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BOUNDARY PENALTY FINITE ELEMENT METHODS FOR BLENDING SURFACES, I BASIC THEORY*1)

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Abstract

When parametric functions are used to blend 3D surfaces, geometric continuity of displacements and derivatives until to the surface boundary must be satisfied. By the traditional blending techniques, however, arbitrariness of the solutions arises to cause a difficulty in choosing a suitable blending surface. Hence to explore new blending techniques is necessary to construct good surfaces so as to satisfy engineering requirements. In this paper, a blending surface is described as a flexibly elastic plate both in partial differential equations and in their variational equations, thus to lead to a unique solution in a sense of the minimal global surface curvature. Boundary penalty finite element methods (BP-FEMs) with and without approximate integration are proposed to handle the complicated constraints along the blending boundary. Not only have the optimal convergence rate $O(h^2)$ of second order generalized derivatives of the solutions in the solution domain been obtained, but also the high convergence rate $O(h^4)$ of the tangent boundary condition of the solutions can be achieved, where h is the maximal boundary length of rectangular elements used. Moreover, useful guidance in computation is discovered to deal with interpolation and approximation in the boundary penalty integrals. A numerical example is also provided to verify perfectly the main theoretical analysis made. This paper yields a framework of mathematical modelling, numerical techniques and error analysis to the general and complicated blending problems.

Key words: Blending surfaces, parametric surfaces, plate, mathematical modelling, variational equations, finite element methods, boundary penalty method, computer geometric aided design

1. Introduction

Blending surfaces is said if when two frame surfaces (or bodies) are located already, a smoothly transferring surface is sought to connect the two frame surfaces along certain boundary. Usually, the terminology "smoothness" means that the blending surface belongs to geometric continuity C^1 (Foley et al. (90) [14), i.e., the blending surface and its tangent plane are continuous until the joint boundary. Many literatures are reported on this subject. We merely mention a few of them relevant to this paper. Uniform

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algebraic polynomials of low order have been discussed in Hoffmann and Hopcroft(92), and Ohkura and Kakazu(92), to blend simple frames, such as those with quadratic and cubic surfaces. Piecewise spline functions can also be used to obtain rather complicated blending surfaces (see Kosters(89), Bajaj and Ihn(92)). This paper is, however, devoted to find efficient approaches to construct good surfaces to blend general, complicated frame-surfaces (or frame-bodies), which may be used in airplane, ships, grand buildings, and astronautic shuttle-station. The existing blending techniques become awkward in handling arbitrary joining locations and boundary conditions. To manages complicated blending, we solicit partial differential equations (PDEs) of order four describing elastic plates, and seek additional conditions of unique solutions. Note that techniques using PDEs are given in Bloor and Wilson(90,91) but still to deal with simple cases.

A plate in algebraic functions, e.g., $z = f(x, y)$ may serve well as a blending surface, applicable to simple surface modelling. However, parametric functions are more advantageous to represent general and complicated 3D surfaces. When the blending surfaces are connected to the frame boundary satisfying the displacement and tangent conditions, there occur multiple parametric surfaces, unfortunately. A simple case in 2D blending curves is illustrated in Foly et. al (90, p.486) [14]. There arises a question how to choose a suitable, unique blending surface. This is important to computer aided design. As far as our current knowledge (referring to Choi(91), Farin(90), Fisher(94), Koenderink(90), Nutbourne and Martin(89), Su and Liu(89), Warren(89), as well as the recent fourth SIAM Conference on Geometric Design, Nashville, Tennessee, Nov., 6-9, 1995, it seems to exist no literatures to address this problem. This paper is, therefore, intended to study such a challenging topic.

The organism of this paper is as follows. In the next section, mathematical modelling of blending surfaces is given with PDEs and their variational forms, thus to yield unique solutions. In Section 3, three kinds of boundary penalty finite element methods (BP-FEMs) are presented, to simplify the algorithms involving the complicated boundary conditions. Error analysis is then made in Section 4, accompanied with comparison; a simple numerical example is given in Section 5 to verify the optimal convergence rates.

