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ORTHOGONALITY CORRECTION TECHNIQUE IN SUPERCONVERGENCE ANALYSIS

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Abstract. A technique of orthogonality correction in an element is introduced and applied to superconvergence analysis in finite element method. Ultraconvergence results for rectangular elements of odd degree $n \geq 3$ are derived in the case of variable coefficients.

Key Words. Finite elements, rectangular element, orthogonality correction, superconvergence and ultraconvergence.

1. Introduction

Consider a second order elliptic problem with Dirichlet condition (BV1)

$$Au \equiv -D_j(a_{ij}D_iu) + a_0u = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma, \tag{1}$$

where Ω is a planar polygonal domain with the boundary Γ. Denote by $W^{k,p}(\Omega)$ Sobolev space with norm

$$||u||_{k,p,\Omega} = (\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u(x)|^p dx)^{1/p}.$$

If $p = 2$, the subscript p is often omitted and we simply use $H^k(\Omega)$ and $||u||_{k,\Omega}$. Assume that the domain Ω is subdivided into a finite number of elements τ (with h the largest diameter of all τ) and its mesh J^h is quasiuniform.

Introduce the following subspace

$$V = \{u :\ u \in H^1(\Omega), u = 0 \text{ on } \Gamma\}$$

and the n-degree finite element subspace by

$$S^h = \{v :\ v \in C(\Omega),\ v|_{\tau_j} \in P_n,\quad \tau_j \in J^h, v|_\Gamma = 0\}.$$

Define the bilinear form and inner product

$$A(u, v) = \int_\Omega (a_{ij}D_iuD_jv + a_0uv)dx,\ f(v) = (f, v)$$

and assume that $A(u, u)$ is V-coercive. We know that the true solution $u \in V$ and its finite element approximation $u_h \in S^h$ satisfy the following orthogonal relation

$$A(u - u_h, v) = 0,\quad v \in S^h. \tag{2}$$

It is well known that under some conditions there are some basic error estimates

$$\|u - u_h\|_{j,\Omega} \leq ch^{n+1-j}\|u\|_{n+1,\Omega},\quad j = 0, 1;\ n \geq 1. \tag{3}$$

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and negative norm estimates (if $n \geq 2,\ 2 \leq s \leq n+1$)

$$(4) \qquad \|u - u_h\|_{-l,\Omega} = \sup_{v \in H^l(\Omega)} \frac{|(u - u_h, v)|}{||v||_l} \leq ch^{s+l}\|u\|_{s,\Omega},\ 1 \leq l \leq n-1,$$

in which $||u - u_h||_{1-n} = O(h^{2n})$ is of superconvergence of the highest order known.

(4) seems to imply two kinds of ideas to study superconvergence. First, (4) means the approximate orthogonality of $e = u - u_h$, because of the arbitrariness of v, which is a local property (generally, the approximate orthogonality is invalid in an element) and leads to various local averaging. Second, (4) also means that (e, v) is of higher accuracy than $||e||$. As a result, the error $e = u - u_h$ has to change rapidly its sign in Ω in order that the cancellation of the positive and negative values makes the integral less. We want to know whether the distribution of zeroes of e has certain regular patterns, and whether it is possible to find its approximate points independent of the coefficients of A and the concrete behavior of u. This is the study of superconvergence points.

Up to now, there are five main methods:

1. The local averaging method. Motivated by the negative norm estimates and interior estimates, applying the splines on a uniform mesh to construct the kernel function K_h^α with small support and α-order difference quotient $\partial_h^\alpha u_h$, Bramble and Schatz [2] (1974-77) and Thomee [28](1977) obtained the high accurate convolution

$$K_h^\alpha * \partial_h^\alpha u_h - D^\alpha u = O(h^{2n}),\ \text{in } \Omega_0 \subset\subset \Omega.$$

Here both function and derivatives of any order are of optimal order superconvergence $O(h^{2n})$. This is incompatible for other methods. Later these results were extended to parabolic equation (Thomee [29] 1980) and nonlinear problems (Chen [6] 1983).

