Journal of Computational Mathematics, Vol.15, No.1, 1997, 23–35.

ON NONLINEAR GALERKIN APPROXIMATION*

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Abstract

Nonlinear Galerkin methods are numerical schemes adapted well to the long time integration of evolution partial differential equations. The aim of this paper is to discuss such schemes for reaction diffusion equations. The convergence results are proved.

1. Introduction

In order to solve the problem of long time integration of evolution partial differential equations, nonlinear Galerkin methods are introduced in recent years. Such methods stem from the theory of inertial manifolds and approximate inertial manifolds. We recall an inertial manifold is a finite dimensional smooth manifold which contains the global attractor and attracts every orbit at an exponential rate[1,2]. However, there are still many dissipative partial differential equations for which the existence of inertial manifolds is not known; there are even in some cases nonexistence results. These problems have lead to introduce the weak concept of approximate inertial manifolds. These manifolds are finite dimensional smooth manifolds such that all orbits enter their a thin neighborhood after a certain time. The existence of such manifolds can be found in Foias, Manley and Temam [3] ; Marion [4].

The algorithms which produce an approximate solution lying on an approximate inertial manifold are introduced by Marion and Temam[5]. They have been called nonlinear Galerkin methods, opposite to the usual Galerkin method which produces an approximate solution in the linear space spanned by the first m's functions of the Galerkin basis.

The improvenments of the nonlinear Galerkin methods over the usual Galerkin method are evidenced by theoretical results and numerical computations that show a significant gain in computing time[5,6,7]. In this paper, we study such nonlinear Galerkin methods for reaction diffusion equations, and prove the convergence rusults.

In section 1, we recall some known results for reaction diffusion equations, and introduce an approximate inertial manifold Σ, where Σ is first given by Wang[8]. Section 2 contains the nonlinear Galerkin methods based on Σ and our main results. The convergence results are obtained in this section.

* Received August 12, 1994.

2. An Approximate Inertial Manifold

We consider the following problem with a real valued function $u(x,t)$ defined on $R^+ \times \Omega$, where Ω denotes a regular bounded set of $R^n (n \le 4)$:

$$\frac{\partial u}{\partial t} - d\Delta u + g(u) = 0 \ , \qquad \text{in } R^+ \times \Omega \tag{2.1}$$

The equation is supplemented with the initial condition

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{2.2}$$

and one of the three following boundary conditions :

$$\begin{cases} \text{Dirichlet} & u = 0 \text{ on } \Gamma = \partial\Omega, \\ \text{Neumann} & \frac{\partial u}{\partial \gamma} = 0 \ \text{ on } \Gamma, \\ \text{Periodic} & \Omega = \prod_{i=1}^{n} (0, L_i) \text{ and } u \text{ is } \Omega \text{ periodic.} \end{cases} \tag{2.3}$$

where $d > 0$ is a diffusion coefficient. We assume that $g \in C^1(R,R)$ satisfies

$$g'(s) \ge -c_1, \qquad \forall\, s \in R \tag{2.4}$$

$$c_2|s|^k - c_4 \le g(s)\cdot s \le c_3|s|^k + c_4, \quad \forall\, s \in R, \tag{2.5}$$

where $k > 2$ is a integer and $c_i > 0$ is constant.
Let $Au = -d\Delta u + u$, then A is a unbounded self-adjoint positive operator on $H = L^2(\Omega)$ with domain

$$D(A) = \{u \in H^2(\Omega) : \text{u satisfies (2.3)}\}$$

Let $|\cdot|$ be the norm of H with scalar product $(\cdot,\cdot)$; and $\|\cdot\| = |A^{\frac{1}{2}}\cdot|$ be the norm of $V = D(A^{\frac{1}{2}})$ with scalar product $((\cdot))$. Denote by $|\cdot|_p$ the norm of $L^p(\Omega)$ for $1 \le p < \infty.(|\cdot|_2 = |\cdot|)$.

