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L^∞ CONVERGENCE OF TRUNC ELEMENT FOR THE BIHARMONIC EQUATION*[1)]

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Abstract

The paper considers the L^∞ convergence for TRUNC finite elements solving the boundary value problems of the biharmonic equation. The nearly optimal L^∞ estimates for the error of first order derivatives are given.

The TRUNC plate element is proposed and developed by Argyris et al.. The numerical experiences show that the element has very good results[1,2]. The mathematical proof of convergence of the TRUNC element is given by Shi Zong-ci in paper [3]. This paper will consider the L^∞ convergence for the TRUNC plate element.

1. The TRUNC Plate Element

Given a triangle T with vertices $a_i = (x_i, y_i), i = 1,2,3$, we denote by λ_i the area coordinates for the triangle T and put

$$\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2,$$
$$\eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1, \quad \eta_3 = y_1 - y_2.$$

The nodal parameters of the element are the function values and the values of the two first derivatives at the vertices of the triangle T. According to paper [3], on the triangle T the shape function is an incomplete cubic polynomial,

$$\begin{aligned} w = {}& b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 \\ &+ b_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + b_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + b_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{1.1}$$

which is uniquely determined by the nine nodal parameters $w_i, w_x(i), w_y(i), i = 1,2,3$. The coefficients b_i are determined as follows,

$$\begin{cases} b_i = w_i, \quad i = 1,2,3, \\ b_4 = -\frac{1}{2}\{(w_x(1) - w_x(2))\xi_3 + (w_y(1) - w_y(2))\eta_3\}, \\ b_5 = -\frac{1}{2}\{(w_x(2) - w_x(3))\xi_1 + (w_y(2) - w_y(3))\eta_1\}, \\ b_6 = -\frac{1}{2}\{(w_x(3) - w_x(1))\xi_2 + (w_y(3) - w_y(1))\eta_2\}, \\ b_7 = w_1 - w_2 - \frac{1}{2}(w_x(1) + w_x(2))\xi_3 - \frac{1}{2}(w_y(1) + w_y(2))\eta_3, \\ b_8 = w_2 - w_3 - \frac{1}{2}(w_x(2) + w_x(3))\xi_1 - \frac{1}{2}(w_y(2) + w_y(3))\eta_1, \\ b_9 = w_3 - w_1 - \frac{1}{2}(w_x(3) + w_x(1))\xi_2 - \frac{1}{2}(w_y(3) + w_y(1))\eta_2. \end{cases} \tag{1.2}$$

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The shape form (1.1) with (1.2) is another one of Zienkiewicz's element. This element is a C^0 element, nonconforming for plate bending problems, which converges to the true solution only for very special meshes. The TRUNC element is obtained by modifying the variational formulation.

Let Ω be a convex polygon in $R^2, f \in L^2(\Omega)$. Consider the plate bending problem with the clamped boundary conditions,

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}|_{\partial\Omega} = 0 \end{cases} \tag{1.3}$$

The weak form of the problem (1.3) is to find $u \in H_0^2(\Omega)$ such that,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where

$$a(u,v) = \int_\Omega (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\, dxdy,$$

$$(f,v) = \int_\Omega fv\, dxdy, \tag{1.5}$$

and $0 < \sigma < \frac{1}{2}$ is the Poisson ratio.

Dividing Ω into a regular family T_h of triangular elements T with diameters $h_T \leq h$, and defining on each triangle T the shape function in the form (1.1) and (1.2), we obtain the finite element space V_h. Then, the standard finite element approximation of problem (1.4) is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h, \tag{1.5}$$

where

$$a_h(u,v) = \sum_T \int_T (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\, dxdy. \tag{1.6}$$

The modification of variational formulation (1.6) is carried out as follows. Every function $v_h \in V_h$ can be splited into two parts,

$$v_h = \bar{v}_h + v_h', \tag{1.7}$$

where

$$\bar{v}_h|_T = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1, \tag{1.8}$$

representing a full quadratic polynomial on T, and

$$v_h' = a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \tag{1.9}$$

being a cubic polynomial. Define a new discrete bilinear form,

$$b_h(v_h, w_h) = a_h(\bar{v}_h, \bar{w}_h) + a_h(v_h', w_h'), \quad \forall v_h, w_h \in V_h. \tag{1.10}$$

