Journal of Computational Mathematics, Vol.16, No.6, 1998, 527–538.

# ON THE CONVERGENCE OF THE RELAXATION METHODS FOR POSITIVE DEFINITE LINEAR SYSTEMS*1)

Zhong-zhi Bai
(*ICMSEC, Chinese Academy of Sciences, P.O.Box 2719, Beijing 100080, China*)
Tin-zhu Huang
(*Department of Applied Mathematics, The University of Electronic Science and Technology of China, Chengdu 610054, China*)

**Abstract**

We establish the convergence theories of the symmetric relaxation methods for the system of linear equations with symmetric positive definite coefficient matrix, and more generally, those of the unsymmetric relaxation methods for the system of linear equations with positive definite matrix.

*Key words*: System of linear equations, Relaxation method, Convergence theory, Positive definite matrix.

## 1. Introduction

The classical iterative methods, such as the Jacobi method, the Gauss-Seidel method and the SOR method, as well as their symmetrized variants, play an important role for solving the large sparse system of linear equations

$$Ax = b, \tag{1.1a}$$

where

$$\begin{aligned} &A = (a_{mj}) \in L(R^n) \quad \text{is a given nonsingular matrix;} \\ &x = (x_1, x_2, \cdots, x_n)^T \in R^n \quad \text{is the unknown vector; and} \\ &b = (b_1, b_2, \cdots, b_n)^T \in R^n \quad \text{is a given vector.} \end{aligned} \tag{1.1b}$$

In accordance with the basic extrapolation principle of the linear iterative method, Hadjidimos[1] further proposed a class of accelerated overrelaxation (AOR) method for solving the linear system (1.1) in 1978. This method includes two arbitrary parameters, and their suitable choices not only can naturally recover the Jacobi, the Gauss-Seidel and the SOR methods, etc., but also can considerably improve the convergence property of this AOR method. After many authors' extensive and deepened researches, the convergence theories of the afore-mentioned relaxation methods have been established in a more complete manner when the coefficient matrix of the linear system (1.1) is

* Received February 9, 1996.
1) Project 19601036 Supported by the National Natural Science Foundation of China.

an $L$-matrix, an $M$-matrix, an $H$-matrix, and a symmetric positive definite matrix, respectively. For details one can refer to [1]–[7] and references therein.

Based on Hadjidimos' work[1], many researchers have designed the symmetrized and, more generally, the unsymmetrized versions of the AOR method, called as the SAOR method and the UAOR method, respectively, and discussed in detail the convergence properties of these methods under the conditions that the coefficient matrix of the linear system (1.1) is either an $L$-matrix, or an $M$-matrix, or an $H$-matrix. For more details one can see [4] and references therein. These studies not only afford efficient algorithm choices for the linear system (1.1), but also establish systematical convergence theories for the relaxation methods.

However, to our knowledge, except for the symmetric positive definite matrix with property–A, there is no convergence result about either the SAOR method or the UAOR method for general (symmetric) positive definite matrix class. The difficulty seems to be that the commutativity as in the SSOR method does not still hold in these methods. In this paper, we will emphatically establish the convergence theory of the SAOR method for the symmetric positive definite matrix class, or more generally, that of the UAOR method for the positive definite matrix class.

## 2. Reviews of the Relaxation Methods

More generally, from now on, we will turn to consider the system of linear equations (1.1) which has the following partitioned form:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}, \tag{2.1a}$$

where

$$A_{i,j} \in L(R^{n_j}, R^{n_i}), \quad x_i, b_i \in R^{n_i}, \quad i, j = 1, 2, \cdots, N \tag{2.1b}$$

and $n_i$ $(i = 1, 2, \cdots, N)$ are positive integers satisfying

$$n_1 + n_2 + \cdots + n_N = n. \tag{2.1c}$$

Also, we will stipulate that $A_{i,i}$ $(i = 1, 2, \cdots, N)$ are nonsingular matrices.

If we take

$$A_D = \text{diag }(A_{1,1}, A_{2,2}, \cdots, A_{N,N}),$$
$$A_L = \begin{pmatrix} 0 & & & & & \\ -A_{2,1} & 0 & & & & \\ \vdots & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & & \\ -A_{N-1,1} & \cdots & \cdots & -A_{N-1,N-2} & 0 & \\ -A_{N,1} & \cdots & \cdots & -A_{N,N-2} & -A_{N,N-1} & 0 \end{pmatrix},$$

and

$$A_U = \begin{pmatrix} 0 & -A_{1,2} & -A_{1,3} & \cdots & \cdots & -A_{1,N} \\ & 0 & -A_{2,3} & \cdots & \cdots & -A_{2,N} \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & -A_{N-1,N} \\ & & & & & 0 \end{pmatrix},$$

then the UAOR method can be expressed as

$$x^{p+1} = \mathcal{H}_{UAOR}(\gamma_1, \omega_1; \gamma_2, \omega_2)x^p + d_{UAOR}(\gamma_1, \omega_1; \gamma_2, \omega_2), \qquad p = 0, 1, 2, \cdots, \tag{2.2}$$

