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# ON THE COUPLING OF BOUNDARY INTEGRAL AND FINITE ELEMENT METHODS FOR SIGNORINI PROBLEMS*

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**Abstract**

In this paper, a exterior Signorini problem is reduced to a variational inequality on a bounded inner region with the help of a coupling of boundary integral and finite element methods. We established a equivalence between the original exterior Signorini problem and the variational inequality on the bounded inner region coupled with two integral equations on an auxiliary boundary. We also introduce a finite element approximation of the variational inequality and a boundary element approximation of the integral equations. Furthermore, the optimal error estimates are given.

*Key words*: Boundary element, finite element, Signorini problems.

## 1. Introduction

Partial differential equations subject to unilateral boundary conditions are usually called Signorini problems in the literature. These problems have been studied by many authods since the appearence of the historical paper by A. Signorini in 1933 [25]. Signorini problems arose in many areas of applications e.g., the elasticity with unilateral conditions[10], the fluid mechnics problems in media with semipermeable boundaries[8,12], the electropaint process[1] etc. For the existence, uniqueness and regularity results for Signorini type problems we refer the reader to [3, 11]. Furthermore, the numerical solution of the Signorini problems by the finite element method has been discussed[4,13]. Boundary element method for solving Signorini problems has been presented in [14, 15].

In this paper, we will present a coupling of boundary integral and finite element methods for solving a exterior Signorini problem, which is reduced to a equivalent new variational inequality on a bounded inner region coupled with two integral equations on an auxiliary boundary. The bilinear form arising in this variational inequality is continuous and coercive on suitable subspaces of some Sobolev space. This leads to existence and uniqueness for the solution of the variational inequality. Furthermore, a coupling of boundary element and finite element methods for numerical solution of the variational inequality is proposed and optimal error estimates are derived.

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## 2. The New Variational Inequality on a Bounded Inner Region

Let $\Omega^c$ be the complement of a bounded regular region in $R^2$ with boundary $\Gamma$. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$ (as shown in Fig.1), with $\Gamma_0 \cap \Gamma_1 = \phi$, $\Gamma_0 \neq \phi$, we consider the following Signorini problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega^c, \\ u = 0, & \text{on } \Gamma_0, \\ u \geq 0,\ \dfrac{\partial u}{\partial n} \geq 0, & \text{on } \Gamma_1, \\ u\dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_1, \\ u \text{ is bounded}, & \text{when } \ |x| \to \infty, \end{cases} \tag{2.1}$$

Fig. 1

where $f$ has its support in a bounded subregion $\Omega_1$ of $\Omega^c$. In case of $\Gamma_0 = \phi$, $f$ satisfies a compatibility condition[8]

$$\int_{\Omega_1} f dx \geq 0. \tag{2.2}$$

Let $\Omega_2 = \Omega^c \backslash \bar{\Omega}_1$, $\Gamma_2 = \partial\Omega_2$ (see Fig.2). We will solve the exterior Signorini problem (2.1) by using the coupling of boundary element and finite element methods. Consider the equivalent system of Signorini problem:

$$\begin{cases} -\Delta u_1 = f, & \text{in } \Omega_1, \\ -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_1 = u_2, & \text{on } \Gamma_2, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \sigma, & \text{on } \Gamma_2, \\ u_1 = 0, & \text{on } \Gamma_0, \\ u_1 \geq 0,\ \dfrac{\partial u_1}{\partial n} \geq 0, & \text{on } \Gamma_1, \\ u_1\dfrac{\partial u_1}{\partial n} = 0, & \text{on } \Gamma_1, \\ u_2 \text{ is bounded}, & \text{when } \ |x| \to \infty, \end{cases} \tag{2.3}$$

Fig. 2

where $u_i = u|_{\Omega_i}$, $i = 1, 2$, and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative to the boundary $\partial\Omega_1 = \Gamma \cup \Gamma_2$ (see Fig.2).

We note that $u_1$ will be completely determined, if $\sigma$ is known. On the other hand, the function $\sigma$ depends linearly on $u_1|_{\Gamma_2}$ via the solution of the following exterior boundary value problem

$$\begin{cases} -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_2 = u_1, & \text{on } \Gamma_2, \\ u_2 \text{ is bounded}, & \text{when } |x| \to \infty. \end{cases} \tag{2.4}$$

Hence the next step in the coupling procedures is to derive equations on $\Gamma_2$ which will relate $\sigma$ and $u_1|_{\Gamma_2}$ (i.e. $\dfrac{\partial u_1}{\partial n}|_{\Gamma_2}$ and $u_1|_{\Gamma_2}$) directly to that the original exterior Signorini problem (2.1) may be reformulated as an equivalent Signorini problem in bounded inner region $\Omega_1$.