Obviously,

$$b_h(v_h,w_h)=a_h(v_h,w_h)-a_h(\bar v_h,w_h')-a_h(v_h',\bar w_h). \tag{1.11}$$

The solution, produced by TRUNC element, is exactly the one of a variational problem to find $u_h\in V_h$ such that

$$b_h(u_h,v_h)=(f,v_h),\qquad \forall v_h\in V_h. \tag{1.12}$$

2. Some Estimates

Before going to discuss the L^∞ convergence of the TRUNC element, we need some estimates about it. For function $v_h\in L^2(\Omega)$ and $v_h|_T\in H^m(T),\forall T\in \mathsf{T}_h$, define a seminorm as follows,

$$|v_h|_{m,h}=(\sum_{T\in\mathsf{T}_h}|v_h|^2_{m,T})^{1/2}. \tag{2.1}$$

Throughout the paper, C is a generic constant independent of h.

From paper [3], we have the following lemmas.

Lemma 1. *Let $u\in H^3(\Omega)\cap H^2_0(\Omega)$ be the solution of problem* (1.4) *and $u_h\in V_h$ be the solution of problem* (1.12)*. Then*

$$\|u-u_h\|_{1,\Omega}+h^{-1}|u-u_h|_{2,h}\le Ch^2|u|_{3,\Omega}. \tag{2.2}$$

Lemma 2. *According to representations* (1.7) *to* (1.9)*, the shape function is $w_h=\bar w_h+w_h'$. Then*

$$|\bar w_h|_{2,T}+|w_h'|_{2,T}\le C|w_h|_{2,T}, \tag{2.3}$$

for each $T\in\mathsf{T}_h$.

For $T\in\mathsf{T}_h$, we define the interpolation operator Π_T such that for $\forall v\in H^3(T)$, $\Pi_T v$ is in the form of shape function (1.1) and $\Pi_T v,(\Pi_T v)_x$ and $(\Pi_T v)_y$ are equal to v,v_x and v_y at the vertices of T respectively. For $w\in L^2(\Omega)$ and $w|_T\in H^3(T),\forall T\in\mathsf{T}_h$, we define $\Pi_h w\in L^2(\Omega)$ such that $\Pi_h w|_T=\Pi_T w$.

Let $(x_0,y_0)\in\overline{\Omega}$ be a fixed point, define the weight function ρ as follows,

$$\rho(x,y)=(x-x_0)^2+(y-y_0)^2+h^2.$$

For integer β and $v\in H^m(T)$ and $T\in\mathsf{T}_h$, define

$$|v|_{m,(\beta),T}=\left(\sum_{i+j=m}\int_T\rho^{-\beta}\left|\frac{\partial^m v}{\partial x^i\partial y^j}\right|^2dxdy\right)^{1/2}. \tag{2.4}$$

When $v\in L^2(\Omega)$ and $v|_T\in H^m(T)$ for $\forall T\in\mathsf{T}_h$, define

$$|v|_{m,(\beta)}=\left(\sum_{T\in\mathsf{T}_h}|v|^2_{m,(\beta),T}\right)^{1/2}. \tag{2.5}$$

In the estimates involing the weight function ρ, the constant C is also independent of (x_0, y_0). For the weight function, the following inequalities,

$$|v|_{m,(\gamma)} \le h^{-(\gamma-\beta)}|v|_{m,(\beta)}, \quad \gamma > \beta, \tag{2.6}$$

are true for $v \in L^2(\Omega)$ and $v|_T \in H^m(T), \forall T \in \mathsf{T}_h$. And

$$|v|_{0,(1)} \le C|\ln h|^{1/2}\|v\|_{0,\infty,\Omega}, \qquad \forall v \in L^\infty(\Omega), \tag{2.7}$$

$$|(v,w)| \le |v|_{0,(\beta)}\,|v|_{0,(-\beta)}, \quad v,w \in L^2(\Omega). \tag{2.8}$$

For the interpolation operator Π_T,

$$|v - \Pi_T v|_{k,(\beta),T} + |v - \overline{\Pi_T v}|_{k,(\beta),T} + |(\Pi_T v)'|_{k,(\beta),T} \le Ch^{3-k}|v|_{3,(\beta),T}, \tag{2.9}$$

hold for $0 \le k \le 3,\ v \in H^3(T),\ T \in \mathsf{T}_h$.

