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ON A THEOREM OF BERNSTEIN AND ITS APPLICATIONS TO WEIGHTED MINIMAX SERIES*

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Abstract

In this paper, some results about approximation in a norm S induced by the minimax series are studied. Then a Bernstein-type theorem for the norm S is established. Finally the Bernstein theorem is applied to prove the existence of certain equalities with minimax series and weighted minimax series.

Key words: Approximation theory, polynomials, Bernstein theorem, minimax series.

1. Introduction

Let f be a continuous function on $[a, b]$. Π_n will designate the set of all polynomials of degree less or equal than n and Π the set of all polynomials. As is well known, for each n the minimax of f is given by:

$$E_n(f) = \|f - p_n\|_\infty = \inf_{p \in \Pi_n} \|f - p\|_\infty,$$

where p_n is the best uniform approximation of f in Π_n.

Let us also consider the minimax series given by the expression

$$S(f) \equiv \sum_{k=0}^{\infty} E_k(f) \tag{1.1}$$

The set of functions for which $S^*(f) = \sum_{k=0}^{\infty} E_k^*(f) < \infty$, where $E_k^*(f)$ denotes the error of best approximation of $f \in C[0, 2\pi]$ with trigonometric polynomials was already studied by S.N. Bernstein. He proved that such functions are of class $C^1[a, b]$.

The series (1.1) can be seen as a measuare of "how good" the funtion f can be approximated by polynomials, in the next sense. If f and $g \in C[a, b]$ and $\|f\|_\infty = \|g\|_\infty$ we will say that f is better approximated by polynomials than g on $[a, b]$ if and only if $S(f) < S(g)$.

On the other hand let $x_0 \in [a, b]$ be fixed. We set:

$$M_0 = \{f \in C[a, b] : f(x_0) = 0\},$$

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$$\Pi^0 = \{\text{polynomials } p : p(x_0) = 0\},$$
$$\Pi_n^0 = \{\text{polynomials } p \in \Pi_n : p(x_0) = 0\},$$

and

$$C_0 = \{f \in M_0 : S(f) < \infty\}.$$

By introducing,

$$S : C_0[a,b] \to \mathbb{R}$$
$$f \to S(f),$$

it can be proved that (C_0, S) is a normed space. Furthermore, $\forall f \in C[a,b]$ such that $S(f) < \infty$ there exists $g = f - f(x_0)$ such that $g \in C_0[a,b]$ and $S(f) = S(g)$. The approximation of a function $f \in (C_0, S)$ by polynomials in Π_n, is studied in [5].

(i) For a given $f \in C_0$, let $p_n \in \Pi_n$ be a best approximation of f in the norm S. Who is p_n?.

(ii) Is the space of all polynomials Π dense in (C_0, S)?.

The answer to these questions is contained in [5]. We recall in the next section some results proved in [5] in order to make this paper selfcontained. Also the convergence in the space (C_0, S) is analized in [5] and it is proved that it is a Banach space.

2. Approximation by Polynomials in the Space (C_0, S)

Let f be a function in (C_0, S). We consider the best approximation of f in Π_n $n = 0, 1, \cdots$ with respect to the norm S. That is, find $q_n \in \Pi_n$, such that:

$$S(f - q_n) = \inf_{p \in \Pi_n} S(f - p)$$

Let $p \in \Pi_n$. Then

$$E_k(f - p) = E_n(f), \quad (k \geq n)$$

and

$$E_k(f - p) \geq E_n(f), \quad (k < n).$$

Then,

$$\inf_{p \in \Pi_n} S(f - p) = \inf_{p \in \Pi_n} \sum_{k=0}^{n-1} E_k(f - p) + C(f),$$

where $C(f) = \sum_{k \geq n} E_k(f)$.

The existence of q_n can be deduced from the fact that (Π_n^0, S) is a normed space of finite dimension. Let us solve the following question, who is the approximant q_n?.