There are various forms of equivalent nonlocal boundary problems for boundary value problems by adopting non-symmetric scheme or symmetric (or positive definite) scheme or quasi-symmetric schemes[4,6,7,9,16,18,19,20,24]. However, the difference among these schemes will become more pronounced in their weak forms, even some coupling procedure will not work without more strong restriction or modifiction e.g. in elasticity (see Wendland [26]).

We adopt a coupling of boundary element and finite element methods[16], which preserves the coercive property of the bilinear form for the original variational inequality. Therefore, the strong restriction for the coupling operator required by Wendland[26] will no longer be needed. Using Green's formula we represent the solution of (2.3) in the form

$$u_2(x) = -\int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u_2(y) ds_y + \int_{\Gamma_2} G(x,y) ds_y + \alpha, \quad x \in \Omega_2, \tag{2.5}$$

where $\alpha$ is a constant, and $G(x,y)$ is the fundmental solution for the two-dimensional Laplacian,

$$G(x,y) = \frac{1}{2\pi} \log|x-y|. \tag{2.6}$$

Letting $x$ approach $\Gamma_2$, we arrive at the boundary integral equation

$$\Big(\frac{1}{2}\mathbf{I} + \mathbf{T}\Big) u_2(x) - \mathbf{F}\frac{\partial u_2}{\partial n}(x) - \alpha = 0,\ x \in \Gamma_2, \tag{2.7}$$

where $\mathbf{I}$ denotes the identity operator, $\mathbf{T}$ and $\mathbf{F}$ are the double and simgle - layer potentials defined by

$$\mathbf{T}u_2(x) \equiv \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u_2(y) ds_y,\ x \in \Gamma_2, \tag{2.8}$$

$$\mathbf{F}\frac{\partial u_2}{\partial n}(x) \equiv \int_{\Gamma_2} G(x,y) \frac{\partial u_2(y)}{\partial n_y} ds_y,\ x \in \Gamma_2. \tag{2.9}$$

We also arrive at the boundary integral equation for $\dfrac{\partial u_2}{\partial n}|_{\Gamma_2}$

$$\frac{\partial u_2(x)}{\partial n} = -\mathbf{W}u_2(x) + \Big(\frac{1}{2}\mathbf{I} + \mathbf{T}'\Big)\frac{\partial u_2}{\partial n}(x),\ x \in \Gamma_2, \tag{2.10}$$

where the boundary integral operators $\mathbf{T}'$ and $\mathbf{W}$ are defined by

$$\mathbf{T}'\frac{\partial u_2}{\partial n}(x) \equiv \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x} \frac{\partial u_2}{\partial n_y}(y) ds_y,\ x \in \Gamma_2, \tag{2.11}$$

$$\mathbf{W}u_2(x) \equiv \int_{\Gamma_2} \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u_2(y) ds_y,\ x \in \Gamma_2, \tag{2.12}$$

$\mathbf{W}u_2(x)$ includes a hypersingular integral. Using the properties of the double - layer potential[17,23], we get

$$\mathbf{W}u_2(x) = \frac{d}{ds_x}\int_{\Gamma_2} G(x,y)\frac{du_2(y)}{ds_y}ds_y,\ x\in\Gamma_2. \tag{2.13}$$

In terms of the transmission conditionin (2.3), we have from (2.3), (2.7) (2.10) a nonlocal boundary Signorini problem: Find $(u_1,\sigma)$ such that

$$\begin{cases} -\Delta u_1 = f, & \text{in } \Omega_1,\\ u_1 = 0, & \text{on } \Gamma_0,\\ u_1 \ge 0, \dfrac{\partial u_1}{\partial n} \ge 0, & \text{on } \Gamma_1,\\ u_1\dfrac{\partial u_1}{\partial n} = 0, & \text{on } \Gamma_1,\\ \Big(\dfrac{1}{2}\mathbf{I}+\mathbf{T}\Big)u_1 - F\sigma - \alpha = 0, & \text{on } \Gamma_2,\\ \dfrac{\partial u_1}{\partial n} = -\mathbf{W}u_1 + \Big(\frac{1}{2}\mathbf{I}+\mathbf{T}'\Big)\sigma, & \text{on } \Gamma_2. \end{cases} \tag{2.14}$$