For $v, w \in L^2(\Omega)$ and $v|_T, w|_T \in H^3(T)$, we define

$$E(v,w) = \sum_{T\in\mathsf{T}_h}\int_{\partial T}[\triangle v - (1-\sigma)\frac{\partial^2 v}{\partial^2 s}]\frac{\partial w}{\partial n}ds. \tag{2.10}$$

Lemma 3. *For each integer β, the inequalities*

$$|(\triangle^2 v, w_h) - b_h(\Pi_h v, w_h)| \le Ch|v|_{3,(\beta)}|w_h|_{2,(-\beta)}, \tag{2.11}$$

$$|E(v,\bar{w}_h)| \le Ch|v|_{3,(\beta)}|w_h|_{2,(-\beta)}, \tag{2.12}$$

$$|E(v-\bar{v}_h, w_h')| \le Ch(|v-\bar{v}_h|_{2,(\beta)} + h|v|_{3,(\beta)})|w_h|_{3,(-\beta)}, \tag{2.13}$$

$$|E(v, \overline{\Pi_h w} - w)| \le Ch^2|v|_{3,(\beta)}|w|_{3,(-\beta)}, \tag{2.14}$$

are true for $\forall v, w \in H_0^2(\Omega) \cap H^3(\Omega)$ *and* $\forall v_h, w_h \in V_h$.

Proof. By the way used in [3] directly or alternatively, we get

$$|(\triangle^2 v, w_h) - b_h(\Pi_h v, w_h)| \le Ch \sum_{T\in\mathsf{T}_h}|v|_{3,T}|w_h|_{2,T}$$

$$|E(v,\bar{w}_h)| \le Ch \sum_{T\in\mathsf{T}_h}|v|_{3,T}|w_h|_{2,T},$$

$$|E(v-\bar{v}_h, w_h')| \le Ch \sum_{T\in\mathsf{T}_h}(|v-\bar{v}_h|_{2,T} + h|v|_{3,T})|w_h|_{3,T},$$

$$|E(v, \overline{\Pi_h w} - w)| \le Ch^2 \sum_{T\in\mathsf{T}_h}|v|_{3,T}|w|_{3,T}.$$

On the other hand, for $\forall T \in \mathsf{T}_h$,

$$\max_{(x,y)\in T} \rho(x,y) \le C \min_{(x,y)\in T} \rho(x,y). \tag{2.15}$$

Then the lemma follows.

3. The L^∞ Estimates

In this section, we will prove the following theorem.

Theorem 1. *Let u be the solution of problem* (1.4) *a nd u_h the solution of problem* (1.12)*. Then*

$$|u-u_h|_{1,\infty,\Omega} \le Ch^2|\ln h|^{5/4}|u|_{3,\infty,\Omega}, \tag{3.1}$$

when $u \in W^{3,\infty}(\Omega)$.

By the interpolation result[6], we have

$$\begin{aligned}|u-u_h|_{1,\infty,\Omega} &\le |u-\Pi_h u|_{1,\infty,\Omega} + |\Pi_h u - u_h|_{1,\infty,\Omega}\\ &\le Ch^2|u|_{3,\infty,\Omega} + |\Pi_h u - u_h|_{1,\infty,\Omega}.\end{aligned} \tag{3.2}$$

So we must estimate $|\Pi_h u - u_h|_{1,\infty,\Omega}$. Let $T' \in \mathsf{T}_h$ be the element such that $|\Pi_h u - u_h|_{1,\infty,\Omega} = |\Pi_h u - u_h|_{1,\infty,T'}$. Without loss the generality, suppose that

$$|\Pi_h u - u_h|_{1,\infty,T'} = |\frac{\partial(\Pi_h u - u_h)}{\partial x}|_{0,\infty,T'}.$$

Let $(x_0, y_0) \in T'$ be the point such that

$$|\frac{\partial(\Pi_h u - u_h)}{\partial x}|_{0,\infty,T'} = |\frac{\partial(\Pi_h u - u_h)}{\partial x}(x_0,y_0)|.$$

To prove (3.1), we need some results about the regular Green function. Let $q \in P_3(T')$ satisfy

$$\int_{T'} qp\, dxdy = \frac{\partial}{\partial x} p(x_0,y_0), \qquad \forall p \in P_3(T'). \tag{3.3}$$

Define $\delta_h \in L^2(\Omega)$ such that,

$$\delta_h(x,y) = \begin{cases} q(x,y) & (x,y)\in T' \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

From Lemma 3 in [5], we have

$$h\|\delta_h\|_{0,\Omega} + \|\delta_h\|_{-1,\Omega} + \|\delta_h\|_{0,(-1)} \le Ch^{-1}. \tag{3.5}$$