Since A^{-1} is compact, there exists an orthonormal basis of H consisting of eigenvectors w_j of A

$$Aw_j = \lambda_j w_j, \ j = 1, 2, \cdots$$

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to +\infty \text{ as } j \to +\infty.$$

Under assumptions (2.4) and (2.5), it follows from [4] that for u_0 given in H, the problem (2.1)-(2.3) possesses a unique solution u defined on R^+ such that

$$u \in C(R^+; H) \bigcap L^2(0,T;V), \forall T > 0.$$

Furthermore, if $u_0 \in V \cap L^k(\Omega)$, then

$$u \in C(R^+; H) \bigcap L^2(0,T;D(A)), \ \forall T > 0.$$

It is useful here to recall several time uniform estimates satisfied by the solution u of (2.1)-(2.3) borrowed from [4]. Let u_0 be given in a ball $B(0,R)$ of H of center 0 and of radius R. Then there exists a time t_0 depending the data (Ω,d,g) and R such that

$$\|u(t)\| \le M_0, |u(t)|_\infty \le M_0, \qquad \text{for } t \ge t_0 \tag{2.6}$$

where M_0 is a constant depending on the data (Ω,d,g).

Let $\phi : R^+ \to R$ be a C^∞ true cation function such that

$$\phi(s) = 1, \quad \text{for } 0 \le s \le 1; \quad \phi(s) = 0, \quad \text{for } \ s \ge 2.$$

Set $f(s) = \phi(\frac{s^2}{M_0^2})(g(s)-s)$ for all $s \in R$, then when $t \ge t_0$, u satisfies

$$\frac{\partial u}{\partial t} - d\Delta u + u + f(u) = 0.$$

This equation is rewrited as the following abstract differential equation in H

$$\frac{du}{dt} + Au + f(u) = 0. \tag{2.7}$$

Since we are only interested in the long time behaviours, we will consider from now on (2.7) instead of (2.1).

For all m, denote by $P_m : H \to \text{span}\{w_1, \cdots, w_m\}$ the projector and $Q_m = I - P_m$. In order to construct an approximate inertial manifold Σ, we introduce

$$A_m = \{y \in P_m H : \|y\| \le 2M_0\}$$
$$A_m^* = \{z \in Q_m H : \|z\| \le 2M_0\}$$

where M_0 is the constant in (2.6).

For $y \in A_m$ from [8] we know that the following implicit system possesses a unique solution $y_1 \in P_m H, z_1 \in Q_m H$ and $z \in A_m^*$:

$$y_1 + Ay + P_m f(y+z) = 0 \tag{2.8}$$
$$Az_1 + Q_m f'(y+z)y_1 = 0 \tag{2.9}$$
$$z_1 + Az + Q_m f(y+z) = 0. \tag{2.10}$$

Then we can define a mapping $\Phi : A_m \to A_m^*$ such that $\Phi(y) = z$ for all $y \in A_m$. Wang[8] proves that $\Sigma = \text{graph}(\Phi)$is an approximate inertial manifold of (2.1)-(2.3) such that every orbit enters its a thin neightborhood after a finite time, and the thickness of this neightborhood is bounded by $N(\frac{\lambda_1}{\lambda_{m+1}})^3$. Here and after, N denote any constant which depends on the data (Ω,d,g) and λ_1.

3. Nonlinear Galerkin Methods

In this section, we consider the nonlinear Galerkin methods associated to the approximate inertial manifold Σ. FOr $M > 0$, denote by

$$B_m = \{y \in P_m H : \|y\| \le M\}$$
$$B_m^* = \{z \in (P_{2m} - P_m)H : \|z\| \le M\}.$$

For $y \in B_m$, we introduce the following system which is the discretizations of (2.8)-(2.10).

$$y_1 + Ay + P_m f(y+z) = 0 \tag{3.1}$$
$$Az_1 + (P_{2m} - P_m) f'(y+z) y_1 = 0 \tag{3.2}$$
$$z_1 + Az + (P_{2m} - P_m) f(y+z) = 0 \tag{3.3}$$

where the knows y_1, z_1 and z are such that

$$y_1 \in P_m H, z_1 \in (P_{2m} - P_m)H, \quad z \in B_m^*.$$

Obviously, the system (3.1)-(3.3) can reduce to an implicit equation for z. We first have

Lemma 3.1. *FOr all $M > 0$, there exists a m_0 depending on the data (Ω, d, g) and M such that when $m \geq m_0$, for all $y \in B_m$,* (3.1)-(3.3) *have a unique solution $y_1(y) \in P_m H, z_1(y) \in (P_{2m} - P_m)H$, and $z(y) \in B_m^*$.*

Proof. We show this lemma by a fixed point argument. Let $y \in B_m$ be fixed, we define a mapping $F : B_m^* \to (P_{2m} - P_m)H$, as follows. For $w \in B_m^*, z = F(w)$ is determined by the resolution of the equations

$$y_1 + Ay + P_m f(y+w) = 0 \tag{3.4}$$
$$Az_1 + (P_{2m} - P_m) f'(y+w) y_1 = 0 \tag{3.5}$$
$$z_1 + Az + (P_{2m} - P_m) f(y+w) = 0 \tag{3.6}$$

which give successively $y_1 \in P_m H, z_1, z \in (P_{2m} - P_m)H$.