Let us introduce two bilinear forms and the following function spaces:

$$\begin{aligned} a(u,v) &= \int_{\Omega_1}\int_{\Omega_1} \nabla u \nabla v dxdy,\\ b(\sigma,\mu) &= -\int_{\Gamma_2}\int_{\Gamma_2} G(x,y)\sigma(x)\mu(y)ds_x ds_y,\\ \overset{*}{H}{}^{1}(\Omega_1) &= \{u\in H^1(\Omega_1),\ u|_{\Gamma_0}=0\},\\ K &= \{v\in \overset{*}{H}{}^{1}(\Omega_1),\ v\ge 0,\ \text{a.e. on }\ \Gamma_1\},\\ \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2) &= \Big\{\mu\in H^{-\frac{1}{2}}(\Gamma_2),\ \int_{\Gamma_2}\mu ds = 0\Big\},\\ W_\alpha &= \overset{*}{H}{}^{1+\alpha}(\Omega_1)\times \overset{*}{H}{}^{-\frac{1}{2}+\alpha}(\Gamma_2), \end{aligned}$$

with norm $\|(v,\mu)\|^2_{W_\alpha} = \|v\|^2_{1+\alpha,\Omega_1} + \|\mu\|^2_{-\frac{1}{2}+\alpha,\Gamma_2}$, where $H^m(\Omega_1)$ and $H^\beta(\Gamma_2)$ denote the usual Sobolev spaces, $m,\beta$ are two real numbers. We denote by $<\cdot,\cdot>$ and $(\cdot,\cdot)_0$ the duality pairings on $H^{-\frac{1}{2}}(\Gamma_2)\times H^{\frac{1}{2}}(\Gamma_2)$ and $H^1(\Omega_1)\times H^{-1}(\Omega_1)$ respectively. Then the variational inequality of (2.14) reads: Given $f\in H^{-1}(\Omega_1)$, find $(u,\sigma)\in K\times \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2)$, such that

$$\begin{cases} a(u_1, v-u_1) + b\Big(\frac{du_1}{ds}, \frac{d(v-u_1)}{ds}\Big) - \Big\langle\Big(\dfrac{1}{2}\mathbf{I}+\mathbf{T}'\Big)\sigma, v-u_1\Big\rangle \\ \qquad \ge (f, v-u_1)_0, \quad \forall v\in K,\\ \Big\langle \mu, \Big(\dfrac{1}{2}\mathbf{I}+\mathbf{T}\Big)u_1\Big\rangle - \langle\mu, \mathbf{F}\sigma\rangle = 0,\ \forall \mu\in \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2), \end{cases} \tag{2.15}$$

where we know

$$\langle \mathbf{W}u_1, v\rangle = b\Big(\frac{du_1}{ds}, \frac{dv}{ds}\Big), \tag{2.16}$$

in terms of (2.13).

Alternatively, we may rewrite (2.15) in the form: Find $(u_1,\sigma)\in K\times \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2)$, such that

$$A(u_1,\sigma;v-u_1,\mu)\geq (f,v-u_1)_0,\ \forall (v,\mu)\in K\times \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2), \tag{2.17}$$

where

$$\begin{aligned}A(u_1,\sigma;v-u_1,\mu)=&a(u_1,v)+b\Big(\frac{du_1}{ds},\frac{dv}{ds}\Big)-\Big\langle\Big(\frac{1}{2}\mathbf{I}+\mathbf{T}'\Big)\sigma,v\Big\rangle\\ &+\Big\langle\mu,\Big(\frac{1}{2}\mathbf{I}+\mathbf{T}\Big)u_1\Big\rangle-\langle\mu,\mathbf{F}\sigma\rangle,\end{aligned}$$

is a bilinear form on $W_0\times W_0$.

We have following Lemmas.