Let g be the regular Green function determined by

$$\begin{cases} \triangle^2 g = \delta_h, & \text{in } \Omega \\ g|_{\partial\Omega} = \dfrac{\partial g}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{3.6}$$

and g_h be its finite element solution by the TRUNC element, i.e.,

$$b_h(g_h, v_h) = (\delta_h, v_h). \qquad \forall v_h \in V_h. \tag{3.7}$$

From (2.2) and (3.5), we get

$$\|g - g_h\|_{1,\Omega} + h|g - g_h|_{2,h} + h^2|g|_{3,\Omega} \le Ch. \tag{3.8}$$

By the way used in [5], we have

$$\|g\|_{2,\Omega} + |g|_{3,(-1)} \le C|\ln h|^{1/2}. \tag{3.9}$$

Lemma 4.

$$|g - \Pi_h g|_{2,(-1)} \le Ch|\ln h|^{1/2} \tag{3.10}$$

$$|\Pi_h g - g_h|_{2,(-1)} \le Ch|\ln h|^{3/4}. \tag{3.11}$$

Proof. Inequaltity (**??**) follows from (**??**) and (2.9). Now we prove (**??**). For $v, w \in H^2(T)$ and integer β, we define

$$a_T(v,w) = \int_T (\triangle v \triangle w + (1-\sigma)(2v_{xy}w_{xy} - v_{xx}w_{yy} - v_{yy}w_{xx}))\, dxdy$$

$$a_{T,\beta}(v,w) = \int_T \rho^{-\beta}(\triangle v \triangle w + (1-\sigma)(2v_{xy}w_{xy} - v_{xx}w_{yy} - v_{yy}w_{xx}))\, dxdy.$$

By Leibniz' formula, (2.8) and the inequality

$$|\frac{\partial \rho}{\partial x}|^2 + |\frac{\partial \rho}{\partial y}|^2 \le 4\rho, \tag{3.12}$$

we obtain

$$a_{T,-1}(v,v) \le a_T(v, \rho v) + C|v|_{2,T,(-1)}|v|_{1,T} - 2(1+\sigma)\int_T \triangle v\, v\, dxdy \tag{3.13}$$

It is easy to prove that for $v_h \in V_h$ the following inequality is true,

$$|v_h|^2_{2,T,(\beta)} \le C[a_{T,\beta}(\bar v_h, \bar v_h) + a_{T,\beta}(v'_h, v'_h)], \quad \forall T \in \mathsf{T}_h. \tag{3.14}$$

From (**??**), (**??**), (2.3) and (2.15), we derive that

$$\begin{aligned}
|v_h|^2_{2,(-1)} &\le C \sum_{T \in \mathsf{T}_h} [a_{T,-1}(\bar v_h, \bar v_h) + a_{T,-1}(v'_h, v'_h)] \\
&\le C\Big\{ a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h) + |\bar v_h|_{2,(-1)}|\bar v_h|_{1,\Omega} + |v'_h|_{2,(-1)}|v'_h|_{1,\Omega} \\
&\qquad + |\sum_{T \in \mathsf{T}_h} \int_T \triangle \bar v_h \bar v_h dxdy + \sum_{T \in \mathsf{T}_h} \int_T \triangle v'_h v'_h dxdy|\Big\} \\
&\le C\Big\{ a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h) + |v_h|_{2,(-1)}|v_h|_{1,\Omega} \\
&\qquad + |\sum_{T \in \mathsf{T}_h} \int_T \triangle \bar v_h v_h dxdy| + |\sum_{T \in \mathsf{T}_h} \int_T (\triangle v'_h - \triangle \bar v_h) v'_h dxdy|\Big\}.
\end{aligned} \tag{3.15}$$

Then we get, from (2.3) and (2.9),

$$
\begin{aligned}
|v_h|^2_{2,(-1)} \le C\Big\{ & a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h) + |v_h|_{2,(-1)}|v_h|_{1,\Omega} \\
& + h^2|v_h|^2_{2,h} + |\sum_{T\in \mathsf{T}_h}\int_T \triangle\, \bar v_h v_h dxdy|\Big\}.
\end{aligned} \tag{3.16}
$$

By similar way used in the proof of lemma 3.5 in [3] and the fact that $v_h \in H^1_0(\Omega)$, we have

$$
|\sum_{T\in \mathsf{T}_h}\int_T \triangle\, \bar v_h v_h dxdy| \le Ch|v_h|_{1,\Omega}|v_h|_{2,h}, \tag{3.17}
$$

and then

$$
\begin{aligned}
|v_h|^2_{2,(-1)} \le C\Big\{ & a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h) + |v_h|_{2,(-1)}|v_h|_{1,\Omega} \\
& + h^2|v_h|^2_{2,h} + h|v_h|_{1,\Omega}|v_h|_{2,h}\Big\}.
\end{aligned} \tag{3.18}
$$