2. Quasi-projection and Tensor Product Method. It is adopted by Douglas, Dupont and Wheeler [17](1974). This method requires the coefficient $a_{12} = 0$ and the use of tensor product polynomials. The tensor product idea is clear and powerful. We remark that it can be applicable to the time-space full discrete problems.

3. Element Orthogonality Analysis (EOA). It was started by Zlamal [39][40](1977-78), and independently found by Chen [3][4][5] (1978-81) at el. Many other scholars worked in this aspect, such as Zhu[36], Lin-Xu [23],Chen and Huang [13], Krizek- Neittaamäki [19], et al. EOA is based on orthogonal approximations in the bilinear inner product sense and it doesn't depend directly on the estimates for PDEs. Therefore, this method is applicable to more general equations, and the corresponding conclusions are often valid up to the boundary (mainly under BV1). To solve the general equations, Chen proposed three important techniques: cancellation technique between elements, orthogonal expansion and orthogonality correction in an element. Chinese scholars have finished the systematical work in this approach, see Chen [12], Chen-Huang [13] and Lin-Yan [24] and so on.

4. Computer-based research. Babuska-Strouboulis et al. in 1995 (see [1]) finished a systematical computational search for superconvergence points of derivatives and drew a lot of valuable conclusions. In particular, a surprising structure of superconvergence for triangular elements of degree $1 \sim 7$ is first exhibited. Their research has shown a very promising future for new approaches.

5. Locally symmetric theory. If the mesh is locally symmetric with respect to a point x_0 in some $O(h^\beta)$-neighborhood of x_0, Schatz-Sloan-Wahlbin [26][31](1995) affirmed superconvergence at $x_0 \subset\subset \Omega$ as follows:

(1) The function $u - u_h = O(h^{n+1+\alpha}), 0 < \alpha < 1$, for even $n \geq 2$;

(2) The average gradient $\bar{D}(u - u_h) = O(h^{n+\alpha})$, for odd $n \geq 1$,

where the index α sometimes is close to 1. As these conclusions are valid for any type of elements and the structure of the mesh in a neighborhood of locally symmetric point possibly is quite general, they are called "general principles" by Wahlbin [32].

The works of Babuska and Wahlbin et al are so exciting that they can encourage us to advance our method. So the orthogonality correction technique in an element was proposed. In many cases, such as high degree triangular elements [10](1997-99), serendipity rectangular elements[11], beam bending problem and L^2-projection and so on, a lot of superconvergence results are derived. The purpose in this paper is to study ultraconvergence of rectangular finite elements for elliptic problems with variable coefficients and a famous result of Douglas-Dupont-Wheeler[17] (1974) is extended.

2. Orthogonality Correction Techniques (OCT) in EOA

The basic idea of EOA is to construct a better projection interpolant (or approximation) $u_I \in S^h$ of u, such that u_I is super-close to the finite element solution $u_h \in S^h$. From (2), it is equivalent to require the approximate orthogonality of $R = u - u_I$ to S^h (see [4, 5, 6]),

$$A(u_h - u_I, v) = A(R, v) = O(h^{n+\alpha})\|v\|_{1,p',\Omega}, \text{ if } n \geq 1; \tag{5}$$

$$or \ = O(h^{n+1+\alpha})\|v\|^*_{2,p',\Omega}, \text{ if } n \geq 2, \ v \in S^h, \tag{6}$$

where the index $\alpha > 0$ and $\|v\|^*_{2,p',\Omega} = (\sum_\tau \|v\|^{p'}_{2,p',\tau})^{1/p'}$ is the mesh norm, and $p' = p/(p-1)$. Taking $v = u_h - u_I$ and $p = 2$ in (5), we immediately get a superconvergence estimate in H^1,

$$||u_h - u_I||_1 = O(h^{n+\alpha}), \ n \geq 1.$$

Constructing a $w \in V$ such that $Aw = u_h - u_I$ in Ω, by the duality argument and (6), we have