Proposition 1. *Let $f \in C[a,b]$. Then $S(f - q_n) = \inf_{p \in \Pi_n} S(f - p)$ iff there exists a constant C such that*

$$q_n = p_n + C \text{ where } \|f - p_n\|_\infty = \inf_{q \in \Pi_n} \|f - q\|_\infty \tag{2.1}$$

(*i.e. the best approximations of f in Π_n in the uniform norm and in the norm S coincide module an additive constant*).

Proof. If p_n is the best uniform approximation of f in Π_n, then:

$$E_k(f-p_n)=E_n(f) \quad (k<n),$$

hence

$$\sum_{k=0}^{n-1} E_k(f-p_n)=nE_n(f),$$

and

$$\sum_{k=0}^{n-1} E_k(f-p)\geq nE_n(f)\ \forall p\in\Pi_n.$$

(i.e. p_n is a best approximation of f on Π_n in the norm S).

Let q_n be another best approximation of f in Π_n in the norm S, then

$$\sum_{k=0}^{n-1} E_k(f-q_n)=nE_n(f)$$

hence

$$E_k(f-q_n)=E_n(f) \quad (k<n)$$

(i.e. there exists $r_k\in\Pi_k$ such that:

$$\|f-(q_n+r_k)\|_\infty=E_n(f)).$$

Note that $q_n+r_k\in\Pi_n$ and, by the uniqueness of the best approximation P_n, we have

$$P_n=q_n+r_k, \quad 0\leq k<n,$$

for $k=0$, $r_0=C$ and $q_n=p_n+r_0$.

Remark 1. Since $S(f+\lambda)=S(f)$ for any continuous function f on $[a,b]$, and $\lambda\in\mathbb{R}$, then from (2.1) it follows $q_n=p_n+C'$, C' being any arbitrary real constant.

Therefore in the space C_0 both approximations coincide and the uniqueness of best approximation in Π_n^0 with the norm S holds.

We now prove that the polynomials Π^0 are dense in (C_0,S).

Proposition 2. *For $n=0,1,2,\cdots$; let $\{p_n\}$ be the sequence of best polynomial approximation of degree n for a function $f\in C_0$ in the uniform norm. Then p_n converges to f in the norm S.*

Proof. Let $f\in C_0$ be. With the previous notations:

$$S(f-p_n)=S(f)-\Big(\sum_{k<n}E_k(f)-E_n(f)\Big)=S(f)-\sum_{k<n}E_k(f)+nE_n(f),$$

then:

$$0\leq S(f-p_n)\leq S(f), \quad \forall n,$$

and $S(f-p_n)$ is a decreasing sequence. Then there exists $\lim S(f-p_n)$ and:

$$\lim S(f-p_n)=\lambda, \quad 0\leq \lambda \leq S(f).$$

Taking into account that $\sum_{k<n} E_k(f)$ is a partial sum of $S(f)$, we have

$$\lim S(f-p_n)=\lim nE_n=\lambda.$$

If $\lambda\neq 0$, then the series $S(f)$ and the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ have the same convergence character, the harmonic series diverges and $f\in C_0$ implies $S(f)<\infty$. Hence $\lambda=0$ and the $\lim S(f-p_n)=0$ □

We have immediately the following

Corollary 1. *The set of polynomials Π^0 is dense in (C_0,S).*

Corollary 2. *If $S(f)<\infty$ then $\lim nE_n(f)=0$.*

Now we will prove a Walsh's type theorem for (C_0,S). (i.e. it is possible the simultaneus interpolation and approximation in the norm S).

Theorem 1. *Let $f\in C_0[a,b]$ and let $x_1,x_2,\cdots,x_n$ be n distinct nodes on $[a,b]$. Let $\varepsilon>0$. Then there exists a polynomial p such that:*

$$p(x_i)=f(x_i), \quad (1\leq i\leq n)$$

and

$$S(f-p)<\varepsilon$$

Proof. Let $f\in C_0$ and $\varepsilon>0$. By Proposition 1, there exists a polynomial $p\in\Pi_m$ such that:

$$S(f-p)<\varepsilon,$$

and p being the best approximation to f in the norm S and in the uniform norm as in the proof of Proposition 1.