where

$$\begin{cases} \mathcal{H}_{UAOR}(\gamma_1, \omega_1; \gamma_2, \omega_2) = \mathcal{U}_{AOR}(\gamma_2, \omega_2)\mathcal{L}_{AOR}(\gamma_1, \omega_1) \\ d_{UAOR}(\gamma_1, \omega_1; \gamma_2, \omega_2) = \mathcal{U}_{AOR}(\gamma_2, \omega_2)(A_D - \gamma_1 A_L)^{-1}(\omega_1 b) + (A_D - \gamma_2 A_U)^{-1}(\omega_2 b), \end{cases} \tag{2.3}$$

and

$$\begin{cases} \mathcal{L}_{AOR}(\gamma_1, \omega_1) = (A_D - \gamma_1 A_L)^{-1}[(1 - \omega_1)A_D + (\omega_1 - \gamma_1)A_L + \omega_1 A_U] \\ \mathcal{U}_{AOR}(\gamma_2, \omega_2) = (A_D - \gamma_2 A_U)^{-1}[(1 - \omega_2)A_D + (\omega_2 - \gamma_2)A_U + \omega_2 A_L]. \end{cases} \tag{2.4}$$

Clearly, this method covers a lot of known practical relaxation methods, for example,
(1) when $\gamma_1 = \gamma_2 = \gamma$, $\omega_1 = \omega_2 = \omega$, it gives the SAOR method:

$$x^{p+1} = \mathcal{H}_{SAOR}(\gamma, \omega)x^p + d_{SAOR}(\gamma, \omega), \quad p = 0, 1, 2, \cdots, \tag{2.5}$$

where

$$\begin{cases} \mathcal{H}_{SAOR}(\gamma, \omega) = \mathcal{U}_{AOR}(\gamma, \omega)\mathcal{L}_{AOR}(\gamma, \omega) \\ d_{SAOR}(\gamma, \omega) = \mathcal{U}_{AOR}(\gamma, \omega)(A_D - \gamma A_L)^{-1}(\omega b) + (A_D - \gamma A_U)^{-1}(\omega b); \end{cases} \tag{2.6}$$

(2) when $\gamma_1 = \omega_1 = \omega$, $\gamma_2 = \omega_2 = \bar{\omega}$, it turns to the unsymmetric SOR (USOR) method: $x^{p+1} = \mathcal{H}_{USOR}(\omega, \bar{\omega})x^p + d_{USOR}(\omega, \bar{\omega})$, $p = 0, 1, 2, \cdots$, where

$$\begin{cases} \mathcal{H}_{USOR}(\omega, \bar{\omega}) = \mathcal{U}_{SOR}(\bar{\omega})\mathcal{L}_{SOR}(\omega) \\ d_{USOR}(\omega, \bar{\omega}) = \mathcal{U}_{SOR}(\bar{\omega})(A_D - \omega A_L)^{-1}(\omega b) + (A_D - \bar{\omega} A_U)^{-1}(\bar{\omega} b), \end{cases}$$

and

$$\begin{cases} \mathcal{L}_{SOR}(\omega) = (A_D - \omega A_L)^{-1}[(1 - \omega)A_D + \omega A_U] \\ \mathcal{U}_{SOR}(\bar{\omega}) = (A_D - \bar{\omega} A_U)^{-1}[(1 - \bar{\omega})A_D + \bar{\omega} A_L]; \end{cases}$$

(3) when $\gamma_1 = \gamma_2 = \omega_1 = \omega_2 = \omega$, it becomes to the SSOR method: $x^{p+1} = \mathcal{H}_{SSOR}(\omega)x^p + d_{SSOR}(\omega)$, $p = 0, 1, 2, \cdots$, where $\mathcal{H}_{SSOR}(\omega) = \mathcal{U}_{SOR}(\omega)\mathcal{L}_{SOR}(\omega)$, $d_{SSOR}(\omega) = \mathcal{U}_{SOR}(\omega)(A_D - \omega A_L)^{-1}(\omega b) + (A_D - \omega A_U)^{-1}(\omega b)$;

(4) when $\gamma_1 = \gamma_2 = \omega_1 = \omega_2 = 1$, it reduces to the symmetric Gauss-Seidel (SGS) method: $x^{p+1} = \mathcal{H}_{SGS}x^p + d_{SGS}$, $p = 0, 1, 2, \cdots$, where $\mathcal{H}_{SGS} = \mathcal{U}_{SGS}\mathcal{L}_{SGS}$, $d_{SGS} = \mathcal{U}_{SGS}(A_D - A_L)^{-1}b + (A_D - A_U)^{-1}b$, and $\mathcal{L}_{SGS} = (A_D - A_L)^{-1}A_U$, $\mathcal{U}_{SGS} = (A_D - A_U)^{-1}A_L$;