For $v_h \in V_h$, $\Pi_h(\rho v_h) \in V_h$. Then from (2.3), (2.8) and (2.15), we get

$$
\begin{aligned}
|a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h)| = & |b_h(v_h, \Pi_h(\rho v_h)) + a_h(\bar v_h, \rho\bar v_h - \overline{\Pi_h(\rho v_h)}) \\
& + a_h(v'_h, \rho v'_h - (\Pi_h(\rho v_h))')| \\
\le & |b_h(v_h, \Pi_h(\rho v_h))| + |\bar v_h|_{2,(-1)}|\rho\bar v_h - \overline{\Pi_h(\rho v_h)}|_{2,(1)} \\
& + |v'_h|_{2,(-1)}|\rho v'_h - (\Pi_h(\rho v_h))'|_{2,(1)} \\
\le & |b_h(v_h, \Pi_h(\rho v_h))| + |v_h|_{2,(-1)}(|\rho\bar v_h - \overline{\Pi_h(\rho v_h)}|_{2,(1)} \\
& + |\rho v'_h - (\Pi_h(\rho v_h))'|_{2,(1)}).
\end{aligned}
$$

Set $P^0_T : L^2(T) \to P_0(T)$ be the $L^2(T)$ orthogonal projection operator and $P_h : L^2(\Omega) \to L^2(\Omega)$ defined such that for $v \in L^2(\Omega)$, $P_h v|_T = P^0_h(v|_T), \forall T \in \mathsf{T}_h$. Then from (2.?) and the inverse inequality of polynomials,

$$
\begin{aligned}
|\rho\bar v_h - \overline{\Pi_h(\rho v_h)}|_{2,(1)} = & |(\rho - P_h\rho)\bar v_h - \overline{\Pi_h((\rho - P_h\rho)v_h)}|_{2,(1)} \\
\le & |(\rho - P_h\rho)(\bar v_h - v_h)|_{2,(1)} + |(\rho - P_h\rho)v_h - \overline{\Pi_h((\rho - P_h\rho)v_h)}|_{2,(1)} \\
\le & C(h|v_h|_{2,h} + |v_h|_{1,\Omega}),
\end{aligned}
$$

$$
\begin{aligned}
|\rho v'_h - (\Pi_h(\rho v_h))'|_{2,(1)} = & |(\rho - P_h\rho)v'_h - (\Pi_h((\rho - P_h\rho)v_h))'|_{2,(1)} \\
\le & |(\rho - P_h\rho)v'_h|_{2,(1)} + |(\Pi_h((\rho - P_h\rho)v_h))'|_{2,(1)} \\
\le & |(\rho - P_h\rho)v'_h|_{2,(1)} + Ch|(\rho - P_h\rho)v_h|_{3,(1)} \\
\le & C(h|v_h|_{2,h} + |v_h|_{1,\Omega}).
\end{aligned}
$$

Thus we get

$$
|a_h(\bar v_h, \rho \bar v_h) + a_h(v'_h, \rho v'_h)| \le |b_h(v_h, \Pi_h(\rho v_h))| + C|v_h|_{2,(-1)}(h|v_h|_{2,h} + |v_h|_{1,\Omega}). \tag{3.19}
$$

By (2.11), we have

$$\begin{aligned}|b_h(\Pi_h g - g_h, \Pi_h(\rho v_h)| =& |b_h(\Pi_h g, \Pi_h(\rho v_h)) - (\triangle^2 g, \Pi_h(\rho v_h))| \\ \le& Ch|g|_{3,(-1)}|\Pi_h(\rho v_h)|_{2,(1)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\Pi_h(\rho v_h)|_{2,(1)} &\le C|\rho v_h|_{2,(1)} \\ &\le C(|v_h|_{2,(-1)} + |v_h|_{1,\Omega} + |v_h|_{0,(1)}) \\ &\le C(|v_h|_{2,(-1)} + |v_h|_{1,\Omega} + |\ln h|^{1/2}|v_h|_{0,\infty,\Omega}).\end{aligned}$$

From [4], we know that

$$|v_h|_{0,\infty,\Omega} \le C|\ln h|^{1/2}|v_h|_{1,\Omega},$$

therefore,

$$|b_h(\Pi_h g - g_h, \Pi_h(\rho v_h)| \le Ch|g|_{3,(-1)}(|v_h|_{2,(-1)} + (1 + |\ln h|)|v_h|_{1,\Omega}). \tag{3.20}$$