$$\begin{aligned} ||u_h - u_I||^2 &= A(u_h - u_I, w) = A(u - u_I, w_I) + A(u_h - u_I, w - w_I) \\ &= O(h^{n+1+\alpha})(||w_I||^*_2 + ||w||_2) = O(h^{n+1+\alpha})||u_h - u_I|| \end{aligned}$$

and get

$$||u_h - u_I|| = O(h^{n+1+\alpha}), \ n \geq 2.$$

From these estimates, we have the expressions in L^2-sense,

$$D^s(u_h - u_I) = O(h^{n+1+\alpha-s}), \ s = 0, 1, \ n + s \geq 2, \tag{7}$$

$$D^s(u - u_h)(x) = D^s R(x) + O(h^{n+1+\alpha-s}), \ s = 0, 1, \ n + s \geq 2. \tag{8}$$

Based on L^1-estimate of the regularized Green function g, g_h, $||g||_{2,1,\Omega} + ||g_h||^*_{2,1,\Omega} \leq c|\ln h|$, (see Frehse and Rannacher [18]1975), and the gradient type Green function G_h, $\|G_h\|_{1,1,\Omega} \leq c|lnh|$ (see Rannacher and Scott [25] 1982), it is shown that superconvergence estimates (7) and (8) hold in $L^\infty(\Omega)$ (with a factor $\ln h$).

We see that the change of $D^sR(x)$ approximately describes the behavior of $D^s\rho(x)$ with $\rho(x) = u - u_h$ in an element. In particular, at the zero point x_0 of D^sR we can obtain the useful superconvergence results $D^s\rho(x_0) = O(h^{n+1+\alpha-s}), s = 0, 1$.

Therefore, the EOA consists of two main ingredients. The first one is to construct the desired superclose approximation $u_I \in S^h$ by the orthogonal expansion in an element.The next one is to prove (5) or (6) by the cancellation techniques between the elements and orthogonality correction in an element. Actually how to construct the desired superclose interpolant u_I is an important technique in FEM. And it is known that various orthogonal expansions in an element are basic tools.

We start with one-dimensional case. Consider a quasiuniform mesh in an interval $I = (0, 1)$, $J^h :\ x_0 = 0 < x_1 < x_2 < ... < x_N = 1$, with the element $e_j = (x_{j-1}, x_j)$, its midpoint $\bar{x}_j = (x_{j-1} + x_j)/2$ and semi-steplenght $h_j = (x_j - x_{j-1})/2$. Consider a standard element $\tau = (-h, h)$ and the transform $x = ht, t \in E = (-1, 1)$. Denote still by $u(t)$ the $u(x) = u(ht)$, by Du (or u_x) the derivation in x and by ∂u (or u_t) the derivation in t. Obviously, $\partial^j u = h^j D^j u = O(h^j)$.

Introduce Legendre polynomials in E, i.e.,

$$l_0 = 1, l_1 = t, l_2 = (3t^2 - 1)/2, l_3 = (5t^3 - 3t)/2, ..., l_n = \partial^n (t^2 - 1)^n/(2n)!!,$$

Integrating it in t, we get M-type polynomials [5][6]:

$$M_0 = 1, M_1 = t, M_2 = (t^2 - 1)/2, ..., M_{n+1} = \partial^{n-1}(t^2 - 1)^n/(2n)!!. \tag{9}$$

It is quasi-orthogonal in the following sense: $(M_i, M_j) \neq 0$, if $i - j = 0$ or $= \pm 2$, else $(M_i, M_j) = 0$. Obviously, $M_n(\pm 1) = 0$ for $n \geq 2$.