2. Mathematical Modelling of Blending Surfaces

Consider that a surface is sought to join two given frame bodies V_1 and V_2 at the left boundary ∂V_1 and the right boundary ∂V_2. Suppose that ∂V_1 and ∂V_2 are disjointed to each other (see Fig.1). Since the algebraic function $y = f(x, y)$ is difficult to represent the closed surface shown in Fig.1, we solicit parametric functions instead. Choose two parameters r and t in a unit solution area $\Omega\{(r,t),\ 0 < r < 1,\ 0 < t < 1\}$, and use three parametric functions

$$x = x(r,t), \quad y = y(r,t), \quad z = z(r,t), \quad (r,t) \in \Omega \tag{2.1}$$

to represent the blending surface in Fig.1. Naturally, we denote the left boundary $\partial\Omega|_{r=0}$, and the right boundary $\partial\Omega|_{r=1}$, to represent ∂V_1 and ∂V_2, respectively. Denote the boundary of Ω as (see Fig.2) by $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \overline{AB} \cup \overline{CD}$, $\Gamma_2 = \overline{AC} \cup \overline{BD}$, and the vector of three parametric functions

$$U = U(r,t) = (x,y,z)^T = (x(r,t), y(r,t), z(r,t))^T \tag{2.2}$$

Therefore, the displacement and tangent conditions of blending surfaces along the joint boundaries ∂V_1 and ∂V_2 can be written as

$$U|_{\overline{AC}} = U|_{r=0} = U_0, \qquad U|_{\overline{BD}} = U|_{r=1} = U_1 \tag{2.3}$$

$$(U_n)_{\overline{AC}} = (U_n)_{r=0} = \alpha_0 U_0', \quad (U_n)_{\overline{BD}} = (U_n)_{r=1} = \alpha_1 U_1' \tag{2.4}$$

where $U_n = \dfrac{\partial}{\partial n} U$, and n is the outside normal to the boundary $\partial\Omega$. The vectors U_0, U_1, U_0' $(\neq 0)$ and $U_1'(\neq 0)$ are known, but the functions $\alpha_0(r)(\neq 0)$ and $\alpha_1(r)(\neq 0)$ are arbitrary real functions. We may express (2.4) as

$$y_n = b_{10}x_n,\ z_n = b_{20}x_n \text{ on } \overline{AC}; \quad y_n = b_{11}x_n,\ z_n = b_{21}x_n \text{ on } \overline{BD} \tag{2.5}$$

or simply

$$y_n = b_1 x_n, \quad z_n = b_2 x_n \text{ on } \Gamma_2 \tag{2.6}$$

where b_{01}, b_{02}, b_{11}, and b_{12} (or b_1 and b_2) are obtained from the ratios of derivatives in (2.4). For the closed surface along direction t, the following periodical conditions on Γ_1 will be satisfied.

$$U(r,0) = U(r,1),\ U_n(r,0) = U_n(r,1),\ 0 \le r \le 1 \tag{2.7}$$

Fig.1 A blending surface connecting V_1 and V_2 along ∂V_1 and ∂V_2

Fig.2 A division of Ω by rectangles

The functions of blending surfaces of geometry C^1 satisfy

$$x(r,t), y(r,t), z(r,t) \in C^1(\Omega), \tag{2.8}$$

where $C^k(\Omega)$ denotes a space of functions having continuous derivatives of order k. By considering the continuity of U and U_n on $\partial\Omega$ described in (2.3), (2.6) and (2.7) we may assume that the solutions $x, y, z(\in C^4(\Omega))$ satisfy the following partial differential equations with the fourth order, describing placements in elastic plates.

$$\Delta^2 U = F,\ \text{where } F = (fx, fy, fz)^T \tag{2.9}$$

where the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and the biharmonic operator $\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$. The functions fx, fy and fz play a role of the loading forces on the thin plate; they can be chosen suitably based on practical requirements in engineering.