(5) when $\gamma_1=\gamma_2=0$, $\omega_1=\omega_2=1$, it changes to the symmetric Jacobi (SJ) method: $x^{p+1}=\mathcal{H}_{SJ}x^p+d_{SJ}$, $p=0,1,2,\cdots$, where $\mathcal{H}_{SJ}=\mathcal{L}_{SJ}^2$, $d_{SJ}=\mathcal{L}_{SJ}A_D^{-1}b+A_D^{-1}b$, and $\mathcal{L}_{SJ}=A_D^{-1}(A_L+A_U)$;

(6) when $\gamma_1=\gamma$, $\omega_1=\omega$ and $\gamma_2=\omega_2=0$, it gives the AOR method: $x^{p+1}=\mathcal{L}_{AOR}(\gamma,\omega)x^p+d_{AOR}(\gamma,\omega)$, $p=0,1,2,\cdots$, where $d_{AOR}(\gamma,\omega)=(A_D-\gamma A_L)^{-1}(\omega b)$;

(7) when $\gamma_1=0$, $\omega_1=1$ and $\gamma_2=\omega_2=0$, it reduces to the Jacobi method: $x^{p+1}=\mathcal{L}_Jx^p+d_J$, $p=0,1,2,\cdots$, where $\mathcal{L}_J=A_D^{-1}(A_L+A_U)$, $d_J=A_D^{-1}b$.

## 3. Convergence Theory of the UAOR Method

In this section, we will mainly discuss the convergence property of the UAOR-method when the coefficient matrix of the system of linear equations (1.1a) with (2.1) is positive definite. More concretely, we will deduce sufficient conditions that can guarantee the convergence of the UAOR method, and thereby, in particular, the USOR method, for this matrix class. First of all, we demonstrate the following equivalent expression of the iteration formula (2.2).

**Theorem 3.1.** *Let*

$$G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)=(\omega_1+\omega_2)A_D-\omega_2\gamma_1A_L-\omega_1\gamma_2A_U-\omega_1\omega_2A. \tag{3.1}$$

*Then the iteration formula* (2.2) *is equivalent to*

$$\begin{cases} x^{p+1}=x^p+(A_D-\gamma_2A_U)^{-1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)(A_D-\gamma_1A_L)^{-1}(b-Ax^p)\\ p=0,1,2,\cdots. \end{cases} \tag{3.2}$$

*Hence, if* $G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ *is nonsingular, then*

$$x^{p+1}=x^p+B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}(b-Ax^p),\quad p=0,1,2,\cdots, \tag{3.3}$$

*where*

$$B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)=(A_D-\gamma_1A_L)G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}(A_D-\gamma_2A_U). \tag{3.4}$$

*Proof.* (3.3) is obviously a direct corollary of (3.2). So, we only need to verify the validity of (3.2). By (2.4) it clearly holds that

$$\begin{cases} \mathcal{L}_{AOR}(\gamma_1,\omega_1)=I-(A_D-\gamma_1A_L)^{-1}(\omega_1A)\\ \mathcal{U}_{AOR}(\gamma_2,\omega_2)=I-(A_D-\gamma_2A_U)^{-1}(\omega_2A). \end{cases}$$

Therefore, according to (2.3) we can obtain

$$\begin{aligned}\mathcal{H}_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)=&[I-(A_D-\gamma_2A_U)^{-1}(\omega_2A)][I-(A_D-\gamma_1A_L)^{-1}(\omega_1A)]\\ =&I-(A_D-\gamma_1A_L)^{-1}(\omega_1A)-(A_D-\gamma_2A_U)^{-1}(\omega_2A)\\ &+(A_D-\gamma_2A_U)^{-1}(\omega_2A)(A_D-\gamma_1A_L)^{-1}(\omega_1A)\\ =&I-(A_D-\gamma_2A_U)^{-1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)(A_D-\gamma_1A_L)^{-1}A.\end{aligned}$$

In addition, we can analogously get

$$\begin{aligned} d_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2) =&[I-(A_D-\gamma_2 A_U)^{-1}(\omega_2 A)](A_D-\gamma_1 A_L)^{-1}(\omega_1 b)\\ &+(A_D-\gamma_2 A_U)^{-1}(\omega_2 b)\\ =&(A_D-\gamma_2 A_U)^{-1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)(A_D-\gamma_1 A_L)^{-1}b. \end{aligned}$$

Substituting these two equalities into (2.2) we can immediately obtain (3.2).

