 120 vertices has valence 12 (like each node in the given tetrahedralization $\mathcal{T}$). Moreover, we see that $5E = 6S$, which means that each edge is shared by 5 tetrahedra from $\mathcal{C}$ (like in $\mathcal{T}$ again).

Now we shall recursively construct a continuous piece-wise linear mapping f from the compact three-dimensional surface of the regular 600-cell into R^3. Let C' be an arbitrary tetrahedron from the surface of the 600-cell and let $T' \in \mathcal{T}$ be arbitrary. Then there exists a linear affine one-to-one mapping $f|_{C'}$ that maps C' onto T' (vertices on vertices and edges on edges). For each neighbour C of C' contained in the set

$$Q' = \bigcup_{\substack{C' \cap C \neq \emptyset \\ C \in \mathcal{C}}} C \tag{19}$$

we can again define a linear affine one-to-one mapping $f|_C$ that maps C onto a tetrahedron T which is the corresponding neighbour of $T' \subset P'$ (where P' is defined similarly as in (16)), so that f remains continuous on interelement boundaries. The 20 regular tetrahedra that surround a given vertex of C' are thus continuously mapped on 20 tetrahedra that form the icosahedron P.

Further let

$$Q'' = \bigcup_{\substack{Q' \cap C \neq \emptyset \\ C \in \mathcal{C}}} C,$$

which is composed of all tetrahedra from $\mathcal{C}$ that have a common point with Q'. Similarly, we appropriately define f on neighbours of Q' and define Q''' consisting of all tetrahedra from $\mathcal{C}$ that have a common point with Q'', etc.

After a finite number of steps we obtain a continuous and piecewise linear function f that maps 600 cells from $\mathcal{C}$ onto 600 tetrahedra from $\mathcal{T}$. The "last" cell in this procedure is mapped on a large tetrahedron, which contains 599 tetrahedra from $\mathcal{T}$ (like in Figure 3, where left tetrahedron is composed from 4 tetrahedra and the right tetrahedron from 15 tetrahedra). Therefore, the associated graphs are isomorphic and thus finite, which is a contradiction. $\square$

Remark 5.8. Consider a uniform tetrahedralization (cf. Figure 1). Then all edges are 4-valent or 6-valent with ratio 3:4. The valence cannot be an odd number, since the associated patch of tetrahedra would not be point-symmetric. It is easy to see that the valence of each node is 14.

Remark 5.9. Recall that (see [33]) a tetrahedron is said to be *acute* if all six of its dihedral angles are acute (i.e., less than 90°). By [33, p. 162] all faces of an acute tetrahedron are acute triangles. A tetrahedralization is said to be *acute* if it contains only acute tetrahedra. It is obvious that no edge in an acute tetrahedralization can have the valence 3 or 4 and, moreover, by [33] each interior node has to be surrounded by at least 20 tetrahedra. In 2004 Eppstein, Sullivan, and Üngör found that there exist acute tetrahedralizations of R^3, such that every edge has valence 5 or 6, and no triangle has two 6-valent edges. In [18] they present several algorithms to accomplish this. An algorithm for an acute tetrahedralization of a slab is given as well. The valence of each node is at least 12. Due to Proposition 5.7, we have:

Corollary 5.10. *In any acute tetrahedralization of R^3 there exists a dihedral angle not greater than $60°$.*

Remark 5.11. *By Proposition 5.2, in any tetrahedralization of R^3 there exists a dihedral angle not less than $72°$. The acuteness assumption in Corollary 5.10 can be removed under the following hypothesis.*

Conjecture 5.12. *There is no tetrahedralization of R^3 such that each edge is surrounded by at most 5 tetrahedra.*