It is noted that the equations (2.9), accompanied with the boundary conditions (2.3)–(2.6) (or (2.4)) and (2.7) will lead to many solutions[14]. Below let us first derive the additional boundary conditions in order to yield a unique solution.

Denote two spaces H and H_0 of U such that

$$H = \{(x, y, z)|x, y, x \in H^2(\Omega), \text{ satisfying (2.3), (2.6) and (2.7)}\} \tag{2.10}$$

$$H_0 = \{(x, y, z)|x, y, z \in H^2(\Omega), \text{ satisfying } U|_{\Gamma_2} = 0, \text{ (2.6) and (2.7)}\} \tag{2.11}$$

where $H^2(\Omega)$ is the Sobolev space (see Marti(86)). By referring the variational equations of biharmonic functions in Courant and Hilbert (53) and Fong and Shi(91), a solutions $U \in H$ can be expressed in a weak form

$$A(U, W) = F(W), \quad \forall W \in H_0 \tag{2.12}$$

where

$$A(U, W) = \iint_\Omega \{\Delta U \cdot \Delta W + (1-\mu)(2U_{rt} \cdot W_{rt} - U_{rr} \cdot W_{tt} - U_{tt} \cdot W_{rr})\} d\Omega \tag{2.13}$$

$$F(U) = \iint_\Omega F \cdot W d\Omega \tag{2.14}$$

and $U_{rr} = \frac{\partial^2 U}{\partial r^2}$, $U_{rt} = \frac{\partial^2 U}{\partial r \partial t}$, $W = (\xi, \eta, \zeta)^T$, and μ is the Poisson ratio satisfying $0 < \mu < \frac{1}{2}$. Also the notation $U \cdot W$ means the scalar product of vectors. Denote

$$M(U) = -\Delta U + (1-\mu)(U_{rr} r_s^2 + 2U_{rt} r_s t_s + U_{tt} t_s^2) \tag{2.15}$$

$$P(U) = \frac{\partial}{\partial n} \Delta U + (1-\mu) \frac{\partial}{\partial n} \{U_{rr} r_n r_s + U_{rt}(r_n t_s + r_s t_n) + U_{tt} t_n t_s\} \tag{2.16}$$

where r_n, t_n and r_s, t_s are the direction cosines of the outnormal and tangent vectors, respectively. By the Green theory, we have

$$\iint_\Omega (\Delta^2 U - F) \cdot W d\Omega + \int_{\partial\Omega} P(U) \cdot W d\Gamma + \int_{\partial\Omega} M(U) \cdot W_n d\Gamma = 0 \tag{2.17}$$

Hence equation (2.9) is obtained due to arbitrary W in Ω, from the first term of the right side of (2.17).

Next on Γ_2, by applying the boundary conditions (2.3), then $W|_{\Gamma_2} = 0$, $\forall W \in H_0$, to get

$$\int_{\Gamma_2} P(U) \cdot W|_{\Gamma_2} d\Gamma = 0, \quad \forall W \in H_0 \tag{2.18}$$

From the boundary condition (2.6), the third term in (2.17) leads to

$$\int_{\Gamma_2} M(U) \cdot W_n d\Gamma = \int_{\Gamma_2} (M(U) \cdot B) x_n d\Gamma = 0 \tag{2.19}$$

where the vector

$$B = (1, b_1, b_2)^T \text{ on } \Gamma_2 \tag{2.20}$$

Since the derivatives x_n are arbitrary on the boundary Γ_2, we obtain an additional boundary condition.