**Lemma 2.1.** *$A(u,\sigma;v,\mu)$ is a bounded bilinear form on $W_0\times W_0$; that is there is a constant $M>0$, such that*

$$|A(u,\sigma;v,\mu)|\leq M\|(u,\sigma)\|_{W_0}\|(v,\mu)\|_{W_0},\ \forall (u,\sigma),(v,\mu)\in W_0. \tag{2.18}$$

*Furthermore, there is a constant $\beta>0$, such that*

$$A(v,\mu;v,\mu)\geq\beta\|(v,\mu)\|^2_{W_0},\ \forall (v,\mu)\in W_0. \tag{2.19}$$

*Proof.* We recall that case (see [21]) $b(\sigma,\mu)$ is a bounded bilinear form on $W_0\times W_0$, i.e. there exists a constant $C_1>0$, such that

$$b(\mu,\mu)\geq C_1\|\mu\|^2_{-\frac{1}{2},\Gamma_2},\ \forall\mu\in\overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2). \tag{2.20}$$

Then it is straightforward to check that $A(u,\sigma;v,\mu)$ is a bounded form on $W_0\times W_0$, i.e. there exists a constant $M>0$, such that $|A(u,\sigma;v,\mu)|\leq M\|(u,\sigma)\|_{W_0}\|(v,\mu)\|_{W_0}$, $\forall (u,\sigma),(v,\mu)\in W_0$. Furthermore[5], we have

$$\begin{aligned}A(v,\mu;v,\mu)=&a(u,v)+b\Big(\frac{du}{ds},\frac{dv}{ds}\Big)+b(\mu,\mu)\geq|v|^2_{1,\Omega_1}+C_1\|\mu\|^2_{-\frac{1}{2},\Gamma_2}\\ \geq&\beta\{\|v\|^2_{1,\Omega_1}+\|\mu\|^2_{-\frac{1}{2},\Gamma_2}\},\ \forall (v,\mu)\in W_0,\end{aligned}$$

with the constant $\beta>0$, i.e. $A(v,\mu;v,\mu)\geq\beta\|(v,\mu)\|^2_{W_0}$, $\forall (v,\mu)\in W_0$.

**Lemma 2.2.** *Suppose $(u_1,\sigma)\in W_\alpha$ with $0\leq\alpha\leq 1$, then there exists a constant $M_\alpha>0$, such that $|A(u_1,\sigma;v,\mu)|\leq M_\alpha\|(u_1,\sigma)\|_{W_\alpha}\|(v,\mu)\|_{W_{-\alpha}}$, $\forall (v,\mu)\in W_0$.*

The proof is omited here, which is similar to the proof of lemma 2.3 in [15].

By Lemma 2.1, an application of existence and uniqueness results for elliptic variational inequality of the first kind by Lions and Stampacchia[22] yields the following result.

**Theorem 2.1.** *Suppose* $f \in H^{-1}(\Omega_1)$, *then the variational inequality problem* (2.17) *has a unique solution* $(u_1, \sigma) \in K \times \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$.

Suppose $u(x)$ is the solution of the Signorini problem (2.1), then we know that $u_1(x) \equiv u(x)|_{\Omega_1} \in K$, $\sigma \equiv \dfrac{\partial u}{\partial n}|_{\Gamma_2} \in \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$. Moreover, $(u_1, \sigma)$ is a solution of the variational inequality problem (2.17). By the uniqueness of the variational inequality problem (2.17), let

$$u = \begin{cases} u_1, & \text{in } \Omega_1, \\ u_2, & \text{in } \Omega_2, \end{cases}$$

we know that the exterior Signorini problem (2.1) is equivalent to the variational inequality problem (2.17).

Furthermore, we have

**Theorem 2.2.** *The variational inequality problem* (2.17) *is equivlent the following saddle point problem: Find* $(u_1, \sigma) \in K \times \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$, *such that*

$$L(u_1, \mu) \le L(u_1, \sigma) \le L(v, \sigma),\ \forall v \in K, \mu \in \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2), \tag{2.21}$$

*where*

$$L(v, \mu) = \frac{1}{2}a(v, v) + \frac{1}{2}b\Big(\frac{dv}{ds}, \frac{dv}{ds}\Big) - \Big\langle \mu, \Big(\frac{1}{2}\mathbf{I} + \mathbf{T}\Big)v \Big\rangle - \frac{1}{2}b(\mu, \mu) - \langle f, v\rangle_0.$$

*Proof.* Suppose that $(u_1, \sigma) \in K \times \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$ is the solution of (2.21).