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References

[1] Du Ciel Aristote, text établi et traduit par Paul Moraux, Les Belles Lettres, Paris, 1965.
[2] I. Babuška, M. Práger and E. Vitásek, Numerical Processes in Differential Equations, John Wiley, New York, 1966.
[3] I. Babuška, T. Strouboulis, C. S. Upadhyay and S. K. Gangaraj, Computer-based proof of the existence of superconvergence points in the finite element method; superconvergence of the derivatives in finite element solutions of Laplace's, Poisson's, and the elasticity equations, Numer. Methods Partial Differential Equations, 12(1996), 347–392.
[4] M. Bakker, A note on C^0 Galerkin methods for two-point boundary problems, Numer. Math., 38(1981/82), 447–453.
[5] H. Blum, Q. Lin and R. Rannacher, Asymptotic error expansion and Richardson extrapolation for linear finite elements, Numer. Math., 49(1986), 11–38.
[6] J. H. Bramble and A. H. Schatz, Higher order local accuracy by averaging in the finite element method, Math. Comp., 31(1977), 94–111.
[7] J. Brandts and M. Křížek, History and future of superconvergence in three-dimensional finite element methods, Proc. Conf. Finite Element Methods: Three-dimensional Problems, Univ. of Jyväskylä, 2000, GAKUTO Internat. Ser. Math. Sci. Appl., vol. 15, Gakkotosho, Tokyo, 2001, 22–33.
[8] J. Brandts and M. Křížek, Gradient superconvergence on uniform simplicial partitions of polytopes, IMA J. Numer. Anal., 23(2003), 489–505.
[9] J. Brandts and Křížek, Superconvergence of quadratic tetrahedral elements, J. Comput. Math. 23(2005), No.1.
[10] C. Carstensen, Some remarks on the history and future of averaging techniques in a posteriori finite element error analysis, Z. Angew. Math. Mech., 84(2004), 3–21.
[11] C. M. Chen, Optimal points of stresses for tetrahedron linear element, Natur. Sci. J. Xiangtan Univ., 3(1980), 16–24.
[12] C. M. Chen, Finite Element Method and Its Analysis in Improving Accuracy, (Chinese), Hunan Science and Technology Press, Changsha, 1982.
[13] C. M. Chen, Structure Theory of Superconvergence of Finite Elements, (Chinese), Hunan Science and Technology Press, Changsha, 2001.
[14] C. M. Chen and Y. Q. Huang, High Accuracy Theory of Finite Element Methods, (Chinese), Hunan Science and Technology Press, Changsha, 1995.
[15] H. S. M. Coxeter, Introduction to geometry, New York, John Wiley, 1969.
[16] J. Douglas Jr. and T. Dupont, Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary problems, Topics in Numerical Analysis (Proc. Roy. Irish Acad. Conf., Univ. Coll., Dublin, 1972), Academic Press, London, 1973, 89–92.

[17] T. Dupont and P. T. Keenan, Superconvergence and postprocessing of fluxes from lowest-order mixed methods on triangles and tetrahedra, SIAM J. Sci. Comput., 19(1998), 1322-1332.

[18] D. Eppstein, J. M. Sullivan and A. Üngör, Tiling space and slabs with acute tetrahedra, Comput. Geom., 27(2004), 237–255.

[19] G. Goodsell, Pointwise superconvergence of the gradient for the linear tetrahedral element, Numer. Methods Partial Differential Equations, 10(1994), 651–666.

[20] I. Hlaváček and M. Křížek, Optimal interior and local error estimates of a recovered gradient of linear elements on nonuniform triangulations, J. Comput. Math., 14(1996), 345–362.

[21] I. Hlaváček, M. Křížek and V. Pištora, How to recover the gradient of linear elements on nonuniform triangulations, Appl. Math., 41(1996), 241–267.

[22] V. Kantchev, Superconvergent L^∞ error estimates for linear finite element approximations of quasilinear boundary value problems, C. R. Acad. Bulg. Sci., 46(1993), 25–28.

[23] V. Kantchev and R. D. Lazarov, Superconvergence of the gradient of linear finite elements for 3D Poisson equation, Proc. Internat. Conf. Optimal Algorithms, B. Sendov (ed.), Blagoevgrad, 1986, Izd. Bulg. Akad. Nauk, Sofia, 1986, 172–182.

[24] P. Keast, Moderate-degree tetrahedral quadrature formulas, Comput. Methods Appl. Mech. Engrg., 55(1986), 339–348.

[25] K. Kolman, A note on superconvergence by Steklov averaging in the finite element method, submitted in 2004, 1–9.

[26] M. Křížek, Higher order global accuracy of a weighted averaged gradient of the Courant elements on irregular meshes, Proc. Conf. Finite Element Methods: Fifty Years of the Courant Element, Univ. of Jyväskylä, 1993, Marcel Dekker, New York, 1994, 267–276.

[27] M. Křížek, Superconvergence phenomena in the finite element method, Comput. Methods Appl. Mech. Engrg., 116(1994), 157–163.

[28] M. Křížek and P. Neittaanmäki, Superconvergence phenomenon in the finite element method arising from averaging gradients, Numer. Math., 45(1984), 105–116.

[29] M. Křížek and P. Neittaanmäki, Finite Element Approximation of Variational Problems and Applications, Pitman Monographs and Surveys in Pure and Applied Mathematics vol. 50, Longman Scientific & Technical, Harlow, 1990.