First making the L-type expansion of u_t

$$u_t = \sum_{j=0}^{\infty} b_{j+1} l_j(t), \quad b_{j+1} = j_0(u_t, l_j) = O(h^{j+1}), \quad j_0 = j + 1/2,$$

and then integrating in t, we get M-type expansion, the partial sum and its remainder

$$u(t) = \sum_{j=0}^{\infty} b_j M_j(t),\ u_n = \sum_{j=0}^{n} b_j M_j(t),\ R = u - u_n = \sum_{j=n+1}^{\infty} b_j M_j(t), \tag{10}$$

respectively, where the coefficients $b_0 = (u(1) + u(-1))/2, b_1 = (u(1) - u(-1))/2$. Obviously $R(\pm 1) = u(\pm 1) - u_n(\pm 1) = 0$. Consequently, we may construct u_n in each element respectively, which automatically forms a continuous piecewise n-degree polynomial function $u_n \in S^h$. Its remainder has good orthogonality properties

$$R(t) = b_{n+1} M_{n+1}(t) + O(h^{n+2}) \perp P_{n-2},\ R_t(t) = b_{n+1} l_n(t) + O(h^{n+2}) \perp P_{n-1}.$$

The roots of $M_{n+1}(t)$ and $l_n(t)$ are called as $n+1$-order Lobatto points $t_j \in L_{n+1}$ and n-order Gauss points $t'_j \in G_n$, respectively. At these specific points, u_n and $\partial_t u_n$ have higher accuracy, respectively.

It is easy to estimate the following integrals with smooth coefficients $a(x), b(x)$,

$$\alpha_{ij} = \int_{-1}^{1} a l_i(t) l_j(t) dt = O(h^{|i-j|}), \quad \beta_{ij} = \int_{-1}^{1} b M_i M_j dt = O(h^{d(i,j)}), \tag{11}$$

where the index $d(i, i) = 0, d(i, i+1) = 1$ and $d(i, j) = |j - i| - 2$, if $|i - j| \geq 2$.

Early Douglas-Dupont [15][16](1973) proved that n-degree finite element u_h has the optimal error estimate at each node x_j

$$(u-u_h)(x_j)=O(h^{2n})||u||_{n+1,\Omega}.$$

Below, as a simple example of OCT(Orthogonality Correction Technique), we shall give another proof of this result.

To improve the superclose projection $u_I\in S^h$, we want to add

$$u_n^*=\sum_{j=2}^{n}b_j^*M_j(t)$$

into u_n, such that the remainder R^* of the new projection $u_I=u_n+u_n^*\in S^h$

$$R^*=u-u_I=(u-u_n)-u_n^*=R-u_n^* \tag{12}$$

satisfies more orthogonal conditions in an element, where u_n^* and $b_j^*, 2\le j\le n$ are to be defined. To this end, in a standard element $\tau=(-h,h)$, taking a transform $x=th$ and any test function $v=\sum_{i=0}^n\beta_iM_i(t)$, it is easy to calculate an element integral

$$J=\int_{-h}^{h}(aR'v'+bRv)dx=h^{-1}\int_{-1}^{1}(aR_tv_t+bh^2Rv)dt.$$

$$J=h^{-1}\sum_{i=0}^{n}\beta_i(B_i-B_i^*), \tag{13}$$

where the coefficients are given as

$$B_i^*=\sum_{j=2}^{n}c_{ij}b_j^*,\ B_i=\sum_{j=n+1}^{\infty}c_{ij}b_j,$$

$$c_{ij}=\int_{-1}^{1}(al_{i-1}l_{j-1}+bh^2M_iM_j)dt=O(h^{|i-j|}),i\ge1,j\ge2,$$

$$c_{0j}=\int_{-1}^{1}bh^2M_jdt=O(h^j),\ j\ge2.$$

Now we require that all coefficients b_j^* satisfy the conditions

$$B_i^*=B_i=O(h^{2n+2-i}),\ i=2,3,...,n. \tag{14}$$

This is a linear system, whose diagonal coefficients $c_{ii}=O(1)$, and other coefficients $O(h)$. Therefore its solutions can be expressed by a linear combination of b_j, and

$$b_i^*=h^{2n+2-i-1/p}||u||_{n+1,p,\tau},\ 2\le i\le n.$$

With this choice of b_i^*, J is deduced to

$$J=h^{-1}(\beta_0(B_0-B_0^*)+\beta_1(B_1-B_1^*))=O(h^{2n-1/p})||u||_{n+1,p,\tau}(h|\beta_0|+|\beta_1|). \tag{15}$$