Then for any $\mu \in \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$ and real number $\varepsilon$, $\sigma + \varepsilon\mu \in \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2)$, we have $L(u_1, \sigma + \varepsilon\mu) \le L(u_1, \sigma)$, that is

$$-\varepsilon\Big[\Big\langle \mu, \Big(\frac{1}{2}\mathbf{I} + \mathbf{T}\Big)u_1 \Big\rangle + b(\sigma, \mu)\Big] - \frac{\varepsilon^2}{2}b(\mu, \mu) \le 0.$$

Since $\varepsilon$ is an arbitrary constant, we obtain

$$\Big\langle \mu, \Big(\frac{1}{2}\mathbf{I} + \mathbf{T}\Big)u_1 \Big\rangle + b(\sigma, \mu) = 0,\ \forall \mu \in \overset{*}{H}^{-\frac{1}{2}}(\Gamma_2).$$

On the other hand, for any $u_1, v \in K$, we know that $u_1 + t(v - u_1) \in K (0 \le t \le 1)$, then $K$ is convex, and we get

$$L(u_1, \sigma) \le L(u_1 + t(v - u_1, \sigma),\ \forall v \in K,$$

that is

$$\begin{aligned} &t\Big[a(u_1, v - u_1) + b\Big(\frac{du_1}{ds}, \frac{d(v - u_1)}{ds}\Big) - \Big\langle \Big(\frac{1}{2}\mathbf{I} + \mathbf{T}'\Big)\sigma, v - u_1 \Big\rangle \\ &- (f, v - u_1) + \frac{t^2}{2}\Big[a(v - u_1, v - u_1) + b\Big(\frac{d(v - u_1)}{ds}, \frac{d(v - u_1)}{ds}\Big)\Big] \ge 0. \end{aligned}$$

Since $t(0 \le t \le 1)$ is an arbitrary constant and

$$a(v-u_1, v-u_1) + b\Big(\frac{d(v-u_1)}{ds}, \frac{d(v-u_1)}{ds}\Big) \ge 0,$$

we get

$$a(u_1, v-u_1) + b\Big(\frac{du_1}{ds}, \frac{d(v-u_1)}{ds}\Big) - \Big\langle\Big(\frac{1}{2}\mathbf{I}+\mathbf{T}'\Big)\sigma, v-u_1\Big\rangle \ge (f, v-u_1)_0.$$

This means that $(u_1, \sigma) \in K \times \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2)$ is the solution of (2.15). Each of the above steps is reversible, hence we conclude that the variational inequality (2.15) is equivalent to the saddle point problem (2.21).

## 3. The Discrete Approximation of the Variational Inequality (2.17)

Suppose that $S_{h_1}$ and $S_{h_2}$ are two dimensional subspaces of $\overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2)$, respectively. Let $K_{h_1} = \{v_h, v_h \in S_{h_1} \cap K\}$. Moreover, we assume that $K_{h_1}$ is a closed convex subset of $S_{h_1}$.

We consider the approximation problem: Find $(u_{h_1}, \sigma^h) \in K_{h_1} \times S_{h_2}$ such that

$$A(u_1^h, \sigma^h; v^h - u_1^h, \mu^h) \ge (f, v^h - u_1^h)_0, \ \forall v^h \in K_{h_1}, \mu^h \in S_{h_2}, \tag{3.1}$$

which is equivalent to the following problem: Find $(u_{h_1}, \sigma^h) \in K_{h_1} \times S_{h_2}$ such that

$$\begin{cases} a(u_1^h, v^h - u_1^h) + b\Big(\frac{du_1^h}{ds}, \frac{d(v^h-u_1^h)}{ds}\Big) - \Big\langle\Big(\frac{1}{2}\mathbf{I}+\mathbf{T}'\Big)\sigma^h, v^h - u_1^h\Big\rangle \\ \qquad \ge (f, v^h - u_1^h)_0, \ \forall v^h \in K_{h_1}, \\ \Big\langle \mu^h, \Big(\frac{1}{2}\mathbf{I}+\mathbf{T}\Big)u_1^h\Big\rangle - \langle \mu^h, \mathbf{F}\sigma^h\rangle = 0, \ \forall \mu^h \in S_{h_2}. \end{cases} \tag{3.2}$$

and the problem: Find $(u_1^h, \sigma^h) \in K_{h_1} \times S_{h_2}$, such that

$$L(u_1^h, \mu^h) \le L(u_1^h, \sigma^h) \le L(v^h, \sigma^h), \ \forall (v^h, \mu^h) \in K_{h_1} \times S_{h_2}, \tag{3.3}$$

It is straightforward to check that for $f \in H^{-1}(\Omega_1)$, the problem (3.1) has a unique solution $(u_1^h, \sigma^h) \in K_{h_1} \times S_{h_2}$. Furthermore, we obtain the following abstract error estimate.