**Theorem 3.2.** *Assume that the matrix $G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ defined by* (3.1) *is nonsingular. Then*

(*i*) $B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)=A+F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$,
*where the matrix $B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is given by* (3.4), *and*

$$F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)=(A_D-\gamma_2 A_U-\omega_2 A)G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}(A_D-\gamma_1 A_L-\omega_1 A);$$

(*ii*) *if $A$ is a positive definite matrix and $F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is a nonnegative definite matrix, it holds that*

$$\sup_{x\neq 0}\frac{x^TAx}{x^TB_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)x}\leq 1.$$

*Furthermore, if $F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is also a positive definite matrix, then*

$$\sup_{x\neq 0}\frac{x^TAx}{x^TB_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)x}< 1.$$

*Proof.* We first test (i). By direct calculations we have

$$\begin{aligned} &(A_D-\gamma_1 A_L)G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}(A_D-\gamma_2 A_U)\\ &\quad=\Big[\frac{1}{\omega_2}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)+\Big(A_D-\gamma_1 A_L-\frac{1}{\omega_2}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)\Big]\\ &\quad\quad\times G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}\\ &\quad\quad\times\Big[\frac{1}{\omega_1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)+\Big(A_D-\gamma_2 A_U-\frac{1}{\omega_1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)\Big]\\ &\quad=\frac{1}{\omega_1\omega_2}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)+\frac{1}{\omega_2}\Big(A_D-\gamma_2 A_U-\frac{1}{\omega_1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)\\ &\quad\quad+\frac{1}{\omega_1}\Big(A_D-\gamma_1 A_L-\frac{1}{\omega_2}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)\\ &\quad\quad+\Big(A_D-\gamma_1 A_L-\frac{1}{\omega_2}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}\\ &\quad\quad\times\Big(A_D-\gamma_2 A_U-\frac{1}{\omega_1}G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)\Big)\\ &\quad=A+(A_D-\gamma_2 A_U-\omega_2 A)G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}(A_D-\gamma_1 A_L-\omega_1 A), \end{aligned}$$

which is just the equality of (i).

Clearly, (ii) is a direct corollary of (i).

Based on Theorem 3.2, we can now give the following sufficient condition that can ensure the convergence of the UAOR-method.

**Theorem 3.3.** *Assume that the matrices $A \in L(R^n)$ is nonsingular and definite, $G_{UAOR}(\gamma_1,\ \omega_1; \gamma_2, \omega_2)$ is nonsingular. Then the UAOR-method is convergent to the unique solution of the system of linear equations* (1.1*a*) *with* (2.1) *if it holds that*

$$\inf_{y\neq 0}\Big\{Re\Big(\frac{y^H F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y}{y^H Ay}\Big)\Big\} > -\frac{1}{2},$$

*where $Re(\bullet)$ denotes the real part and $Im(\bullet)$ the imaginary part of the corresponding complex, respectively.*

*Proof.* Clearly, under the conditions of this theorem, we know that $B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is nonsingular, and for all $y \in C^n\setminus\{0\}$ there hold $y^H Ay \neq 0$. Let $\lambda$ be an eigenvalue of the matrix $B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)^{-1}A$, and $y$ be the corresponding eigenvector. Then, there has $\lambda \neq 0$ and it holds that $Ay = \lambda B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y$. Hence,

$$\lambda = \frac{y^H Ay}{y^H B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y} = \frac{y^H Ay}{y^H Ay + y^H F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y}.$$

Now, if we denote

$$\zeta(y) = \frac{y^H F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y}{y^H Ay}$$

and

$$\xi(y) = Re(\zeta(y)), \quad \eta(y) = Im(\zeta(y)),$$

then $\lambda = 1/(1+\zeta(y))$.

Let $\mu$ be an eigenvalue of the matrix $\mathcal{H}_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$. Obviously, there holds

$$\mu = 1-\lambda = \frac{\zeta(y)}{1+\zeta(y)}, \quad \forall y \in \mathcal{E}^n,$$

where

$$\mathcal{E}^n = \{y \in C^n \setminus \{0\} \mid Ay = \lambda B_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)y\}.$$

Hence, the convergence of the UAOR-method is equivalent to

$$|\mu| = \frac{\sqrt{\xi(y)^2+\eta(y)^2}}{\sqrt{(1+\xi(y))^2+\eta(y)^2}} < 1, \quad \forall y \in \mathcal{E}^n.$$

Noticing that this inequality is equivalent to $\xi(y) > -\frac{1}{2}$ $(\forall y \in \mathcal{E}^n)$, we thereby fulfill the proof of this theorem.

Theorem 3.3 immediately implies the following sufficient condition for guaranteeing the convergence of the UAOR-method.

**Theorem 3.4.** *If the matrix $A$ is positive definite, the matrix $G_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is nonsingular, and the matrix $F_{UAOR}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is nonnegative definite, then the UAOR-method is convergent.*