$$M(U) \cdot B = 0 \tag{2.21}$$

Third, on the boundary Γ_1, by applying the periodical boundary conditions (2.7) we write the second and third terms in (2.17) as

$$\int_0^1 (P(U(r,0)) + P(U(r,1))) \cdot W(r,0)dr + \int_0^1 (M(U(r,0)) + M(U(r,1))) \cdot W_n(r,0)dr = 0 \tag{2.22}$$

Since the function $W(r,0)$ and $W_n(r,0)$ are also arbitrary, other additional boundary conditions on Γ_1 are found as

$$P(U(r,0)) + P(U(r,1)) = 0,\ M(U(r,0)) + M(U(r,1)) = 0,\ 0 \le r \le 1 \tag{2.23}$$

In fact, the boundary conditions (2.21) and (2.23) are called the natural conditions; and Eqs. (2.3), (2.6) and (2.7) the essential conditions. Both the essential and the natural boundary conditions should be implemented to the differential equation (2.9) to yield a unique solution. Note that the variational equation (2.12) requires the essential boundary conditions only, where $x, y, z \in H^2(\Omega)$, less smooth than $x, y, z \in C^4(\Omega)$ required in (2.9). The true solution U can also be restated as follows.

$$I(U) = \min_{W \in H} I(W), \quad I(U) = \frac{1}{2}A(U,U) - F(U), \tag{2.24}$$

which also indicates the minimal, global curvature of blending surfaces (also see Carmo(76)).

3. Boundary Penalty Finite Element Methods

Since true solutions can not be obtained for general functions F, numerical solutions should be sought. Finite element methods are efficient methods to yield approximate solutions, based on the variational equations (2.12) or (2.24).

We choose piecewise bi-cubic Hermite interpolatory functions due to (2.8), although other conforming finite elements $\in C^1$ can also be used in Ciarlet(90). To deal with conditions (2.6) and (2.7), a direct treatment is introduced in Li(91) by eliminating some unknowns. This technique may, however, cause complexity in programming. We will here follow the penalty techniques described in Li(92, 98), to simplify algorithms; other kinds of penalty techniques for essential boundary conditions can also be found in Babuska(73), Utku and Carey(82), Barrett and Elliott(86), Shi(84), etc.

The basic Hermite functions on [0,1] are given in Carey and Oden(83).

$$\begin{aligned} \phi_0(\theta) &= 2\theta^3 - 3\theta^2 + 1, \quad \phi_1(\theta) = -2\theta^3 + 3\theta^2 \\ \psi_0(\theta) &= \theta^3 - 2\theta^2 + \theta, \quad \psi_1(\theta) = \theta^3 - \theta^2 \end{aligned} \tag{3.1}$$

Then a cubic polynomial on [0,1] is defined by

$$f(\theta) = x_0\phi_0(\theta) + x_1\phi_1(\theta) + x'_0\phi_0(\theta) + x'_1\phi_1(\theta) \tag{3.2}$$

satisfying the boundary functions and derivatives:

$$f(\theta)|_{\theta=0} = x_0,\ f(\theta)|_{\theta=1} = x_1,\ f'(\theta)|_{\theta=0} = x'_0,\ f'(\theta)|_{\theta=1} = x'_1 \tag{3.3}$$

Let the square solution area Ω be divided into small rectangular elements by the coordinate lines $r = r_i$ and $t = t_j$, where

$$\begin{aligned} &0 = r_0 < r_1 < ... < r_i < r_{i+1} < ... < r_n = 1,\ n \geq 1 \\ &0 = t_0 < t_1 < ... < t_i < t_{i+1} < ... < t_m = 1,\ m \geq 2 \end{aligned} \tag{3.4}$$

Denote the stepsize $\delta r_i = r_{i+1} - r_i$, $\delta t_j = t_{j+1} - t_j$ and the small rectangular element $\square_{ij}$ by

$$\square_{ij} = \{(r,t), r_i < r < r_{i+1},\ t_j < t < t_{j+1}\} \tag{3.5}$$

We assume that the small rectangles $\square_{ij}$ are quasiuniform, i.e., there exists a bounded constant C independent of δr_i and δt_j such that $\dfrac{(\delta r_i, \delta t_j)}{(\delta r_i, \delta t_j)} \leq C$.