Using the inverse estimates $|\beta_0|\le c\int_{-1}^{1}|v|dt\le ch^{-1}\int_\tau|v|dx$ and $|\beta_1|\le c\int_{-1}^{1}|v_t|dt\le c\int_\tau|v_x|dx$, we have

$$|J|\le Ch^{2n}||u||_{n+1,p,\tau}||v||_{1,p',\tau}.$$

Summing over all elements, we get a sharp superconvergence estimate

$$|A(u_h-u_I,v)|=|A(R,v)|\le Ch^{2n}||u||_{n+1,p}||v||_{1,p'}. \tag{16}$$

Taking $p=2, v=u_h-u_I$ and using the embedding theorem, $H^1 \hookrightarrow L^\infty$, we get superconvergence estimates

$$\max_x |u_h-u_I| + ||u_h-u_I||_1 \le Ch^{2n}||u||_{n+1}. \tag{17}$$

So Douglas-Dupont's result is obtained.

It is seen that the OCT is an useful technique to construct the optimal approximation of the function u in bilinear sense $A_\tau(u-u_I, v)$ in an element, which doesn't directly depend on an estimate of differential equations or a high order estimate of a discrete Green's function. Therefore it is possible to apply this technique to high order equations or in multi-dimensional cases.

3. Ultraconvergence of Rectangular Elements For Odd $n \ge 3$

Consider n-degree polynomials in $E=\{-1<s,t<1\}$,

$$Q_\lambda(n) = \sum_{(i,j)\in I_{n,\lambda}} b_{ij}s^it^j,$$

where the index

$$I_{n,\lambda} = \{(i,j):\ 0\le i,j\le n; i+j\le n+\lambda\}.$$

If the index $\lambda \ge 1$, $Q_\lambda(n)$ called regular family. The case $Q_n(n)$ called the tensor product case. Below, we need $I_n = I_{n,2}$ and $Q_2(n)$. The structure of $Q_2(5)$ is listed in Tab.1.

Tab.1 Rectangular family $Q_\lambda(5), \lambda=2$.
The main terms (x^iy^j) of remainder R.

1	y	y^2	y^3	y^4	y^5	(y^6)	(y^7)
x	xy	xy^2	xy^3	xy^4	xy^5	(xy^6)	
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	x^2y^5		
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4			
x^4	x^4y	x^4y^2	x^4y^3				
x^5	x^5y	x^5y^2					
(x^6)	(x^6y)						
(x^7)							

Early, based on a quasi-projection R_hu, Douglas-Dupont-Wheeler [17](1974) studied n-degree rectangular tensor product element $u_h \in Q_n(n)$ for Poisson equation and obtained an ultraconvergence estimate

$$||u_h - R_hu|| \le Ch^{n+3},\ n\ge 3.$$

At present, we are able to get point-wise ultraconvergence at angular nodes $z\in Z_h$,

$$(u-u_h)(z) = O(h^{n+3}\ln h), \quad n\ge 3.$$

So, $(u-u_h)(z) = O(h^{2n}lnh)$, at least, for $n=1,2,3$.

Based on OCT, this work can be extended to the case of variable coefficients (but $a_{12}=0$), if odd $n\ge 3$.

Theorem 1 (Odd n) *Assume that Ω is a rectangle and the rectangular mesh is uniform. All coefficients a_{ij} are suitably smooth, $a_{12}=0$. Assume that $n\ge 3$*

odd, $\lambda \geq 2$, and under BV1, then the finite element solution $u_h \in Q_2(n)$ has ultraconvergence at angular nodal set Z_h

$$\max_{x \in Z_h} |(u - u_h)(x)| = O(h^{n+3} \ln h)||u||_{n+3,\infty}.$$

The result is valid up to the boundary Γ.