[30] M. Křížek and Neittaanmäki, On superconvergence techniques, Acta Appl. Math., 9(1987), 175–198.

[31] M. Křížek and P. Neittaanmäki, Bibliography on superconvergence, Proc. Conf. Finite Element Methods: Superconvergence, Postprocessing and A Posteriori Estimates, Marcel Dekker, New York, 1998, 315–348.

[32] M. Křížek, P. Neittaanmäki and R. Stenberg (eds), Finite Element Methods: Superconvergence, Postprocessing, and A Posteriori Estimates, Proc. Conf., Univ. of Jyväskylä, 1996, LN in Pure and Appl. Math. vol. 196, Marcel Dekker, New York, 1998.

[33] M. Křížek and J. Šolc, Acute versus nonobtuse tetrahedralizations, In: Conjugate Gradient Algorithms and Finite Element Methods, Springer-Verlag, Berlin, 2004, 161–170.

[34] R. D. Lazarov, Superconvergence of the gradient for triangular and tetrahedral finite elements of a solution of linear problems in elasticity theory. (Russian), Computational Processes and Systems, No. 6, "Nauka", Moscow, 1988, 180–191.

[35] B. Li, Lagrange interpolation and finite element superconvergence, submitted in 2003, 1–25.

[36] Z. C. Li and C. S. Chang, Boundary penalty finite element methods for blending surfaces, III. Superconvergence and stability and examples, J. Comput. Appl. Math., 110(1999), 241–270.

[37] Z. C. Li and N. N. Yan, Global superconvergence for blending surfaces by boundary penalty plus hybrid FEMs, Computer Mathematics, Proc. of the 4th Asian Sympos. (ed. X. S. Gao et al.), World Scientific, Singapore, 2000, 61–85.

[38] Z. C. Li and N. N. Yan, Global superconvergence for blending surfaces by boundary penalty plus hybrid FEMs, Appl. Numer. Math., 39(2001), 61–85.

[39] Q. Lin, Tetrahedral or cubic mesh?, Proc. Conf. Finite Element Methods: Three-dimensional Problems, Univ. of Jyväskylä, 2000, GAKUTO Internat. Ser. Math. Sci. Appl., vol. 15, Gakkotosho, Tokyo, 2001, 160–182.

[40] Q. Lin, L. Tobiska and A. Zhou, On the superconvergence of nonconforming low order finite elements applied to the Poisson equation, Preprint 01-17, Fakultät für Mathematik, Otto-von-Guericke-Universität, Magdeburg, 2001.

[41] Q. Lin and N. N. Yan, A rectangle test in R^3, Proc. Syst. Sci. & Syst. Engrg., Great Wall Culture Publ. Co., Hongkong, 1991, 242–246.

[42] Q. Lin and N. N. Yan, The Construction and Analysis for Efficient Finite Elements, Hebei Univ. Publ. House, 1996.

[43] Q. Lin and N. N. Yan, Global superconvergence for Maxwell's equations, Math. Comp., 69(2000), 159–176.

[44] Q. Lin and J. P. Wang, Some expansions of the finite element approximation, Shuli Kexue, Mathematical Sciences. Research Reports IMS 15, Academia Sinica, Inst. of Math. Sciences, Chengdu, 1984, 1–11.

[45] R. Lin and Z. Zhang, Natural superconvergent points in 3D finite elements, Preprint Dept. of Math., Wayne State Univ., 2004, 1–30.

[46] L. Liu, T. Liu, M. Křížek, T. Lin and S. Zhang, Global superconvergence and a posteriori error estimators of finite element methods for a quasilinear elliptic boundary value problem of a nonmonotone type, accepted by SIAM J. Numer. Anal. in 2004, 1–15.

[47] L. A. Lyusternik, Convex Figures and Polyhedra, Dover Publications, Inc., New York, 1963, Moscow, 1956.

[48] G. Matthies and L. Tobiska, Superconvergence of a 3d finite element method for stationary Stokes and Navier-Stokes problems, Preprint 19/2003, Fakultät für Mathematik, Otto-von-Guericke-Universität, Magdeburg, 2003.

[49] L. A. Oganesjan and L. A. Ruhovec, An investigation of the rate of convergence of variational-difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary, Ž. Vyčisl. Mat. i Mat. Fiz., 9(1969), 1102-1120.

[50] L. A. Oganesjan and L. A. Ruhovec, Variational-difference Methods for the Solution of Elliptic Equations, Izd. Akad. Nauk Armjanskoj SSR, Jerevan, 1979.