Clearly, any fixed point of F is a solution of (3.1)-(3.3).

(i) F maps B_m^* into B_m^* for m sufficiently large.

By (3.4) we find that

$$\begin{aligned} |y_1| &\leq |Ay| + |f(y+w)| \\ &\leq \lambda_m^{\frac{1}{2}} \|y\| + N \ \ (\text{by } |Ay| \geq \lambda_m^{\frac{1}{2}} \|y\| \text{ and } |f(s)| \leq N \text{ for } s \in R) \\ &\leq M \lambda_{m+1}^{\frac{1}{2}} + N \ \ (\text{by } y \in B_m \text{ and } \lambda_{m+1} \geq \lambda_m). \end{aligned} \tag{3.7}$$

And thus it comes from (3.5) that

$$\begin{aligned} |Az_1| &\leq |f'(y+w) y_1| \\ &\leq N |y_1| \ \ (\text{by } |f'(s)| \leq N \text{ for } s \in R) \\ &\leq NM \lambda_{m+1}^{\frac{1}{2}} + N \ \ (\text{by}(3.7)). \end{aligned}$$

Due to $|Az_1| \geq \lambda_{m+1} |z_1|$ we infer that

$$|z_1| \leq NM \lambda_{m+1}^{-\frac{1}{2}} + N \lambda_{m+1}^{-1}. \tag{3.8}$$

It follows from (3.6) that

$$\begin{aligned} |Az| &\leq |z_1| + |f(y+w)| \\ &\leq NM \lambda_{m+1}^{-\frac{1}{2}} + N \lambda_{m+1}^{-1} + N \ \ (\text{by (3.8) and } |f(s)| \leq N). \end{aligned}$$

By $|Az| \geq \lambda_{m+1}^{\frac{1}{2}} \|z$, the above implies

$$\|z\| \leq (NM\lambda_{m+1}^{-\frac{1}{2}} + N)\lambda_{m+1}^{-\frac{1}{2}}.$$

Taking into account $\lambda_{m+1} \to +\infty$ as $m \to +\infty$, we deduce that there exists m_1 depending on the data and M such that when $m \geq m_1$

$$\|z \leq M$$

this shows when $m \geq m_1$, F maps B_m^* into itself.
(ii) F is a contraction.

Let $w_1, w_2 \in B_m^*$. In view of (3.4) we find that

$$\begin{aligned} |y_1(w_1) - y_1(w_2)| &\leq |f(y + w_1) - f(y + w_2)| \\ &\leq N|w_1 - w_2| \ \ (\text{by}|f'(s)| \leq N \text{ for } s \in R). \end{aligned} \tag{3.9}$$

Due to (3.5)

$$\begin{aligned} &|Az_1(w_1) - Az_1(w_2)| \leq |f'(y + w_1)y_1(w_1) - f'(y + w_2)y_1(w_2)| \\ \leq &|(f'(y + w_1) - f'(y + w_2))y_1(w_1)| + |f'(y + w_2)(y_1(w_1) - y_1(w_2))| \\ \leq &N|(w_1 - w_2)y_1(w_1)| + N|y_1(w_1) - y_1(w_2)| \\ &\text{by } |f"(s)| \leq N \text{ and } |f'(s)| \leq N \text{ forall } s \in R) \\ \leq &N|(w_1 - w_2)|_{L^4} \cdot |y_1(w_1)|_{L^4} + N|y_1(w_1) - y_1(w_2)| \ \ (\text{by Holder inequality}) \\ \leq &N|(w_1 - w_2)|_{H^1} \cdot |y_1(w_1)|_{H^1} + N|y_1(w_1) - y_1(w_2)| \ \ (\text{by } H^1(\Omega) \subset L^4(\Omega)) \\ \leq &N\|(w_1 - w_2)\|\|y_1(w_1)\| + N|y_1(w_1) - y_1(w_2)| \\ &(\text{by } H^1 - \text{norm is equivalent to the norm}\| \cdot \|) \\ \leq &N\lambda_{m+1}^{\frac{1}{2}}\|(w_1 - w_2)\| \cdot |y_1(w_1)| + N|y_1(w_1) - y_1(w_2)| \\ &(\text{by } \|y_1(w_1)\| \leq \lambda_m^{\frac{1}{2}}|y_1(w_1)| \leq \lambda_{m+1}^{\frac{1}{2}}|y_1(w_1)|) \\ \leq &N(N + M\lambda_{m+1}^{\frac{1}{2}})\lambda_{m+1}^{\frac{1}{2}}\|(w_1 - w_2)\| + |w_1 - w_2| \ \ (\text{by (3.7) and (3.9)}). \end{aligned}$$