Remark. It is a pity that this ultraconvergence result, in general, is invalid for even $n \geq 2$. However, if using the recovery technique or high degree interpolation of gradient in an element patch, ultraconvergence of gradient is still possibly obtained. Also see Zhang [33, 34, 35].

Proof of Theorem 1: Denote $a = a_{11}(x) > 0, b = a_{22}(x) > 0$ (but $a_{12} = 0$) and the polynomial family $Q_2(n), n \geq 3$, whose index set

$$I_n = I_{n,2} = \{(p, q) : 0 \leq p, q \leq n, p + q \leq n + 2\}$$

is divided into four groups: $I_n = I_Q + I_C + I_A + I_B$. The set I_A is important, because it is used to construct a correction polynomials. For odd $n = 2m + 1 \geq 3$, define $I_A = I_{A1} + I_{A2}$, where

$$I_{A1} = \{(p, q) : p = n - 2i, q = 2j, p + q \leq n + 2, 0 \leq i \leq m - 1, 0 \leq j \leq m\}.$$

$$I_{A2} = \{(p, q) : p = 2i, q = n - 2j, p + q \leq n + 2, 0 \leq i \leq m, 0 \leq j \leq m - 1\}.$$

The other three groups are

$$I_Q = \{(0, 0), (0, 1), (1, 0), (1, 1)\},$$

$$I_C = \{(1, 2), ..., (1, n)\} \cup \{(2, 1), ..., (n, 1)\},$$

$$I_B = I_n \backslash \{I_A \cup I_Q \cup I_C\}.$$

The distribution of these indexes for $n = 5$ is listed in Tab.2.

Tab.2 The index I_A for $n = 5$, $m = 2$, $dist(I_Q, I_A) = 2$.

(i, j)	0	1	2	3	4	5
0	Q	Q		A2		A2
1	Q	Q	C	C	C	C
2		C		A2		A2
3	A1	C	A1		A1	
4		C		A2		
5	A1	C	A1			

An important character of the set I_A is that it always includes two corner indexes $(n, 0), (0, n)$ and that the distance $dist(I_A, I_Q) = 2$ for odd $n \geq 3$.

For the simplicity of analysis, assume u is suitably smooth and the high order term $O(h^{n+3})$ will be omitted in the proof. We also assume that there are no lower order terms e.g., Duv, uv in $A(u, v)$, because they have no influences on ultraconvergence.

First denoting $\phi_{pq}(s, t) = M_p(s)M_q(t)$, expanding $u(s, t)$ in $(s, t) \in E$ and taking the part sum,

$$u(s, t) = \sum_{p+q \leq n+2} b_{pq}\phi_{pq}(s, t) + O(h^{n+3}), \quad b_{pq} = O(h^{p+q}),$$

$$u_n = \sum_{(p,q) \in I_{n,\lambda}} b_{pq}\phi_{pq}(s, t) \in Q_\lambda(n),$$

we have the expression of error,

$$R_n(s,t) = u - u_n = g_1(s,t) + g_2(s,t) + R_{n+2}, \quad R_{n+2} = O(h^{n+3}), \tag{18}$$

where the two terms and index sets are

$$g_1 = \sum_{(p,q)\in I_1(n)} b_{pq}\phi_{pq}(s,t), \quad I_1(n) = \{(n+1,0),(n+2,0),(n+1,1)\},$$

$$g_2 = \sum_{(p,q)\in I_2(n)} b_{pq}\phi_{pq}(s,t), \quad I_2(n) = \{(0,n+1),(0,n+2),(1,n+1)\}.$$

By the symmetry between x and y, it is enough to consider $R_n = g_1$.