Now, if we introduce matrices

$$\begin{aligned} &G_{SJ} = G_{UAOR}(0,1;0,1) = 2A_D - A; \\ &G_{SGS} = G_{UAOR}(1,1;1,1) = A_D; \\ &G_{SSOR}(\omega) = G_{UAOR}(\omega,\omega;\omega,\omega) = \omega(2-\omega)A_D; \\ &G_{USOR}(\omega,\bar\omega) = G_{UAOR}(\omega,\omega;\bar\omega,\bar\omega) = (\omega+\bar\omega-\omega\bar\omega)A_D; \\ &G_{SAOR}(\gamma,\omega) = G_{UAOR}(\gamma,\omega;\gamma,\omega) = \omega(2-\gamma)A_D + \omega(\gamma-\omega)A, \end{aligned}$$

and

$$\begin{aligned} F_{SJ} =& F_{UAOR}(0,1;0,1) = (A_D - A)G_{SJ}^{-1}(A_D - A); \\ F_{SGS} =& F_{UAOR}(1,1;1,1) = A_L G_{SGS}^{-1} A_U; \\ F_{SSOR}(\omega) =& F_{UAOR}(\omega,\omega;\omega,\omega) \\ =& (A_D - \omega A_U - \omega A)G_{SSOR}(\omega)^{-1}(A_D - \omega A_L - \omega A); \\ F_{USOR}(\omega,\bar\omega) =& F_{UAOR}(\omega,\omega;\bar\omega,\bar\omega) \\ =& (A_D - \bar\omega A_U - \bar\omega A)G_{USOR}(\omega,\bar\omega)^{-1}(A_D - \omega A_L - \omega A); \\ F_{SAOR}(\gamma,\omega) =& F_{UAOR}(\gamma,\omega;\gamma,\omega) \\ =& (A_D - \gamma A_U - \omega A)G_{SAOR}(\gamma,\omega)^{-1}(A_D - \gamma A_L - \omega A), \end{aligned}$$

then the following convergence conclusions about the SJ-method, SGS-method, SSOR-method, USOR-method and the SAOR-method can be directly got from Theorem 3.4.

**Corollary 3.1.** *Let the matrix $A$ be positive definite. Then*

(1) *if $(2A_D - A)$ is nonsingular and $F_{SJ}$ is nonnegative definite, the SJ-method is convergent;*

(2) *if $F_{SGS}$ is nonnegative definite, the SGS-method is convergent;*

(3) *if $\omega \in R^1 \setminus \{0,2\}$ and $F_{SSOR}(\omega)$ is nonnegative definite, the SSOR-method is convergent;*

(4) *if $\omega + \bar\omega - \omega\bar\omega \neq 0$ and $F_{USOR}(\omega,\bar\omega)$ is nonnegative definite, the USOR-method is convergent;*

(5) *if $G_{SAOR}(\gamma,\omega)$ is nonsingular and $F_{SAOR}(\gamma,\omega)$ is nonnegative definite, the SAOR-method is convergent.*

**Corollary 3.2.** *Let the matrix $A$ be symmetric positive definite. Then*

(1) *if $(2A_D - A)$ is positive definite, the SJ-method is convergent;*

(2) *the SGS-method is convergent;*

(3) *if $\omega(2-\omega) > 0$, the SSOR-method is convergent;*

(4) *if $G_{SAOR}(\gamma,\omega)$ is positive definite, the SAOR-method is convergent.*

## 4. Convergence Theory of the SAOR Method

In this section, we will further investigate conditions for ensuring the convergence of the SAOR-method when the coefficient matrix of the system of linear equations (1.1a) with (2.1) is a symmetric positive definite matrix. Corollary 3.2 in the last section will be our basis through this section. Noticing from the last section we have that

$$G_{SAOR}(\gamma,\omega) = \omega(2-\gamma)A_D + \omega(\gamma-\omega)A,$$

and the SAOR-method is convergent if $G_{SAOR}(\gamma,\omega)$ is positive definite.

Because the matrix $G_{SAOR}(\gamma,\omega)$ is congruent with the matrix

$$\omega(2-\gamma)I+\omega(\gamma-\omega)A_D^{-1/2}AA_D^{-1/2},$$

while this matrix is similar to the matrix

$$\omega(2-\gamma)I+\omega(\gamma-\omega)A_D^{-1}A,$$

or, after using $A_D^{-1}A=I-\mathcal{L}_J$, it is similar to the matrix

$$P_{SAOR}(\gamma,\omega)=\omega(2-\omega)I+\omega(\omega-\gamma)\mathcal{L}_J,$$

we see that $G_{SAOR}(\gamma,\omega)$ is positive definite if and only if all the eigenvalues of the matrix $P_{SAOR}(\gamma,\omega)$ are positive. If we define the functions

$$F_J(x)=\frac{x^T\mathcal{L}_Jx}{x^Tx},\quad \forall x\in R^n\setminus\{0\}, \tag{4.1}$$

then it is clear that the matrix $G_{SAOR}(\gamma,\omega)$ is positive definite if and only if

$$\omega(2-\omega)+\omega(\omega-\gamma)F_J(x)>0,\quad \forall x\in R^n\setminus\{0\}. \tag{4.2}$$

It is much easy to see that either of the following six conditions are sufficient to guarantee the validity of (4.2):

A) $\omega\geq\gamma$, $\omega>0$, $2-\omega+(\omega-\gamma)\min\limits_{x\neq 0}F_J(x)>0$;

B) $0<\omega\leq\gamma$, $2-\omega+(\omega-\gamma)\max\limits_{x\neq 0}F_J(x)>0$;

C) $\omega\leq\gamma$, $\omega<0$, $2-\omega+(\omega-\gamma)\min\limits_{x\neq 0}F_J(x)<0$;

D) $\gamma\leq\omega<0$, $2-\omega+(\omega-\gamma)\max\limits_{x\neq 0}F_J(x)<0$.