Because of

$$|Az_1(w_1) - Az_1(w_2)| \geq \lambda_{m+1}|z_1(w_1) - z_1(w_2)|$$

we find that the above implies

$$|z_1(w_1) - z_1(w_2)| \leq N(N\lambda_{m+1}^{-\frac{1}{2}} + M)\|(w_1 - w_2)\| + N\lambda_{m+1}^{-1}|w_1 - w_2|. \tag{3.10}$$

Taking into account (3.6)

$$\begin{aligned} &|Az_1(w_1) - Az_1(w_2)| \leq |z_1(w_1) - z_1(w_2)| + |f(y + w_1) - f(y + w_2)| \\ \leq &N(N\lambda_{m+1}^{-\frac{1}{2}} + M)\|w_1 - w_2\| + N\lambda_{m+1}^{-1}|w_1 - w_2| + N|w_1 - w_2| \\ &(\text{by (3.10) and } |f'(s)| \leq N) \\ \leq &(N + NM)\|w_1 - w_2\| \ \ (\text{by } |w_1 - w_2| \leq \lambda_1^{-\frac{1}{2}}\|w_1 - w_2\| \text{ and } \lambda_{m+1} \geq \lambda_1). \end{aligned}$$

Due to

$$|Az_1(w_1) - Az_1(w_2)| \geq \lambda_{m+1}^{\frac{1}{2}} \|z(w_1) - z(w_2)\|$$

then we see that

$$\|z(w_1) - z(w_2)\| \leq (N + NM)\lambda_{m+1}^{-\frac{1}{2}} \|w_1 - w_2\|.$$

For $\lambda_{m+1} \to +\infty$, there exists a m_2 depending on the data and M such that: when $m \geq m_2$, F is a contraction.

Let $m_0 = \max\{m_1, m_2\}$, then we deduce that when $m \geq m_0$, F has a unique fixed point in B_m^*, thus we complete the proof of Lemma 3.1.

The nonlinear Galerkin methods based on Σ consists in looking for

$$u_m = y_m + z_m, y_m \in P_m H, z_m \in (P_{2m} - P_m)H$$

such that

$$\frac{dy_m}{dt} + Ay_m + P_m f(y_m + z_m) = 0, \tag{3.11}$$

where z_m is given by

$$y_{1,m} + Ay_m + P_m f(y_m + z_m) = 0 \tag{3.12}$$

$$Az_{1,m} + (P_{2m} - P_m) f'(y_m + z_m) y_{1,m} = 0 \tag{3.13}$$

$$z_{1,m} + Az_m + (P_{2m} - P_m) f(y_m + z_m) = 0 \tag{3.14}$$

$$y_m(0) = P_m u_0. \tag{3.15}$$

By Lemma 3.1 and the general theorems on ordinary differential equations, for any $M > \|u_0\|$ and $m \geq m_0$, (3.11)-(3.15) possess a unique maximal solution $y_m(t)$ defined on some interval $(0, T_m)$. Our aim in the sequel is to show that, for a convenient value of M, $T_m = +\infty$, i.e. (3.11) has a solution $y_m(t)$ on R^+. Furthermore, we prove that u_m converges to the solution u of (2.1)-(2.3) as $m \to +\infty$. This is stated in

Theorem 3.1. *Let $g \in C^2(R, R)$ such that (2.4) and (2.5) hold. $\Omega \subset R^n$ and $n \leq 4$. If $u_0 \in V \cap L^k(\Omega), u(t)$ is the solution of (2.7), (1,2) and (1,3), then there exists a constant $M_1$and m_1 depending on the data and u_0 such that when $m \geq m_1$ (3.11)-(3.15) have a unique solution y_m defined on R^+ with*

$$\|y_m\| \leq M_1, \|z_m\| \leq M_1; \tag{3.16}$$

When $m \to +\infty$, $u_m \to u$ in $L^\infty(R^+; V)$ weak star, in $L^2(0, T; D(A))$ and $L^p(0, T; V)$ strongly for all $T > 0$ and $1 \leq p < \infty$; (3.17)

$y_m \to u$ in $L^\infty(R^+; V)$ weak star, in $L^2(0, t; D(A))$ and $L^p(0, t; V)$ strongly for all $T > 0$ and $1 \leq p < \infty$; (3.18)

$z_m \to 0$ in $L^\infty(R^+; V)$ strongly and $L^2(0, t; D(A))$ for all $T > 0$. (3.19)

Proof. The proof relies on a priori estimates on the solutions of (3.11)-(3.15). The precise choice of M_1 will be done later. For the moment, let M be any constant with $M > \|u_0\|a$ and $m \geq m_0$ so that (3.11) has a unique solution $y_m(t)$ on a maximal interval $(0, T_m)$.