Now adding a polynomial R^* to u_n, we have $u_I = u_n + R^*$ and the new remainder is

$$R = u - u_I = R_n - R^*, \quad R^* = \sum_{(p,q)\in I_A} b^*_{pq}\phi_{pq}(s,t), \tag{19}$$

where the coefficients $b^*_A = \{b^*_{pq} : (p,q) \in I_A\}$ are to be defined. Taking any test function $v = \sum_{(i,j)\in I_n} \beta_{ij}\phi_{ij} \in Q_2(n)$, and calculating an element integral, we have

$$J_\tau = \int_E (a(x)R_s v_s + b(x)R_t v_t)dsdt = \sum_{(i,j)\in I_n} \beta_{ij} r_{ij}, \tag{20}$$

where $r_{ij} = \bar{r}_{ij} - r^*_{ij}$,

$$r^*_{ij} = \sum_{(p,q)\in I_A} c_{ijpq} b^*_{pq}, \quad \bar{r}_{ij} = \sum_{(p,q)\in I_1(n)} c_{ijpq} b_{pq}.$$

We recall the orthogonality of bases $l_j(s), M_i(s)$, and then these coefficients are of

$$c_{ijpq} = A_\tau(\phi_{ij}, \phi_{pq}) = \int_E \{a l_{i-1}(s) l_{p-1}(s) M_j(t) M_q(t)$$
$$+ b M_i(s) M_p(s) l_{j-1}(t) l_{q-1}(t)\} dsdt = O(h^\rho),$$
$$\rho = \min\{|i-p| + d(j,q), |j-q| + d(i,p)\} \geq |i-p| + |j-q| - 2.$$

Now, we require that all coefficients $b^*_{pq}, (p,q) \in I_A$ satisfy

$$r^*_{ij} = \sum_{(p,q)\in I_A} c_{ijpq} b^*_{pq} = \bar{r}_{ij} = O(h^{n+2}), \ (i,j) \in I_A. \tag{21}$$

Its diagonal elements are $O(1)$, whereas other elements are of small quantity at most $O(h)$. This system has a unique solution and it leads to the following estimates

$$|b^*_A| = \sum_{(i,j)\in I_A} |b_{ij}| \leq C \max_{(i,j)\in I_A} |\bar{r}_{ij}| \leq C h^{n+2}.$$

After determining b^*_A, the integral J_τ is deduced to

$$J_\tau = \sum_{(i,j)\in C_n} r_{ij}\beta_{ij} = J_B + J_Q + J_C, \ C_n = I_B + I_Q + I_C,$$

which are bounded as follows.

1) For $(i,j) \in I_B$ and $(p,q) \in I_A$, as $I_B \cap I_A$ is empty, we have at least

$$c_{ijpq} = O(h), \quad r_{ij} = O(hb^*_A) = O(h^{n+3}).$$

By the inverse estimate $|\beta_{ij}| \leq C||v||_{2,1,\tau}$ we have

$$J_B = O(h^{n+3})||v||_{2,1,\tau}.$$

2) For $(i,j) \in I_Q$. As $n \geq 3$ is odd, b^* does not contain b^*_{20}, b^*_{02}, b^*_{22}, and $dist(I_A, I_Q) = 2$. So

$$r_{00} = O(h^{n+5}),\ r_{10}, r_{01} = O(h^{n+4}),\ r_{11} = O(h^{n+3} + h^{2n}),$$

and it leads to

$$|J_Q| \leq Ch^{n+3}(h^2|\beta_{00}| + h|\beta_{10}| + h|\beta_{01}| + h^2|\beta_{11}|) \\ \leq Ch^{n+3}||v||_{2,1,\tau}.$$

3) When $(i,j) \in I_C$, the cancellation techniques [4, 6] are needed. As an example, we only discuss the most difficult term

$$B(\tau) = r_{1,n-1} = O(h^{n+2}).$$

The decomposition of the coefficient is

$$\beta_{1,n-1}(\tau) = c_n \int_{E(\tau)} v_{st}(s,t) l_{n-2}(t) ds dt = \beta_{n-1}(\tau^+) - \beta_{n-1}(\tau^-),$$