Based upon these observations, we immediately have the following conclusions about the convergence of the SAOR-method.

**Theorem 4.1.** *Let the matrix $A$ be symmetric positive definite. Assume $\underline{\lambda},\overline{\lambda}$ be respectively the smallest and the largest eigenvalues of the matrix $\mathcal{L}_J$. Then the SAOR-method is convergent if either of the following* 17 *conditions is satisfied:*

($a_1$) $\underline{\lambda}>1$, $0\leq\gamma<\omega$, $0<\omega<2$;

($a_2$) $\underline{\lambda}>1$, $0\leq\gamma<\dfrac{(\underline{\lambda}-1)\omega+2}{\underline{\lambda}}$, $\omega\geq 2$;

($a_3$) $1>\underline{\lambda}\neq 0$, $0\leq\gamma\leq\min\{2,\ \omega\}$, $0<\omega<\dfrac{\underline{\lambda}\gamma-2}{\underline{\lambda}-1}$;

($a_4$) $\underline{\lambda}=1$, $0\leq\gamma\leq\min\{2,\ \omega\}$;

($a_5$) $\underline{\lambda}>1$, $\gamma\leq 0<\omega$;

($a_6$) $0<\underline{\lambda}<1$, $\gamma\leq 0$, $0<\omega<\dfrac{\underline{\lambda}\gamma-2}{\underline{\lambda}-1}$;

($a_7$) $\underline{\lambda}<0$, $\dfrac{2}{\underline{\lambda}}\leq\gamma\leq 0$, $0<\omega<\dfrac{\underline{\lambda}\gamma-2}{\underline{\lambda}-1}$;

($a_8$) $\underline{\lambda}=1$, $\gamma\leq 0<\omega$;

($b_1$) $0 < \overline{\lambda} \neq 1,\ \omega \le \gamma \le \dfrac{(\overline{\lambda}-1)\omega + 2}{\overline{\lambda}},\ 0 < \omega \le 2$;

($b_2$) $\overline{\lambda} < 0,\ 0 \le \gamma \le 2,\ 0 < \omega \le \gamma$;

($b_3$) $\overline{\lambda} < 0,\ \gamma > 2,\ 0 < \omega < \dfrac{\overline{\lambda}\gamma - 2}{\overline{\lambda} - 1}$;

($b_4$) $\overline{\lambda} = 1,\ 0 \le \gamma \le 2,\ 0 < \omega \le \gamma$;

($c_1$) $\underline{\lambda} > 1,\ \gamma > \dfrac{(\underline{\lambda}-1)\omega + 2}{\underline{\lambda}},\ \dfrac{2}{1-\underline{\lambda}} \le \omega < 0$;

($c_2$) $0 < \underline{\lambda} < 1,\ \gamma \ge \dfrac{2}{\underline{\lambda}},\ \dfrac{\underline{\lambda}\gamma - 2}{\underline{\lambda} - 1} < \omega < 0$;

($c_3$) $\underline{\lambda} = 1,\ \gamma > 2,\ \omega < 0$;

($c_4$) $\underline{\lambda} > 1,\ \dfrac{(\underline{\lambda}-1)\omega + 2}{\underline{\lambda}} < \gamma \le 0,\ \omega \le \dfrac{2}{1-\underline{\lambda}}$;

($d_1$) $\overline{\lambda} < 0,\ \gamma \le \dfrac{2}{\overline{\lambda}},\ \dfrac{\overline{\lambda}\gamma - 2}{\overline{\lambda} - 1} < \omega < 0$.

*Proof.* Noticing that $\underline{\lambda} = \min\limits_{x \neq 0} F_J(x)$, $\overline{\lambda} = \max\limits_{x \neq 0} F_J(x)$, by directly solving the previous inequalities A)-D), we can obtain the conclusion of the theorem at once.

**Corollary 4.1.** *Let the matrix $A$ be symmetric positive definite. Assume $\underline{\lambda}$, $\overline{\lambda}$ be respectively the smallest and the largest eigenvalues of the matrix $\mathcal{L}_J$. Then*

(1) *the SJ-method is convergent if $\underline{\lambda} > -1$;*

(2) *the SGS-method is convergent;*

(3) *the SSOR-method is convergent provided $\omega \in (0, 2)$.*

If we denote $\rho_J = \rho(\mathcal{L}_J)$, then we easily see that the inequalities A)-D) are satisfied if the following inequalities hold, respectively:

A') $\omega \ge \gamma,\ \omega > 0,\ 2 - \omega - (\omega - \gamma)\rho_J > 0$;

B') $0 < \omega \le \gamma,\ 2 - \omega + (\omega - \gamma)\rho_J > 0$;

C') $\omega \le \gamma,\ \omega < 0,\ 2 - \omega - (\omega - \gamma)\rho_J < 0$;

D') $\gamma \le \omega < 0,\ 2 - \omega + (\omega - \gamma)\rho_J < 0$.