Multiply (3.11) by Ay_m in H to obtain

$$\frac{1}{2}\frac{d}{dt}\|y_m\|^2 + |Ay_m|^2 = -(f(y_m + z_m), Ay_m).$$

Due to

$$|(f(y_m + z_m), Ay_m)| \le N|Ay_m| \ \ (by\ |f(s)| \le N) \le \frac{1}{2}|Ay_m|^2 + N$$

then

$$\frac{d}{dt}\|y_m\|^2 + |Ay_m|^2 \le N. \tag{3.20}$$

And thus

$$\frac{d}{dt}\|y_m\|^2 + \lambda_1\|y_m\|^2 \le N \ \ (by\ |Ay_m|^2 \ge \lambda_1\|y_m\|^2).$$

It comes from Gronwall Lemma that

$$\begin{aligned}\|y_m(t)\|^2 &\le \|y_m(0)\|^2 \exp(-\lambda_1 t) + N \\ &\le \|y_m(0)\|^2 + N \le \|u_0\|^2 + N, \quad t \in (0, T_m).\end{aligned}$$

This shows that there exists a constant M_1 depending on the data and u_0 such that

$$\|y_m(t)\| \le M_1, \ \forall\, t \in (0, T_m). \tag{3.21}$$

Then for $M = 2M_1$, we can easily deduce that the corresponding solution $y_m(t)$ of (3.11) is defined for all $t \ge 0$. And thus (3.16) is proved. (3.21) along with $T_m = +\infty$ shows that

$$y_m \text{ is bounded in } L^\infty(R^+; V). \tag{3.22}$$

Next, coming back to (3.20) that we integrate between 0 and T, we see that

$$y_m \text{ is bounded in } L^2(0, T; D(A)) \text{ for all } T > 0. \tag{3.23}$$

By (3.11) we find that

$$\begin{aligned}|\frac{dy_m}{dt}| &\le |Ay_m| + |f(y_m + z_m)| \\ &\le |Ay_m| + N, \ \ (\text{by } |f(s)| \le N, \forall\, s \in R).\end{aligned}$$

Above formula together with (3.23) implies that

$$\frac{dy_m}{dt} \text{ is bounded in } L^2(0, T; D(A)) \text{ for all } T > 0. \tag{3.24}$$

Similar to the proof of Lemma 3.1 (i), from (3.12)-(3.14) we can easily verify that there exists a N depending on the data, λ_1 and u_0 such taht

$$|Az_m| \le N$$

and

$$\|z_m\| \le N\lambda_{m+1}^{-\frac{1}{2}}.$$

In view of $\lambda_{m+1} \to \infty$ as $m \to \infty$, we infer from the above that

$$z_m \to 0 \text{ in } L^\infty(R^+;V) \text{ and } L^\infty(R^+;H) \text{ strongly,} \tag{3.25}$$

$$z_m \text{ is bounded in } L^2(0,T;D(A)) \text{ for all } T > 0. \tag{3.26}$$

Using (3.22)-(3.26) we know that there exists u^* and a subsequence, still denoted by y_m, such that

$$u^* \in L^\infty(R^+;V) \bigcap L^2(0,T;D(A)), \forall T > 0.$$

and

$$y_m \to u^* \text{ in } L^\infty(R^+;V) \text{ weak star, in } L^2(0,T;D(A)) \text{ weakly}, \forall T > 0, \tag{3.27}$$

$$\frac{dy_m}{dt} \to \frac{du^*}{dt} \text{ in } L^2(0,T;D(A)) \text{ weakly}, \forall T > 0, \tag{3.28}$$

$$z_m \to 0 \; in \; L^2(0,T;D(A)) \text{ weakly}, \forall T > 0. \tag{3.29}$$

Due to a classical compactness theorem[9], it follows from (3.27) and (3.28) that