Then

$$E_m(f-p)=E_m(f)=\|f-p\|_\infty$$

and

$$\|f-p\|_\infty\leq S(f-p)<\varepsilon$$

Let q be the interpolation polynomial of $f-p$ at the nodes $\{x_i\}$,

$$q(x)=\sum_{k=1}^{n}(f(x_k)-p(x_k))L_k(x),$$
$$L_k(x)=\frac{\Pi(x)}{(x-x_k)\Pi'(x_k)}, \quad \Pi(x)=\prod_{j=1}^{n}(x-x_j).$$

Taking $p_1=p+q$, results:

$$p_1(x_i)=p(x_i)+q(x_i)=f(x_i) \quad (1\leq i\leq n),$$

and

$$S(f-p_1) \leq S(f-p) + S(q) < \varepsilon + S(q),$$

in addition

$$S(q) \leq \sum_{k=1}^{n} |f(x_k) - p(x_k)| S(L_k) \leq \|f-p\|_\infty \sum_{k=1}^{n} S(L_k) < \varepsilon M$$

where

$$M = \sum_{k=1}^{n} S(L_k).$$

Then

$$S(f-p_1) < \varepsilon(1+M),$$

ans the proof follows since $\varepsilon > 0$ is arbitary. □

3. The Minimax $E_n(f)$ and $F_n(f)$

We have two type of minimax $E_n(f)$ and $F_n(f)$, both corresponding to the same best approximant. Clearly, the minimax $E_n(f)$ does not give us information about how the function f could be approximated by polynomials of degree lesser than n. However, the number $F_n = S(f-p_n)$ contains certain information about this, because it can be expressed in terms of $S(f)$ and the minimaxs $E_k(f)$ $(0 \leq k \leq n)$. Indeed, one has the following

Proposition 3.

$$F_n(f) = S(f) - \sum_{k<n} (E_k(f) - E_n(f))$$

for all $f \in C[a,b]$.

Proof.

$$\begin{aligned} S(f-p_n) &= \sum_{k=0}^{n-1} E_k(f-p_n) + \sum_{k=n}^{\infty} E_k(f) = \sum_{k=0}^{n-1} (E_k(f-p_n) - E_k(f)) + S(f) \\ &= \sum_{k=0}^{n-1} (E_n(f) - E_k(f)) + S(f) = S(f) - \Big(\sum_{k<n} E_k(f) - E_n(f) \Big) \quad \square \end{aligned}$$

Proposition 4. The next recurrence relation holds:

$$\frac{F_{n+1} - F_n}{n+1} = E_{n+1} - E_n, \quad n = 0, 1, 2, \cdots$$

for all $f \in C[a,b]$.

Proof. We have that:

$$\begin{aligned} F_n &= S(f-p_n) = nE_n(f) + \sum_{k \geq n} E_k(f) = (n+1)E_n(f) + \sum_{k \geq n+1} E_k(f), \\ F_{n+1} &= S(f-p_{n+1}) = (n+1)E_{n+1}(f) + \sum_{k \geq n+1} E_k(f). \end{aligned}$$

Hence

$$F_{n+1}(f) - F_n(f) = (n+1)[E_{n+1}(f) - E_n(f)] \quad \square$$

4. A Bernstein's Type Theorem for the Norm Induced by the Minimax Series

The next result is known as the Bernstein's theorem:

If a sequence

$$A_0 \geq A_1 \geq A_2 \geq \cdots, \lim A_n = 0,$$

and an interval $[a, b]$ are given, then there exists a function $f(x) \in C[a, b]$ with the minimax

$$E_n(f) = A_n, \quad (n = 0, 1, 2, \cdots)$$

See e.g. [4, pp.137].