Hence, the SAOR-method is convergent if either of the above inequalities A')–D') is valid. Based upon this fact, we can analogously get the following conclusions about the convergence of the SAOR-method.

**Theorem 4.2.** *Let the matrix $A$ be symmetric positive definite. Assume that $\rho_J = \rho(\mathcal{L}_J)$. Then the SAOR-method is convergent if either of the following four conditions is satisfied:*

(*i*) $\rho_J > 0,\ 0 < \omega < 2,\ \dfrac{(1+\rho_J)\omega - 2}{\rho_J} < \gamma \le \omega$;

(*ii*) $0 < \rho_J \le 1,\ 0 < \omega < 2,\ \omega \le \gamma < \dfrac{2 - (1-\rho_J)\omega}{\rho_J}$;

(*iii*) $\rho_J > 1,\ 0 < \omega \le \gamma,\ \gamma \ge \dfrac{2}{\rho_J}$;

(*iv*) $\rho_J = 0,\ \gamma < 2,\ 0 < \omega < 2$.

*Proof.* Solving the inequalities A')–D') directly, we can obtain (i)–(iv) immediately.

**Corollary 4.2.** *Let the matrix $A$ be symmetric positive definite. Assume that $\rho_J = \rho(\mathcal{L}_J)$. Then*

(1) *the SJ-method is convergent if $\rho_J < 1$;*

(2) *the SGS-method is convergent;*
(3) *the SSOR-method is convergent if* $\omega \in (0, 2)$.

## 5. Some Concrete Applications

We now consider some applications of our novel theoretical results demonstrated in the previous section to two classes of practical systems of linear equations, which are resulted from the discretizations of the one-dimensional and two-dimensional model problems, i.e., the two-point boundary value problems of the Poisson equations:

$$\begin{cases} -x_{tt}(t) = f(t), & \text{for } \ 0 < t < 1 \\ x(t) = x_t(t) = 0, & \text{for } \ t = 0 \end{cases}$$

and

$$\begin{cases} -\Delta x = -x_{ss} - x_{tt} = f(s,t), & \text{for } \ 0 < s,t < 1 \\ x(s,t) = 0, & \text{for } \ s = 0 \text{ or } t = 0 \\ x(s,t) = 1, & \text{for } \ s = 1 \text{ or } t = 1, \end{cases}$$

on equidistant grids by either the finite difference method or the finite element method, respectively. These systems of linear equations are respectively of the following forms:

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \in L(R^n)$$

and

$$A = \begin{pmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & B & -I \\ & & & & -I & B \end{pmatrix} \in L(R^{N^2}),$$

where

$$B = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix} \in L(R^N)$$

and the right-hand sides are chosen suitably. For these special linear systems, if we use the symmetric relaxation methods to get their solutions, we can immediately get the following convergence conclusions for these methods by making use of Theorem 4.1 and Corollary 4.1, as well as Theorem 4.2 and Corollary 4.2.

**Theorem 5.1.** *Let us solve the linear system* (1.1) *corresponding to the one-dimensional model problem by the symmetric relaxation methods. Then*

($i$) *the pointwise SJ-method is convergent;*

($ii$) *the pointwise SGS-method is convergent;*

($iii$) *the pointwise SSOR-method is convergent provided* $\omega \in (0,2)$*;*

($iv$) *the pointwise SAOR-method is convergent in either of the following four cases:*

(1) $0 \le \gamma \le \min\{2,\ \omega\}$, $0 < \omega < \dfrac{2(\gamma \sin^2 \vartheta_n - 1)}{2\sin^2 \vartheta_n - 1}$;

(2) $\omega \le \gamma \le \dfrac{(2\cos^2 \vartheta_n - 1)\omega + 2}{2\cos^2 \vartheta_n}$, $0 < \omega \le 2$;

(3) $\gamma \le 0$, $0 < \omega < \dfrac{2(\gamma \sin^2 \vartheta_n - 1)}{2\sin^2 \vartheta_n - 1}$;

(4) $\gamma \ge \sin^{-2} \vartheta_n$, $\dfrac{2(\gamma\ sin^2\vartheta_n - 1)}{2\sin^2 \vartheta_n - 1} < \omega < 0$,

*where* $\vartheta_n = \dfrac{\pi}{2(n+1)}$. *Moreover, the pointwise SAOR-method is convergent also in either of the following three cases:*

(1′) $0 \le \gamma \le \omega$, $0 < \omega < \dfrac{2(\gamma \cos^2 \vartheta_n + 1)}{2\cos^2 \vartheta_n + 1}$;

(2′) $0 < \omega < 2$, $\omega \le \gamma < \dfrac{2 - (1 - 2\cos^2 \vartheta_n)\omega}{2\cos^2 \vartheta_n}$;

(3′) $0 < \omega < \dfrac{2}{1 + 2\cos^2 \vartheta_n}$, $\dfrac{(1 + 2\cos^2 \vartheta_n)\omega - 2}{2\cos^2 \vartheta_n} < \gamma \le 0$.