For each element node $(i,j) = (r_i, t_j)$, we assign four unknowns, e.g., x_{ij}, $(x_r)_{ij}$, $(x_t)_{ij}$, $(x_{rt})_{ij}$ of function $x(r,t)$. Also denote the basis functions with the nonzero support

$$\phi_{i,l}(r) = \phi_l\Big(\frac{r-r_i}{\delta r_i}\Big),\ \psi_{i,l}(r) = \psi_l(\frac{r-r_i}{\delta r_i}\Big),\ \text{in } [r_0, r_1] \text{ as } i = 0,\ \text{in } (r_i, r_{i+1}] \text{ as } i > 0, \tag{3.6}$$

and

$$\phi_{j,l}(t) = \phi_l\Big(\frac{t-t_j}{\delta t_j}\Big), \psi_{j,l}(t) = \psi_l\Big(\frac{t-t_j}{\delta t_j}\Big),\ \text{in } [t_0, t_1] \text{ as } j = 0,\ \text{in } (t_j, t_{j+1}] \text{ as } j > 0, \tag{3.7}$$

where $l = 0, 1$. By the tensor product, the following piecewise bi-cubic Hermite polynomials can be formulated for $x_h(r,t)$ in two dimensions.

$$x_h(r,t) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Big\{\sum_{k,l=0}^{1} x_{i+k,j+l}\phi_{i,k}(r)\phi_{j,l}(t) + \delta r_i \sum_{k,l=0}^{1}(x_r)_{i+k,j+l}\psi_{i,k}(r)\phi_{j,l}(t) \tag{3.8}$$

$$+ \delta t_j \sum_{k,l=0}^{1}(x_t)_{i+k,j+l}\phi_{i,k}(r)\psi_{j,l}(t) + \delta r_i\, \delta t_j$$

$$+ \delta r_i\, \delta t_j \sum_{k,l=0}^{1}(x_{rt})_{i+k,j+l}\psi_{i,k}(r)\psi_{j,l}(t)\Big\} \tag{3.9}$$

Such functions in (3.8) are the same as those in the Bogner-Fox-Schmit rectangle in Ciarlet(90) and Fong and Shi(81) due to unisolvence. The admissible functions are then written as

$$U_h = U_h(r,t) = (x_h(r,t), y_h(r,t), z_h(r,t))^T \tag{3.10}$$

where $y_h(x,t)$ and $z_h(r,t)$ are also defined in (3.8). Obviously, $x_h, y_h, z_h \in H^2(\Omega) \cap C^1(\Omega)$. Define a finite-dimensional collation of the functions as

$$V = \{U \text{ as (3.10), satisfying (2.3)}\} \tag{3.11}$$

$$V_0 = \{U \text{ as (3.10), satisfying } U|_{\Gamma_2} = 0\} \tag{3.12}$$

The conditions (2.3) still remain in space V, but neither (2.6) nor (2.7) in V and V_0. Hence if compared with the functional spaces H and H_0 in (2.10) and (2.11), we can see

$$V_0 \notin H_0, \quad V \notin H \tag{3.13}$$

For this reason, we introduce boundary penalty techniques to handle these constraints. The solution $U_h^* \in V$ is obtained from the boundary penalty finite element method, called Method I of BP-FEMs,

$$A_P(U_h^*, W_h) = F_h(W_h), \ \forall W_h \in V_0 \tag{3.14}$$

where

$$F_h(U) = \iint_\Omega \bar{F} \cdot W d\Omega \tag{3.15}$$

$\bar{F}$ is the piecewise linear (or bilinear) interpolatory functions of F, and