We proof the next similar result for the minimax $F_n(f)$.

Theorem 2. *If a sequence*

$$B_0 \geq B_1 \geq B_2 \geq \cdots, \lim B_n = 0,$$

and an interval $[a, b]$ *are given, then there exists a function* $f(x) \in C[a, b]$ *such that* $S(f) < \infty$ *and*

$$F_n(f) = B_n, \quad (n = 0, 1, 2, \cdots).$$

Proof. Let us consider

$$\begin{aligned} H_n &= \frac{B_{n+1} - B_n}{n+1}, \quad n = 0, 1, 2, \cdots, \\ A_0 &= \sum_{n=0}^{\infty} \frac{B_n - B_{n+1}}{n+1}, \\ A_{n+1} &= A_n + H_n, \quad n = 0, 1, 2, \cdots. \end{aligned}$$

Note that

$$\sum_{n=0}^{\infty} \frac{B_n - B_{n+1}}{n+1} \leq \sum_{n=0}^{\infty} (B_n - B_{n+1}) = B_0.$$

So $H_n \leq 0$ and $A_{n+1} \leq A_n$, $A_{n+1} = H_n + H_{n-1} + H_{n-2} + \cdots + H_0 + A_0$.

Then

$$\lim A_{n+1} = \Big(\sum_{n=0}^{\infty} H_n \Big) + A_0 = -A_0 + A_0 = 0,$$

and, from Bernstein's theorem there exists a function $f \in C[a, b]$ such that:

$$E_n(f) = A_n, \quad n = 0, 1, 2, \cdots.$$

Let us now prove by induction that the following holds:

$$F_n(f) = B_n, \quad n = 0, 1, 2, \cdots.$$

For $n = 0$,

$$
\begin{aligned}
B_0 &= \sum_{n=0}^{\infty}(B_n - B_{n+1}) = \sum_{n=0}^{\infty}(n+1)(A_n - A_{n+1}) \\
&= -\sum_{n=0}^{\infty}(n+1)A_{n+1} + \sum_{n=0}^{\infty} nA_n + \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} E_n(f) = S(f) = F_0(f),
\end{aligned}
$$

Suppose that $F_n(f) = B_n$ holds. For $n+1$ we have:

$$
\begin{aligned}
F_{n+1}(f) =& F_n(f) + (n+1)(E_{n+1} - E_n) = B_n + (n+1)(A_{n+1} - A_n) \\
=& B_n + H_n(n+1) = B_{n+1} \square
\end{aligned}
$$

5. A Theorem of Bernstein's Type for the Norm Induced by the Weighted Minimax Series

The so called weighted minimax series are considered in [6]. If $\alpha = (n_k)_{k=0}^{\infty}$ is a sequence of real numbers $n > 0$ and $n_k \geq 0$, for all $k \geq 1$, then the weighted minimax series of $f \in C[a,b]$ relative to α, is the expression:

$$
S_\alpha(f) \equiv \sum_{k=0}^{\infty} n_k E_k(f).
$$

We can also consider the space

$$
C_{\alpha,0} = \{f \in C[a,b] : f(x_0) = 0, \text{ and } S_\alpha(f) < \infty\},
$$

where x is a fixed point of $[a,b]$, and

$$
\begin{aligned}
S_\alpha : C_{\alpha,0}[a,,b] &\to \mathbb{R} \\
f &\to S_\alpha(f).
\end{aligned}
$$

Then $(C_{\alpha,0}, S_\alpha)$ is a Banach space. The convergence in these spaces is studied in [6]. Also the approximation by polynomials with the norm S_α is studied in [6]. Similar results for the case $\alpha \equiv 1$, exposed in the second paragraph of this paper are obtained. If p is the best uniform approximation of f on Π_n then p_n is a best approximation of f on Π_n with the norm S_α, and all best approximations of f on Π_n with the norm S_α are of the form $p_n + C$ being C a constant. We can consider the minimaxs $F_{\alpha,n} = S_\alpha(f - p_n)$, $E_n(f) = \|f - p_n\|_\infty$.