*Proof.* By direct calculations we know that the eigenvalues of the matrix $\mathcal{L}_J$ are

$$\lambda_m = 2\cos^2\Big(\frac{m\pi}{2(n+1)}\Big) = 2\cos^2(m\vartheta_n), \quad m = 1, 2, \cdots, n.$$

Therefore,

$$\underline{\lambda} = 2\cos^2\Big(\frac{n\pi}{2(n+1)}\Big) = 2\sin^2 \vartheta_n, \quad \overline{\lambda} = 2\cos^2\Big(\frac{\pi}{2(n+1)}\Big) = 2\cos^2 \vartheta_n$$

and $\rho_J = 2\cos^2 \vartheta_n$. Substituting these quantities into Theorem 4.1 and Theorem 4.2, and making use of Corollary 4.1 and Corollary 4.2, respectively, we can get the results of this theorem.

**Theorem 5.2.** *Let us solve the linear system* (1.1$a$) *with* (2.1) *corresponding to the two-dimensional model problem by the* (*blockwise*) *symmetric relaxation methods. Then*

($i$) *the SJ-method is convergent;*

($ii$) *the SGS-method is convergent;*

($iii$) *the SSOR-method is convergent provided* $\omega \in (0,2)$*;*

($iv$) *the SAOR-method is convergent in either of the following three cases:*

(1) $0 \le \gamma \le \min\{2,\ \omega\}$, $0 < \omega < \dfrac{\gamma \cos(2\vartheta_N) + 2(1 + 2\cos^2 \vartheta_N)}{4\cos^2 \vartheta_N}$;

(2) $\omega \le \gamma \le \dfrac{4 - 4(1-\omega)\sin^2 \vartheta_N}{\cos(2\vartheta_N)}$, $0 < \omega \le 2$;

(3) $-\dfrac{2(1 + 2\cos^2 \vartheta_N)}{\cos(2\vartheta_N)} \le \gamma \le 0$, $0 < \omega < \dfrac{\gamma \cos(2\vartheta_N) + 2(1 + 2\cos^2 \vartheta_N)}{4\cos^2 \vartheta_N}$,

*where* $\vartheta_N = \dfrac{\pi}{2(N+1)}$. *Moreover, the SAOR-method is convergent also in either of the following three cases:*

(1′) $0 \le \gamma \le \omega$, $0 < \omega < \frac{1}{2}\gamma\cos(2\vartheta_N) + 2\sin^2\vartheta_N + 1$;

(2′) $0 < \omega < 2$, $\omega \le \gamma < \dfrac{2 + 4(1-\omega)\sin^2\vartheta_N}{\cos(2\vartheta_N)}$;

(3′) $0 < \omega < 2\sin^2\vartheta_N + 1$, $\dfrac{4\sin^2\vartheta_N - 2(1-\omega)}{\cos(2\vartheta_N)} < \gamma \le 0$.

*Proof.* By direct calculations we know that the eigenvalues of the matrix $\mathcal{L}_J$ are

$$\lambda_{m,j} = \frac{1 - 2\sin^2(j\vartheta_N)}{1 + 2\sin^2(m\vartheta_N)}, \quad m, j = 1, 2, \cdots, N.$$

Therefore,

$$\underline{\lambda} = \frac{1 - 2\sin^2(N\vartheta_N)}{1 + 2\sin^2(N\vartheta_N)} = -\frac{\cos(2\vartheta_N)}{1 + 2\cos^2\vartheta_N}, \quad \overline{\lambda} = \frac{1 - 2\sin^2\vartheta_N}{1 + 2\sin^2\vartheta_N} = \frac{\cos(2\vartheta_N)}{1 + 2\sin^2\vartheta_N},$$

and

$$\rho_J = \frac{\cos(2\vartheta_N)}{1 + 2\sin^2\vartheta_N}.$$

Substituting these quantities into Theorem 4.1 and Theorem 4.2, and making use of Corollary 4.1 and Corollary 4.2, respectively, we can get the results of this theorem.

## References


[1] A. Hadjidimos, Accelerated overrelaxation method, *Math. Comp.,* **32** (1978), 149–157.

[2] A. Hadjidimos, A. Yeyios, The principle of extrapolation in connection with the accelerated method, *Linear Algebra Appl.,* **30** (1980), 115–128.

[3] A. Hadjidimos, A. Yeyios, On some extensions of the accelerated overrelaxation (AOR) theory, *Intern. J. Math. Sci.,* **5** (1982), 49–60.

[4] J.G. Hu, Iterative Methods for Systems of Linear Algebraic Equations, Science Press, Beijing, 1991.

[5] M. Neumann, R.S. Varga, On the sharpness of some upper bound for the spectral radii of S.O.R. iteration matrices, *Numer. Math.,* **35** (1980), 69–79.

[6] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[7] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.