$$y_m \to u^* \text{ in } L^2(0,T;V) \text{ strongly.} \tag{3.30}$$

In the following, we prove $u^* = u$ is the solution of (2.7), (2.2) and (2.3). By (3.11) we infer that $\forall w_j$ with $j \leq m$

$$\frac{d}{dt}(y_m, w_j) + ((y_m, w_j)) + (f(y_m + z_m), w_j) = 0. \tag{3.31}$$

Let ϕ be a continuously differentiable function on $[0,T]$ with $\phi(T) = 0$. We multiply (3.31) by ϕ, and then integrate by parts. This leads to

$$\begin{aligned} &-\int_0^T (y_m, \phi'(t) w_j) dt + \int_0^T ((y_m, \phi(t) w_j)) dt \\ &= (y_m(0), w_j)\phi(0) - \int_0^T (f(y_m + z_m), \phi(t) w_j) dt. \end{aligned} \tag{3.32}$$

By (3.27) we obtain

$$\int_0^T (y_m, \phi'(t) w_j) dt \to \int_0^T (u^*, w_j)\phi'(t) dt \tag{3.33}$$

$$\int_0^T ((y_m, \phi(t) w_j)) dt \to \int_0^T ((u^*, w_j))\phi(t) dt \tag{3.34}$$

$$\begin{aligned} &|\int_0^T (f(y_m + z_m), w_j)\phi(t) dt - \int_0^T (f(u^*), w_j)\phi(t) dt| \\ \leq &\int_0^T |f(y_m + z_m) - f(u^*)||\phi(t) w_j| dt \\ \leq &N \int_0^T |y_m + z_m - u^*||\phi(t) w_j| dt \quad (\text{by} |f'(s)| \leq N, \forall s \in R) \\ \leq &N |y_m + z_m - u^*|_{L^2(0,T;H)} \times |\phi(t) w_j|_{L^2(0,T;H)} \quad (\text{by Holder inequality}) \end{aligned}$$

(3.30) and (3.25) imply that $y_m \to u^*$ and $z_m \to 0$ in $L^2(0,T;H)$ strongly. And then we have

$$\int_0^T (f(y_m + z_m), w_j)\phi(t)dt \to \int_0^T (f(u^*), w_j)\phi(t)dt \tag{3.35}$$

Thanks to (3.33)-(3.35) and $y_m(0) = P_m u_0 \to u(0)$, pass to the limit in (3.32)

$$\begin{aligned} &-\int_0^T (u^*, w_j)\phi'(t)dt + \int_0^T ((u^*, w_j))\phi(t)dt \\ =&(u(0), w_j)\phi(0) - \int_0^T (f(u^*), w_j)\phi(t)dt. \end{aligned}$$

Clearly, by linearity, this equality holds for any finite linear combination of w_j. Since each term of this equality is continuous in V, this equality is still valid, by continuity, for all $v \in V$, i.e.

$$\begin{aligned} &-\int_0^T (u^*, v)\phi'(t)dt + \int_0^T ((u^*, v))\phi(t)dt \\ =&(u(0), v)\phi(0) - \int_0^T (f(u^*), v)\phi(t)dt, \qquad \forall\, v \in V. \end{aligned} \tag{3.36}$$

Now writing in particular (3.36) with $\phi \in C_0^\infty(0,T)$, we find that the following equality which is valid in the distribution sense on $(0,T)$

$$\frac{d}{dt}(u^*, v) + ((u^*, v)) + (f(u^*), v) = 0\ \forall\, v \in V. \tag{3.37}$$

Due to (3.37) and reference [9], we know that $u^*(t)$ is a continuous function in t. And therefore $u^*(0)$ makes sense. Finally we check $u^*(0) = u(0)$.

Let ϕ be a continuously differentiable function with $\phi(T) = 0$. Multiply (3.37) by ϕ and integrate by parts to get

$$\begin{aligned} &-\int_0^T (u^*, v)\phi'(t)dt + \int_0^T ((u^*, v))\phi(t)dt \\ =&(u^*(0), v)\phi(0) - \int_0^T (f(u^*), v)\phi(t)dt. \qquad \forall v \in V \end{aligned}$$

By comparison with (3.36), we see that

$$(u^*(0), v)\phi(0) = (u(0), v)\phi(0). \quad \forall v \in V$$

In particular, we choose ϕ such that $\phi(0) \neq 0$, then

$$(u^*(0) - u(0), v) = 0. \quad \forall v \in V$$

This equality implies that

$$u^*(0) = u(0) \tag{3.38}$$

(3.37) and (3.38) show that $u^* = u$ is the solution of (2.7), (2.2) and (2.3).