A recurrence relation between these minimax is established in the next result (which is a generalization of proposition 4).

Proposition 5.

$$
F_{\alpha,n+1}(f) - F_{\alpha,n}(f) = S_{n+1}(E_{n+1}(f) - E_n(f))
$$

where

$$
S_n = \sum_{k<n} n_k.
$$

Proof. We have that

$$F_{\alpha,n}(f) = S_\alpha(f - p_n) = S_n E_n(f) + \sum_{k\geq n} n_k E_k(f) = S_{n+1}E_n(f) + \sum_{k\geq n+1} n_k E_k(f),$$

and

$$F_{\alpha,n+1} = S(f - p_{n+1}) = S_{n+1}E_{n+1}(f) + \sum_{k\geq n+1} n_k E_k(f).$$

Hence

$$F_{\alpha,n+1}(f) - F_{\alpha,n}(f) = S_{n+1}(E_{n+1}(f) - E_n(f))\square$$

We establish now a Bernstein's type theorem in the norm S_α.

Theorem 3. *If a sequence*

$$C_0 \geq C_1 \geq C_2 \geq \cdots \lim C_n = 0,$$

and

$$\lim \frac{C_n}{S_n} = 0,$$

and an interval $[a, b]$ *are given, then there exists a function* $f(x) \in C[a, b]$ *such that* $S_\alpha(f) < \infty$ *and*

$$F_{\alpha,n}(f) = C_n, \quad (n = 0, 1, 2, \cdots)$$

Proof. Set

$$\begin{aligned} H_n &= \frac{C_{n+1} - C_n}{S_{n+1}} \quad n = 0, 1, 2, \cdots, \\ A_0 &= \sum_{n=0}^{\infty} \frac{C_n - C_{n+1}}{S_{n+1}}, \\ A_{n+1} &= A_n + H_n, \quad n = 0, 1, 2, \cdots, \end{aligned}$$

Note that $\displaystyle\sum_{n=0}^{\infty} \frac{C_n - C_{n+1}}{S_{n+1}} < \infty$ since $\displaystyle\sum_{n=1}^{\infty} \frac{C_n - C_{n+1}}{S_{n+1}} \leq \sum_{n=1}^{\infty} \Big(\frac{C_n}{S_n} - \frac{C_{n+1}}{S_{n+1}}\Big) = \frac{C_0}{S_0}$. Hence $H_n \leq 0$ and $A_{n+1} \leq A_n$, $A_{n+1} = H_n + H_{n-1} + H_{n-2} + \cdots + H_0 + A_0$. Then

$$\lim A_{n+1} = \Big(\sum_{n=0}^{\infty} H_n\Big) + A_0 = -A_0 + A_0 = 0,$$

and from Bernstein's Theorem, there exists a function $f \in C[a, b]$ such that:

$$E_n(f) = A_n, \quad n = 0, 1, 2, \cdots.$$

Then

$$F_{\alpha,n}(f) = C_n, \quad n = 0, 1, 2, \cdots.$$

will be verified by induction.

For $n = 0$,

$$\begin{aligned}C_0 &= \sum_{n=0}^{\infty}(C_n - C_{n+1}) = \sum_{n=0}^{\infty} S_{n+1}(A_n - A_{n+1}) = -\sum_{n=0}^{\infty} S_{n+1}A_{n+1} + \sum_{n=0}^{\infty} S_{n+1}A_n \\ &= -\sum_{n=0}^{\infty} S_{n+1}A_{n+1} + \sum_{n=1}^{\infty} S_nA_n + \sum_{n=0}^{\infty} n_nA_n = \sum_{n=0}^{\infty} n_nE_n(f) = S_\alpha(f) = F_{\alpha,0}(f),\end{aligned}$$