Combining (**??**) to (**??**) with $v_h = \Pi_h g - g_h$, we get

$$\begin{aligned}|\Pi_h g - g_h|^2_{2,(-1)} \le C\Big\{&h^2|\Pi_h g - g_h|^2_{2,h} + h|\Pi_h g - g_h|_{1,\Omega}|\Pi_h g - g_h|_{2,h} \\ &+ h|g|_{3,(-1)}(1 + |\ln h|)|\Pi_h g - g_h|_{1,\Omega} \\ &+ |\Pi_h g - g_h|_{2,(-1)}(h|v_h|_{2,h} + |\Pi_h g - g_h|_{1,\Omega} + h|g|_{3,(-1)})\Big\}.\end{aligned}$$

By (2.7), (2.9), (3.8) and (3.9), we have

$$|\Pi_h g - g_h|^2_{2,(-1)} \le Ch^2(1 + |\ln h|(1 + |\ln h|^{1/2})). \tag{3.21}$$

Inequality (**??**) is proved.

From (3.3) and (3.7), we see that

$$|\Pi_h u - u_h|_{1,\infty,\Omega} = |b_h(g_h, \Pi_h u - u_h)|. \tag{3.22}$$

From (1.11) and (1.12), we have

$$b_h(g_h, \Pi_h u - u_h) = a_h(g_h, \Pi_h u) - a_h(\bar g_h, (\Pi_h u)') - a_h(\overline{\Pi_h u}, g_h') - (\triangle^2 u, g_h).$$

From [3] we have

$$\begin{aligned}&a_h(\bar g_h, (\Pi_h u)') = E(\bar g_h, (\Pi_h u)') \\ &a_h(\overline{\Pi_h u}, g_h') = E(\overline{\Pi_h u}, g_h') \\ &(\triangle^2 u, g_h) = a_h(g_h, u) - E(u, g_h).\end{aligned}$$

Then

$$\begin{aligned} b_h(g_h, \Pi_h u - u_h) =& a_h(g - g_h, u - \Pi_h u) - (\triangle^2 g, u - \Pi_h u) + E(u - \overline{\Pi_h u}, g_h') \\ &+ E(g - \bar{g}_h, (\Pi_h u)') + E(u, \overline{g_h - \Pi_h g}) + E(u, \overline{\Pi_h g} - g) \\ &+ E(g, \overline{\Pi_h u} - u) + E(u, g) + E(g, u), \end{aligned}$$

Noticing $E(u,g) = E(g,u) = 0$, by (2.8) and (2.12) to (2.14) we have,

$$\begin{aligned} |b_h(g_h, \Pi_h u - u_h)| \le C\Big\{ & |g - g_h|_{2,(-1)} |u - \Pi_h u|_{2,(1)} + |\delta_h|_{0,(-1)} |u - \Pi_h u|_{0,(1)} \\ & + h^2 |u|_{3,(1)} |g|_{3,(-1)} + h(|u - \overline{\Pi_h u}|_{2,(1)} + h|u|_{3,(1)}) |g_h'|_{3,(-1)} \\ & + h(|g - \bar{g}_h|_{2,(-1)} + h|g|_{3,(-1)}) |(\Pi_h u)'|_{3,(1)} \Big\}. \end{aligned}$$

From (2.9), (3.5) and (3.9) to (3.11), we derive that

$$|b_h(g_h, \Pi_h u - u_h)| \le C h^2 |u|_{3,(1)} (1 + |\ln h|^{3/4} + |g_h'|_{3,(-1)}). \tag{3.23}$$

Using the inverse inequality of polynomials and (3.9) to (3.11), we have

$$\begin{aligned} |g_h'|_{3,(-1)} &\le C|g_h|_{3,(-1)} \le C(|\Pi_h g - g_h|_{3,(-1)} + |\Pi_h g|_{3,(-1)} \\ &\le C(h^{-1}|\Pi_h g - g_h|_{2,(-1)} + |\ln h|^{1/2}) \\ &\le C|\ln h|^{3/4}. \end{aligned}$$

From (3.23), (2.7), we derive that

$$|b_h(g_h, \Pi_h u - u_h)| \le C h^2 |u|_{3,\infty,\Omega} |\ln h|^{5/4} \tag{3.24}$$

Inequality (3.1) follows from (3.24), (3.22) and (3.2).

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