To complete the proof of Theorem 3.1, it remains to check the strong convergence results in (3.18) and (3.19). Let us introduce the expression

$$X_m = \int_0^T |Az_m|^2 ds \tag{3.39}$$

$$Y_m = \frac{1}{2}\|y_m(T) - u(T)\|^2 + \int_0^T |Ay_m - Au|^2 ds. \tag{3.40}$$

Take the scalar product of (3.14) with Az_m and integrate between 0 and T to get

$$\int_0^T (z_{1,m}, Az_m) ds + \int_0^T |Az_m|^2 ds + \int_0^T (f(y_m + z_m), Az_m) ds = 0. \tag{3.41}$$

Multiplying (3.13) by z_m, we find that

$$\int_0^T (z_{1,m}, Az_m) ds + \int_0^T (f'(y_m + z_m) y_{1,m}, z_m) ds = 0. \tag{3.42}$$

It comes from (3.12) that

$$\begin{aligned} |y_{1,m}| &\leq |Ay_m| + |f(y_m + z_m)| \\ &\leq |Ay_m| + N \ \ (by\ |f(s)| \leq N\ for\ s \in R). \end{aligned} \tag{3.43}$$

Then

$$\begin{aligned} &|\int_0^T (f'(y_m + z_m) y_{1,m}, z_m) ds| \\ \leq &\int_0^T |f'(y_m + z_m) y_{1,m}| \cdot |z_m| ds \\ \leq &N \int_0^T |y_{1,m}| \cdot |z_m| ds \ \ (by |f'(s)| \leq N) \\ \leq &N \int_0^T |Ay_m| \cdot |z_m| ds + N \int_0^T |z_m| ds \ \ (\text{by (3.43)}) \\ \leq &N |y_m|_{L^2(0,T;D(A))} \times |z_m|_{L^2(0,T;H)} + N \int_0^T |z_m| ds. \end{aligned}$$

Due to (3.23) and (3.25), we infer that the right-hand side of the above formula converges to 0 as $m \to +\infty$. And therefore it follows from (3.42) that

$$\int_0^T (z_{1,m}, Az_m) ds \to 0 \ \text{ as } m \to +\infty \tag{3.44}$$

$$\begin{aligned} &|\int_0^T (f(y_m + z_m) - f(u), Az_m) ds| \\ \leq &\int_0^T |f(y_m + z_m) - f(u)| \cdot |Az_m| ds \\ \leq &N \int_0^T |y_m + z_m - u| \cdot |Az_m| ds \ \ (by |f'(s) \leq N) \\ \leq &N |y_m + z_m - u|_{L^2(0,T;H)} \times |z_m|_{L^2(0,T;D(A))}. \end{aligned}$$

By (3.25), (3.26) and (3.30) we derive that the above converges to 0 as $m \to +\infty$. Hence

$$\int_0^T (f(y_m + z_m) - f(u), Az_m)ds \to 0 \text{ as } m \to +\infty. \tag{3.45}$$

In view of (3.29) we see that

$$\int_0^T (f(u), Az_m)ds \to 0 \text{ as } m \to +\infty \tag{3.46}$$

(3.45) and (3.46) imply that

$$\int_0^T (f(y_m + z_m), Az_m)ds \to 0 \ as\ m \to +\infty. \tag{3.47}$$

And thus, (3.41) together with (3.44) and (3.47) yields

$$X_m = \int_0^T |Az_m|^2 ds \to 0. \tag{3.48}$$

Now we prove $Y_m \to 0$ as$m \to \infty$. Taking scalar product of (3.11) with Ay_m in H, and integrating between 0 and T, we find that

$$\frac{1}{2}\|y_m(T)\|^2 = \frac{1}{2}\|y_m(0)\|^2 - \int_0^T |Az_m|^2 ds - \int_0^T (f(y_m + z_m), Ay_m)ds$$

and then it follows from (3.40) that

$$\begin{aligned} Y_m =& \frac{1}{2}\|y_m(T)\|^2 - ((y_m(T), u(T))) + \frac{1}{2}\|u(T)\|^2 + \int_0^T |Ay_m|^2 \\ &- 2\int_0^T (Ay_m, Au) + \int_0^T |Au|^2 \\ =& \frac{1}{2}\|y_m(0)\|^2 - ((y_m(T), u(T))) + \frac{1}{2}\|u(T)\|^2 \\ &- 2\int_0^T (Ay_m, Au) + \int_0^T |Au|^2 - \int_0^T (f(y_m + z_m), Ay_m). \end{aligned} \tag{3.49}$$