Suppose that $F_{\alpha,n}(f) = C_n$ holds. For $n+1$ we have:

$$\begin{aligned}F_{\alpha,n+1}(f) =& F_{\alpha,n}(f) + S_{n+1}(E_{n+1}(f) - E_n(f)) = C_n + S_{n+1}(A_{n+1} - A_n) \\ =& + C_n + H_nS_{n+1} = C_{n+1} \quad \square\end{aligned}$$

6. Application of the Bernstein Theorem to the Existence of Certain Weighted Minimax Series and Certain Series with Minimax

Proposition 6. *If $\alpha = (n_k)_{k=0}^{\infty}$ is a decreasing sequence and converges to zero, then for each $m = 1, 2, 3, \cdots$, there exists f and $g \in C[a,b]$ such that:*

$$S_\alpha(g) = \sum_{k=0}^{\infty}(E_k(f))^m$$

Proof. From Bernstein's theorem, there exists a fonction f such that:

$$E_k(f) = n_k, \quad k = 0, 1, 2, \cdots$$

and there exists a g such that:

$$E_k(g) = (E_k(f))^{m-1}, \quad k = 0, 1, 2, \cdots.$$

Thus,

$$S_\alpha(g) = \sum_{k=0}^{\infty} n_k(E_k(f))^{m-1} = \sum_{k=0}^{\infty} E_k(f)(E_k(f))^{m-1} = \sum_{k=0}^{\infty}(E_k(f))^m \quad \square$$

Proposition 7. *If $\alpha = (n_k)_{k=0}^{\infty}$ is an increasing sequence which tends to infinity, then for each $m = 1, 2, 3, \cdots$ and for each $f \in C[a,b]$ there exists $g \in C[a,b]$ such that:*

$$S_\alpha(g) = \sum_{k=0}^{\infty}(E_k(f))^m$$

Proof. $\dfrac{(E_k(f))^m}{n_k}$ is a decreasing sequence which converges to zero. From Bernstein's theorem there exists g:

$$E_k(g) = \frac{(E_k(f))^m}{n_k}, \quad k = 0, 1, 2, \cdots.$$

then

$$S_\alpha(g) = \sum_{k=0}^{\infty} (E_k(f))^m \quad \square$$

Proposition 8. *If $(n_k)_{k=0}^{\infty}$ is a decreasing sequence which converges to zero, then for each $m = 1, 2, 3, \cdots$ and for each $f \in C[a, b]$ there exists $g \in C[a, b]$ such that:*

$$S(g) = \sum_{k=0}^{\infty} n_k (E_k(f))^m$$

Proof. $n_k(E_k(f))^m$ is a decreasing sequence which converges to zero. Thus there exists g such that:

$$E_k(g) = n_k(E_k(f))^m, \quad k = 0, 1, 2, \cdots$$

so,

$$S(g) = \sum_{k=0}^{\infty} n_k (E_k(f))^m \quad \square$$

Proposition 9. *If $(n_k)_{k=0}^{\infty}$ is an increasing sequence and diverges, then for each $m = 2, 3, \cdots$ there exists f and $g \in C[a, b]$ such that:*

$$S(g) = \sum_{k=0}^{\infty} n_k (E_k(f))^m$$

Proof. $\dfrac{1}{n_k}$ is a decreasing sequence which converges to zero. Then there exists $f \in C[a, b]$ such that:

$$E_k(f) = \frac{1}{n_k}, \quad k = 0, 1, 2, \cdots$$

Also there exists g such that:

$$E_k(g) = E_k^{m-1}(f), \quad k = 0, 1, 2, \cdots.$$

Therefore,

$$\begin{aligned} S(g) &= \sum_{k=0}^{\infty} E_k(g) = \sum_{k=0}^{\infty} E_k^{m-1}(f) = \sum_{k=0}^{\infty} \frac{1}{n_k} n_k E_k^{m-1}(f) \\ &= \sum_{k=0}^{\infty} n_k E_k(f) E_k^{m-1}(f) = \sum_{k=0}^{\infty} n_k (E_k(f))^m \square \end{aligned}$$