$$A_P(U_h, W_h) = A(U_h, W_h) + \bar{D}(U_h, W_h) \tag{3.16}$$

$A(U,W)$ is given in (2.13) already, and h is the maximal length of rectangular elements defined by $h = \max_{ij}(\delta r_i, \delta t_j)$. The boundary penalty integrals $\bar{D}(U,W)$ are given by (also see Li(92) and Li and Bui(92))

$$\begin{aligned}\bar{D}(U,W) =& \frac{P_c}{h^{2\sigma}}\Big\{\int_{\Gamma_2}(y_n - \bar{b}_1 x_n)(\eta_n - \bar{b}_1\xi_n)d\Gamma + \int_{\Gamma_2}(z_n - \bar{b}_2 x_n)(\zeta_n - \bar{b}_2\xi_n)d\Gamma \\ &+ \int_0^1 (U(r,0) - U(r,1))(W(r,0) - W(r,1))dr \\ &+ \int_0^1 (U_n(r,0) - U_n(r,1))(W_n(r,0) - W_n(r,1))dr\Big\}\end{aligned} \tag{3.17}$$

where P_c is a bounded positive constant independent of h, U_h and W_h, and $\sigma > 0$ is a penalty power. Note that $\bar{b}_1$ and $\bar{b}_2$ used in (3.17) are the piecewise q-order interpolatory polynomials of functions b_1 and b_2 respectively, where $1 \le q \le 3$. It is worthy pointing out that the same penalty factor $\frac{P_c}{h^{2\sigma}}$ are used for all the boundary conditions of displacement and derivatives because they are all the essential boundary conditions.

The exact integrals in $\bar{D}(U,W)$ may also be evaluated directly due to polynomial integrands, or from integration rules with accuracy of order up to $6+2q$. Gaussian rules are suggested with integration nodes up to $4+q$. Furthermore, integration rules of order four and six can evaluate exactly integrals $A_h(U_h, W_h)$ and $F_h(U_h)$, respectively (see Davis and Rabinowitz(84)).

Next, we also provide another algorithm of the BP-FEMs involving integration approximation called Method II of BD-FEMs, to seek the solution $\hat{U}_h^* \in V$ such that

$$\hat{A_P}(\hat{U}_h^*, W_h) = \hat{F}(W_h),\ \forall W_h \in V_0 \tag{3.18}$$

where

$$\hat{A_P}(U_h, W_h) = \hat{A}(U_h, W_h) + \hat{D}(U_h, W_h) \tag{3.19}$$

and

$$\begin{aligned}\hat{D}(U, W) =& \frac{P_c}{h^{2\sigma}}\Big\{\hat{\int}_{\Gamma_2}(y_n - b_1 x_n)(\eta_n - b_1\xi_n)d\Gamma + \hat{\int}_{\Gamma_2}(z_n - b_2 x_n)(\zeta_n - b_2\xi_n)d\Gamma \\ &+ \hat{\int}_0^1 (U(r,0) - U(r,1)) \cdot (W_h(r,0) - W(r,1))dr \\ &+ \hat{\int}_0^1 (U_n(r,0) - U_{hn}(r,1)) \cdot (W_n(r,0) - W_n(r,1))dr\Big\}\end{aligned} \tag{3.20}$$

Also $\hat{A}(U_h, W_h)$, $\hat{F}(U_h)$, $\hat{\int}_{\Gamma_2}$ and $\hat{\int}_0^1$ are the integration approximation to $A(U_h, W_h)$, $F(U_h)$, $\int_{\Gamma_2}$ and $\int_0^1$, respectively. Note that functions b_1 and b_2 in $\hat{D}(U, W)$ and F in $F(U_h)$ are chosen as the true functions without interpolatory approximation, contrasted to those in (3.14).

An analysis in Section 4.1 is derived to prove that the solutions U_h^* obtained from (3.14) have the optimal convergence rates $O(h^2)$ of second order generalized derivatives when $q \geq 1$. A further analysis in Section 4.2 shows that integration rules of order two should be chosen for $A(U_H, W_h)$ and $F(W_h)$, but integration rules of order six for $D(U, W)$, in order to maintain the optimal convergence rates $O(h^2)$.