Each term of the above is majorized as follows :

$$\frac{1}{2}\|y_m(0)\|^2 = \frac{1}{2}\|P_m u_0\|^2 \to \frac{1}{2}\|u(T)\|^2. \tag{3.50}$$

It follows from (3.28) that

$$\begin{aligned} ((y_m(T), u(T))) &= ((y_m(0), u(T)) + \int_0^T ((\frac{d}{dt}y_m, u(T)))dt \\ &= ((y_m(0), u(T))) + \int_0^T (\frac{d}{dt}y_m, Au(T))dt \\ &\to ((u(0), u(T))) + \int_0^T (\frac{d}{dt}u(t), Au(T))dt \\ &= ((u(0), u(T))) + \int_0^T ((\frac{d}{dt}u(t), u(T)))dt \\ &= \|u(T)\|^2. \end{aligned} \tag{3.51}$$

(3.27) gives that

$$\int_0^T (Ay_m, Au) \to \int_0^T |Au|^2. \tag{3.52}$$

In addition

$$\begin{aligned}
&|\int_0^T (f(y_m + z_m) - f(u), Ay_m)| \\
\le &\int_0^T |f(y_m + z_m) - f(u)| \cdot |Ay_m| \\
\le &N \int_0^T |y_m + z_m - u| \cdot |Ay_m| \ \ (\text{by} |f'(s)| \le N) \\
\le &N |y_m + z_m - u|_{L^2(0,T;H)} \times |y_m|_{L^2(0,T;D(A))} \\
\to &0 \ \ (\text{by } (3.23), (3.25) \text{and} (3.30)\ ).
\end{aligned}$$

Hence

$$\int_0^T (f(y_m + z_m) - f(u), Ay_m) \to 0.$$

In view of (3.27) we see that

$$\int_0^T (f(u), Ay_m) \to \int_0^T (f(u), Au).$$

And thus we obtain

$$\int_0^T (f(y_m + z_m), Ay_m) \to \int_0^T (f(u), Au). \tag{3.53}$$

(3.49)-(3.53) yield when $m \to \infty$

$$Y_m \to \frac{1}{2}\|u(0)\|^2 - \frac{1}{2}\|u(T)\|^2 - \int_0^T |Au|^2 - \int_0^T (f(u), Au). \tag{3.54}$$

Since u is the solution of (2.7) and (2.2). Multiplying (2.7) by Au and integrating between 0 and T, we find that

$$\lim_{m\to\infty} Y_m = 0 \tag{3.55}$$

(3.40) and (3.55) show that

$$\int_0^T |Ay_m - Au|^2 ds \to 0 \tag{3.56}$$

$$y_m(T) \to u(T) \text{ in } V \text{strongly}, \ \ \forall T > 0. \tag{3.57}$$

(3.22), (3.57) and the Lebesgue dominated convergence theorem yield

$$y_m \to u \ \text{ in } L^p(0, T; V) \text{ strongly}, \ \ \forall T > 0, 1 \le p < \infty. \tag{3.58}$$

Obviously (3.27), (3.56) and (3.58) prove (3.18) ; (3.25) and (3.48) prove (3.19) ; moreover, (3.18) and (3.19) imply (3.17). And then Theorem 3.1 isproved.

Similar to Theorem 3.1, if $u_0 \in H$, then we can show

Theorem 3.2. *Let* $g \in C^2(R, R)$ *such that* (2.4) *and* (2.5) *hold,* $\Omega \subseteq R^n$ *and* $n \leq 4$. *If* $u_0 \in H$, $u(t)$ *is the solution of* (2.7), (2.2) *and* (2.3), *then when* $m \to +\infty$,

1. $u_m \to u$ *in* $L^\infty(R^+; H)$ *weak star, in* $L^2(0, T; V)$ *and* $L^p(0, T; H)$ *strongly for all* $T > 0$ *and* $1 \leq p < \infty$.

2. $y_m \to u$ *in* $L^\infty(R^+; H)$ *weak star, in* $L^2(0, T; V)$ *and* $L^p(0, T; H)$ *strongly for all* $T > 0$ *and* $1 \leq p < \infty$.

3. $z_m \to 0$ *in* $L^\infty(R^+; H)$ *and* $L^2(0, T; V)$ *strongly for all* $T > 0$.

The proof of this theorem is analogous to that of Theorem 3.1. So we omit it.

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