$$\beta_{n-1}(\tau^{\pm}) = c_n \int_{-1}^{1} v_t(\pm 1, t) l_{n-2}(t) dt,$$

where τ^+ and τ^- are right and left side in τ respectively. By inverse estimate, $\beta_{n-1}(\tau^{\pm}) = O(1)||v||_{2,1,\tau}$ for $n \geq 3$. Denote by $S_p = \tau_1 + \tau_2 + ... + \tau_l$ a long slip of elements arranged from left to right. And then we combine the linear integrals on the same side to obtain,

$$K = \sum_{j=1}^{l} \beta_{1,n-1}(\tau_j) B(\tau_j) = \beta_{n-1}(\tau_l^+) B(\tau_l) \\ - \sum_{j=1}^{l-1} \beta_{n-1}(\tau_j)(B(\tau_{j+1}) - B(\tau_j)) - \beta_{n-1}(\tau_1^-) B(\tau_1),$$

where the difference $B(\tau_{j+1}) - B(\tau_j) = O(h^{n+3})$ is of high order. As $v = 0$ on τ_1^- and τ_l^+, then the corresponding integral $\beta_{n-1} = 0$. Thus high order estimates follow:

$$K = O(h^{n+3})||v||_{2,1,S_p},\ n \geq 3.$$

Summarizing the above estimates, we get, for $2 \leq p \leq \infty$,

$$A(R, v) = O(h^{n+2})||u||_{n+3,p}||v||^*_{2,p'},\ \text{for odd } n \geq 3. \tag{22}$$

We should point out that the function $u_I = u_n + R^*_n$ constructed above is discontinuous on the common side between elements (continuous only at angular nodes), because the coefficients b^*_A in R^*_n is determined by $b_R = \{b_{n+1,0}, b_{n+2,0}, ...\}$, whereas these coefficients b_R is defined by the area integrals of u in τ. However, the jump of b_R in two adjacent elements is of high order $O(h^{n+3})$. So, the jumps $\Phi(s)$ and $\Psi(t)$ of R^*_n on sides $t = 1$ and $s = 1$ are of $O(h^{n+3})$. Now we construct a correction

$$\eta = \Phi(s)\frac{1+t}{2} + \frac{1+s}{2}\Psi(t) = O(h^{n+3}),$$

and define a new function $u_I = u_n + R^*_n - \eta$, which is continuous in Ω such that the second weak estimate (6) is still valid.

Finally, we consider whether BV1 is satisfied by u_I. Assume that τ is a boundary element whose side $s = 1$ falls on boundary Γ. Note that the index set I_A is divided into two groups independently each other(see Tab.2). For example, the coefficient

$b^*_{i,j}, (i,j) \in I_{A2}$ is determined by $b_{0,n+1}, ...$ and independent of $b_{n+1,0}, ...$ From $u(1,t) = 0$, we know that $u_t(1,t) = 0$, and then

$$\begin{aligned} b_{0,n+1} &= c_n \int_{-1}^{1} (-u_t(1,t) - u_t(-1,t)) l_n(t) dt \\ &= c_n \int_{-1}^{1} (u_t(1,t) - u_t(-1,t)) l_n(t) dt \\ &= c_n \int_E u_{st}(s,t) l_n(t) ds dt = O(h^{n+2}). \end{aligned}$$

It leads to a high order estimate $b^*(i,j) = O(h^{n+3}), (i,j) \in I_{A2}$. Denote

$$\phi(t) = R(1,t) = \sum_{(0,j)\in I_{A1}} b^*_{0j} M_j(t) = O(h^{n+3}),$$

and $\xi = (1-s)\phi(t)/2$, and then the new function $u_I = u_n - R^* - \eta - \xi \in Q_2(n)$ satisfies boundary condition $u_I = 0$ on $s = 1$. At this point we can still get the desired estimate $A_\tau(\xi, v) = O(h^{n+3})||v||_{2,1,\tau}$, i.e. new remainder $R = u - u_I$ satisfies (6). Finally, using the discrete Green function, we get the desired superconvergence estimate

$$||u_h - u_I||_{0,\infty} = O(h^{n+3} \ln h) ||u||_{n+3,\infty},$$

from which Theorem 1 follows.

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