We may combine Methods I and II, Equations (3.14) and (3.18), to obtain Method III of BP-FEMs. The solution $\hat{\bar{U}}^*{}_h \in V$ is obtained by Method III of BP-FEMs defined as

$$\hat{A}_P(\hat{\bar{U}}^*{}_h, W_h) = \hat{F}_h(W_h), \quad W_h \in V_0 \tag{3.21}$$

where

$$\hat{F}_h(U) = \hat{\int\int}_{\Omega} \bar{F} \cdot W d\Omega \tag{3.22}$$

and

$$\hat{A}_P(U_h, W_h) = \hat{A}(U_h, W_h) + \hat{\bar{D}}(U_h, W_h) \tag{3.23}$$

where $\hat{F}_h$, $\hat{A}$ and $\hat{\bar{D}}$ are the approximation of F_h, A and $\bar{D}$ defined in Method I by using integration rules as in Method II.

4. Error Analysis

In this section, we derive error bounds of numerical solutions obtained from (3.14) and (3.18), Methods I and II of BP-FEMs. Analysis of Method III can be easily obtained. We will focus on error bounds of the solutions in Ω, and in particular on error bounds of the tangent boundary conditions (2.6). Moreover, we also derive useful guidance in choosing suitable interpolation of b_1 and b_2, and suitable integration rules for the boundary penalty integrals.

4.1. Errors Bounds of Solutions from Method I of BP-FEMs

First denote a norm

$$|||U||| = \{\|U\|^2_{(H^2(\Omega))^3} + \bar{D}(U,U)\}^{\frac{1}{2}} \tag{4.1}$$

where $\bar{D}(U,U)$ is given in (3.17), using q $(1 \le q \le 3)$ order interpolatory polynomials of b_1 and b_2, the norm notations are defined by

$$\|U\|_{(H^k(\Omega))^3} = \{\|x\|^2_{H^k(\Omega)} + \|y\|^2_{H^k(\Omega)} + \|z\|^2_{H^k(\Omega)}\}^{\frac{1}{2}} \tag{4.2}$$

$$|U|_{(H^k(\Omega))^3} = \{|x|^2_{H^k(\Omega)} + |y|^2_{H^k(\Omega)} + |z|^2_{H^k(\Omega)}\}^{\frac{1}{2}} \tag{4.3}$$

and $\|x\|^2_{H^k(\Omega)}$ and $|x|^2_{H^k(\Omega)}$ are the Sobolev norms (see Marti(86)). We have the following lemma.

Lemma 4.1. *There exist two bounded positive constants C_0 and C_1 independent of h, U and W such that*

$$|A_P(U,W)| \le C_0|||U||| \times |||W|||, \quad U \in H_0 \quad and \quad W \in V_0 \tag{4.4}$$

and the uniformly V_0- elliptic inequality exists

$$C_1|||U|||^2 \le A_P(U,U), \quad U \in V_0 \tag{4.5}$$

Proof. By noting $\mu \in \left(0, \frac{1}{2}\right)$, it is easy to show (4.4) and

$$C_1|U|^2_{(H^2(\Omega))^3} + \bar{D}(U,U) \le A_P(U,U), \quad U \in V_0 \tag{4.6}$$

Based on the generalized Friedrichs' inequality (see Marti(86, p.82)) and noting $U|_{\Gamma_2} = 0$ due to $U \in V_0$, we obtain

$$\|U\|^2_{(H^2(\Omega))^3} \le C_2\{|U|^2_{(H^2(\Omega))^3} + \int_{\Gamma_2} U^2 d\Gamma\} = C_2|U|^2_{(H^2(\Omega))^3} \tag{4.7}$$

where C_2 is also a positive bounded constant. Combining (4.6) and (4.7) yields the desired inequality (4.5), thus to complete the proof of Lemma 4.1. □