[51] A. Pehlivanov, Superconvergence of the gradient for quadratic 3D simplex finite elements, Proc. Conf. Numer. Methods and Applications (Sofia, 1989), Publ. Bulgar. Acad. Sci., Sofia, 1989, 362–366.

[52] A. I. Pehlivanov, G. F. Carey and V. K. Kantchev, Finite element gradient superconvergence for nonlinear elliptic problems, TICAM Report 98-06, Univ. of Texas at Austin, 1998, 1–28.

[53] L. C. Pontrjagin, Osnovy kombinatornoj topologii, Nauka, Moscow, 1986.

[54] A. H. Schatz, Pointwise error estimates, superconvergence and extrapolation, Proc. Conf. Finite Element Methods: Superconvergence, Post-processing and A Posteriori Estimates, Marcel Dekker, New York, 1998, 237–247.

[55] A. H. Schatz, I. H. Sloan and L. B. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, SIAM J. Numer. Anal., 33(1996), 505–521.

[56] A. H. Schatz and L. B. Wahlbin, Interior maximum norm estimates for finite element methods, Math. Comp., 31(1977), 414–442.

[57] W. Schmid, Adaptive mixed and non-conformal finite element methods and applications to multigroup diffusion equations (German), Augsburger Mathematisch-Naturwissenschaftliche Schriften, 1996.

[58] J. Stillwell, The story of the 120-cell, Notices Amer. Math. Soc., 48(2001), 17–24.

[59] V. Thomée, High order local approximations to derivatives in the finite element method, Math. Comp., 31(1977), 652–600.

[60] V. Thomée, Negative norm estimates and superconvergence in Galerkin methods for parabolic problems, Math. Comp., 34(1980), 93–113.

[61] V. Thomée and B. Westergren, Elliptic difference equations and interior regularity, Numer. Math., 11(1968), 196–210.

[62] L. B. Wahlbin, Local behaviour in finite element methods, Handbook of Numer. Anal., vol. II., P.G. Ciarlet, J.L. Lions (eds), North-Holland, Amsterdam, 1991, 355–522.

[63] L. B. Wahlbin, Lecture Notes on Superconvergence in Galerkin Finite Element Methods, Cornell Univ., 1994, 1–243.

[64] L. B. Wahlbin, Superconvergence in Galerkin Finite Element Methods, LN in Math., vol. 1605, Springer, Berlin, 1995.

[65] L. B. Wahlbin, General principles of superconvergence in Galerkin finite element methods, Proc. Conf. Finite Element Methods: Superconvergence, Postprocessing and A Posteriori Estimates, Marcel Dekker, New York, 1998, 269–285

[66] H. Wang, The convergence of BEM for the stationary Stokes problem in three dimensions, Boundary elements (Beijing, 1986), Pergamon, Oxford-Elmsford, NY, 1986, 143–150.

[67] B. Westergren, Interior Estimates for Elliptic Systems of Difference Equations, Dept. of Math., Univ. of Göteborg, 1982.

[68] J. R. Whiteman and G. Goodsell, A survey of gradient superconvergence for finite element approximations to second order elliptic problems on triangular and tetrahedral meshes, The

mathematics of finite elements and applications, VII (Uxbridge, 1990), Academic Press, London, 1991, 55–74.

[69] G. Xie, Three-dimensional finite elements, (Chinese), Math. Pract. and Know., 1975, 28–41.

[70] Q. D. Zhu, The derivative good points for the finite element method with 2-degree triangular element, (Chinese), Natur. Sci. J. Xiangtan Univ., 4(1981), 36–45.

[71] Q. D. Zhu, A survey of superconvergence techniques in finite element methods, Proc. Conf. Finite Element Methods: Superconvergence, Postprocessing and A Posteriori Estimates, Marcel Dekker, New York, 1998, 287–302.

[72] Q. D. Zhu and Q. Lin, The Superconvergence Theory of the Finite Element Method, (Chinese), Hunan Science and Technology Publishing House, Changsha, 1989.

[73] M. Zlámal, Superconvergence and reduced integration in the finite element method, Math. Comp., 32(1978), 663–685.

[74] A. A. Zlotnik, Some finite-element and finite-difference methods for solving mathematical physics problems with non-smooth data in an n-dimensional cube, Part I, Soviet J. Numer. Anal. Math. Modeling, 6(1991), 421–451.

[75] A. A. Zlotnik, On superconvergence of a gradient for finite element methods for an elliptic equation with the nonsmooth right-hand side, Comput. Methods Appl. Math., 2(2002), 295–321.

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