**Theorem 3.1.** *Suppose that $f \in H^{-1+\alpha}(\Omega_1)$, and that the solution of* (2.17)*, $(u_1, \sigma)$, satisfies $u_1 \in K \cap H^{1+\alpha}(\Omega_1)$, $\sigma \in \overset{*}{H}{}^{-\frac{1}{2}}(\Gamma_2) \cap H^{-\frac{1}{2}+\alpha}(\Gamma_2)$, with $0 \le \alpha \le 1$, then we have*

$$\begin{aligned} \|(u_1 - u_1^h, \sigma - \sigma_h)\|_{W_0}^2 \le & C_\alpha \inf_{(v^h,\mu^h)\in K_{h_1}\times S_{h_2}} \{\|(u_1 - v^h, \sigma - \mu^h)\|_{W_0}^2 \\ & + \|u_1 - v^h, \sigma - \mu^h)\|_{W_{-\alpha}}\}, \end{aligned} \tag{3.4}$$

*where $(u_1^h, \sigma^h)$ is the solution of* (3.1) *and $C_\alpha$ is a constant independent of $h_1$ and $h_2$.*

*Proof.* By Lamma 2.1 we have

$$\begin{aligned}\|(u_1-u_1^h,\sigma-\sigma^h)\|_{W_0}^2 \le & \frac{1}{\beta}A(u_1-u_1^h,\sigma-\sigma^h;u_1-u_1^h,\sigma-\sigma^h)\\ & +\frac{1}{\beta}\{A(u_1-u_1^h,\sigma-\sigma^h;u_1-v^h,\sigma-\mu^h)\\ & +A(u_1,\sigma;v^h-u_1,\mu^h-\sigma)\\ & -A(u_1^h,\sigma^h;v^h-u_1^h,\mu^h-\sigma^h)\\ & -A(u_1,\sigma;u_1^h-u_1,\sigma^h-\sigma)\},\\ & \forall v^h\in K_{h_1},\mu^h\in S_{h_2}.\end{aligned}$$

On the other hand, we take $v=u_1^h,\mu=\sigma^h-\sigma$ in (2.17), then it follows

$$-A(u_1,\sigma;u_1^h-u_1,\sigma^h-\sigma)\le -(f,u_1^h-u_1)_0.$$

Similarly we take $\mu^h-\sigma^h$ instead of $\mu^h$ in (3.1), then we get

$$-A(u_1^h,\sigma^h;v^h-u_1^h,\mu^h-\sigma^h)\le -(f,v^h-u_1^h)_0.$$

Hence we have

$$\begin{aligned}\|(u_1-u_1^h,\sigma-\sigma^h)\|_{W_0}^2 \le & \frac{1}{\beta}\{A(u_1-u_1^h,\sigma-\sigma^h;u_1-v^h,\sigma-\mu^h)\\ & +A(u_1,\sigma;v^h-u_1,\mu^h-\sigma)+(f,u_1-v^h)_0\}\\ \le & \frac{1}{\beta}\{M\|(u_1-u_1^h,\sigma-\sigma^h)\|_{W_0}\|(u_1-v^h,\sigma-\mu^h)\|_{W_0}\\ & +M_\alpha\|(u_1,\sigma)\|_{W_\alpha}\|(u_1-v^h,\sigma-\mu^h)\|_{W_{-\alpha}}\\ & +\|f\|_{-1+\alpha,\Omega_1}\|u_1-v^h\|_{1-\alpha,\Omega_1}\}\\ \le & \frac{1}{2}\|(u_1-u_1^h,\sigma-\sigma^h)\|_{W_0}^2+\frac{m^2}{2\beta^2}\|u_1-v_h,\sigma-\mu^h)\|_{W_0}^2\\ & +\frac{1}{\beta}[M_\alpha\|(u_1,\sigma)\|_{W_\alpha}+\|f\|_{-1+\alpha,\Omega_1}]\\ & \|(u_1-v^h,\sigma-\mu^h)\|_{W_{-\alpha}},\ \forall(v^h,\mu^H)\in K_{h_1}\times S_{h_2}.\end{aligned}$$