Proposition 10. (a) *If $\alpha = (n_k)_{k=0}^{\infty}$ decreases to zero as n tends to infinity, then there exists $f \in C[a, b]$ such that:*

$$S_\alpha(f) = \sum_{k=0}^{\infty} E_k^2(f).$$

(b) If $\alpha = (n_k)_{k=0}^{\infty}$ decreases to zero, then for each $m > 1$ there exists $f \in C[a,b]$ such that:

$$S_\alpha(f) = \sum_{k=0}^{\infty} (E_k(f))^m.$$

Proof. To prove (a) we can use Bernstein's theorem to get a function f such that:

$$E_k(f) = n_k, \quad k = 0, 1, 2, \cdots.$$

Then,

$$S_\alpha(f) = \sum_{k=0}^{\infty} n_k E_k(f) = \sum_{k=0}^{\infty} E_k^2(f) \quad \square$$

As for (b) we have just:

(n_k^r) $(r > 0)$ is a decreasing sequence which converges to zero. Then there exists f such that

$$E_k(f) = n_k^r, \quad k = 0, 1, 2, \cdots.$$

Then

$$n_k = E_k^{\frac{1}{r}}(f), \quad k = 0, 1, 2, \cdots$$

and

$$S_\alpha(f) = \sum_{k=0}^{\infty} n_k E_k(f) = \sum_{k=0}^{\infty} E_k^{\frac{1}{r}}(f) E_k(f) = \sum_{k=0}^{\infty} E_k^{\frac{r+1}{r}}(f).$$

Thus, taking $r = \dfrac{1}{m-1}$, $\dfrac{r+1}{r} = m$ we obtain

$$S_\alpha(f) = \sum_{k=0}^{\infty} (E_k(f))^m \quad \square$$

Proposition 11. *(a) If $\alpha = (n_k)_{k=0}^{\infty}$ is an increasing sequence which diverges to infinity, then there exists $f \in C[a,b]$ such that:*

$$S_\alpha(f) = \sum_{k=0}^{\infty} E_k^{-1}(f).$$

(b) If $\alpha = (n_k)_{k=0}^{\infty}$ is an increasing sequence which diverges to infinity, and $m > -1$, then there exists $f \in C[a,b]$ such that:

$$S_\alpha(f) = \sum_{k=0}^{\infty} E_k^{-m}(f).$$

Proof. (a) $\left\{\dfrac{1}{n_k^{1/2}}\right\}$ is a decreasing which converges to zero. Then there exists f such that:

$$E_k(f) = \frac{1}{n_k^{1/2}}, \quad k = 0, 1, 2, \cdots.$$

Then

$$S_\alpha(f) = \sum_{k=0}^{\infty} n_k \frac{1}{n_k^{1/2}} = \sum_{k=0}^{\infty} n_k^{1/2} = \sum_{k=0}^{\infty} \frac{1}{E_k(f)} \quad \square$$

(b) $\left\{\frac{1}{n_k^r}\right\}$ $(r > 0)$ is a decreasing sequence which converges to zero. Then there exists f such that:

$$E_k(f) = \frac{1}{n_k^r}, \quad k = 0, 1, 2, \cdots.$$

Then

$$n_k = \frac{1}{E_k^{1/r}(f)}, \quad k = 0, 1, 2, \cdots$$

and

$$S_\alpha(f) = \sum_{k=0}^{\infty} n_k E_k(f) = \sum_{k=0}^{\infty} \frac{1}{E_k^{1/r}(f)} E_k(f) = \sum_{k=0}^{\infty} E_k^{(r-1)/r}(f).$$

Hence, by choosing, $r = \dfrac{1}{1+m} > 0$, then $\dfrac{r-1}{r} = -m$,

$$S_\alpha(f) = \sum_{k=0}^{\infty} E_k^{-m}(f) \quad \square$$

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