Thus we derive the error estiamte (3.4) with

$$C_\alpha=\max\Big\{\frac{M^2}{\beta^2},\frac{2}{\beta}[M_\alpha\|(u_1,\sigma)\|_{W_\alpha}+\|f\|_{-1+\alpha,\Omega_1}]\Big\}.$$

Let $J_{h_1}$ denote a regular triangulation of $\Omega_1$, i.e. $\Omega_1$ is written as a union $\bigcup_{T_1\in J_{h_1}} T_1$ of triangular $T_1$ and

$$h_1=\max_{T_1\in J_{h_1}}\{\text{ diameter of } T_1\}.$$

Assume that the boundary $\Gamma_2$ of $\Omega_1$ is represented as $x_1=x_1(s),\ x_2=x_2(s),\ 0\le s\le L$, and $x_j(0)=x_j(L),\ j=1,2$. Furthermore, $\Gamma_2$ is divided into segments $\{T_2\}$ by the points $x^j = (x_1(s_i), x_2(s_i))$, $i = 1, 2, \cdots, N_2$. with $0 = s_0 < s_1 < \cdots < s_{N_2} = L$. We define

$$h_2 = \max_{1 \le i \le N_2} |s_{i+1} - s_i|,$$

and this partition of $\Gamma_2$ is denoted as $J_{h_2}$.

Define

$$S_{h_1} = \{v_h \in C^0(\Omega_1) \; |v_{h_1}|_{T_1} \quad \text{is a linear function on } T_1, \; \forall T_1 \in J_{h_1}\}, \tag{3.5}$$

$$K_{h_1} = \{v^h \in S_{h_1}, \; v^h \ge 0 \;\; \text{on } \Gamma_1\}, \tag{3.6}$$

$$S_{h_2} = \Big\{\mu^h \mid \mu^h|_{T_2} \text{ is a constant, } \forall T_2 \in J_{h_2} \;\; \text{and} \;\; \int_{\Gamma_2} \mu^h ds = 0\Big\}. \tag{3.7}$$

Obviously, $K_{h_1}$ is a closed convex subset of $S_{h_1}$, and it is nonempty. $S_{h_1}$ and $S_{h_2}$ are two regular finite element spaces in the sense by Babuška and Aziz[26] which satisfy the following approximation property:

$$\inf_{v^h \in K_{h_1}} \{\|u_1 - v^h\|^2_{1,\Omega_1} + \|u_1 - v^h\|_{1-\alpha,\Omega_1}\} \le C^1_\alpha h_1^{2\alpha} [\|u_1\|^2_{1+\alpha,\Omega_1} + \|u_1\|_{1+\alpha,\Omega_1}],$$

$$\inf_{\mu^h \in S_{h_2}} \{\|\sigma - \mu^h\|^2_{-\frac{1}{2},\Gamma_2} + \|\sigma - \mu^h\|_{-\frac{1}{2}-\alpha,\Gamma_2}\} \le C^2_\alpha h_2^{2\alpha} [\|\sigma\|^2_{-\frac{1}{2}+\alpha,\Gamma_2} + \|\sigma\|_{-\frac{1}{2}+\alpha,\Gamma_2}].$$

By the abstract error estimate (3.4), we obtain

**Theorem 3.2.** *Suppose that $S_{h_1}$ and $S_{h_2}$ are given by* (3.5) *and* (3.7)*, $f$ and the solution $(u_1, \sigma)$ satisfy the assumptions of Theorem* 3.1*. Then the following error estimate holds:*

$$\begin{aligned} \|(u_1 - u_1^h, \sigma - \sigma^h)\|^2_{W_0} \le & \tilde{C}_\alpha \{h_1^{2\alpha} [\|u_1\|^2_{1+\alpha,\Omega_1} + \|u_1\|_{1+\alpha,\Omega_1}] \\ & + h_2^{2\alpha} [\|\sigma\|^2_{-\frac{1}{2}+\alpha,\Gamma_2} + \|\sigma\|_{-\frac{1}{2}+\alpha,\Gamma_2}]\}, \end{aligned} \tag{3.8}$$

*where $\tilde{C}_\alpha$ is a constant independent of $h_1$ and $h_2$.*

From above it is easy to see that we can choose the subspaces $K_{h_1}$ and $S_{h_2}$ independently. We avoid a faster mesh refinement of the boundary elements than that of the finite elements required by Wendland[26]. For optimal error estimate, we should take $h_1 = h_2$.

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