Lemma 4.2. *Let $M(U) \in (H^0(\Gamma))^3$, $P(U) \in (H^0(\Gamma_1))^3$, $F \in (H^2(\Omega))^3$, and $B \in (H^{q+1}(\Gamma_2))^2$ with $1 \le q \le 3$ be given. There exists a bounded constant C independent of h, U and W such that*

$$\begin{aligned}|||U - U_h^*||| \le & C\{\inf_{W \in V} |||U - W||| + h^\sigma(|M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3}) \\ & + h^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2} \times |M(U)|_{(H^0(\Gamma_2))^3} + h^2|F|_{(H^2(\Omega))^3}\}\end{aligned} \tag{4.8}$$

where $U(\in H)$ are the true solutions, $U_h^*(\in V)$ are the solutions from Method I of BP-FEMs, and the notations are

$$|B|_{(H^k(\Gamma_2))^2} = \{|b_1|^2_{H^k(\Gamma_2)} + |b_2|^2_{H^k(\Gamma_2)}\}^{\frac{1}{2}},$$

$$\|B\|_{(H^k(\Gamma_2))^2} = \{\|b_1\|^2_{H^k(\Gamma_2)} + \|b_2\|^2_{H^k(\Gamma_2)}\}^{\frac{1}{2}} \tag{4.9}$$

Proof. Since the true solutions U satisfy natural boundary conditions (2.21) and (2.23) we can obtain the following equations by applying the Green theorem (also see (2.17)),

$$\begin{aligned} A_P(U,W) =& \iint_\Omega F\cdot W d\Omega + \int_{\Gamma_2} \{m(y)(\eta_n - b_1\xi_n) + m(z)(\zeta_n - b_2\xi_n)\}d\Gamma \\ &+ \int_0^1 P(U(r,0))\cdot(W(r,0) - W(r,0))dr \\ &+ \int_0^1 M(U(r,0))\cdot(W_n(r,0) - W_n(r,1))dr = 0, W\in V_0 \end{aligned} \tag{4.10}$$

where $m(x)$ and $m(y)$ are the components of the vector $M(U)$ defined in (2.15). Since solutions U_h^* satisfy (3.14), we obtain

$$\begin{aligned} |A_P(U-U_h^*,W) \le& \Big|\int_{\Gamma_2}\{m(y)(\eta_n - \bar b_1\xi_n) + m(z)(\zeta_n - \bar b_2\xi_n)\}d\Gamma\Big| \\ &+\Big|\int_{\Gamma_2}\{m(y)(\bar b_1 - b_1)\xi_n + m(z)(\bar b_2 - b_1)\xi_n\}d\Gamma\Big| \\ &+\Big|\int_0^1 P(U(r,0))\cdot(W(r,0)-W(r,1))dr\Big| \\ &+\Big|\int_0^1 M(U(r,0)\cdot(W_n(r,0)-W_n(r,1))dr\Big| + \Big|\iint_\Omega (F-\bar F)\cdot W d\Omega\Big| \\ =& I + II + III + IV + V \end{aligned} \tag{4.11}$$

For the first term in the right side of the above equation, we have from the Schwarz inequality and definition (4.1) of $|||W|||$.

$$\begin{aligned} I =& \Big|\int_{\Gamma_2}\{m(y)(\eta_n - \bar b_1\xi_n) + m(z)(\zeta_n - \bar b_2\xi_n)\}d\Gamma\Big| \\ \le& C|M(U)|_{(H^0(\Gamma_2))^3}\times\Big\{\int_{\Gamma_2}(|\eta_n - \bar b_1\xi_n|^2 + |\zeta_n - \bar b_2\xi_n|^2)d\Gamma\Big\}^{\frac{1}{2}} \\ \le& Ch^\sigma|M(U)|_{(H^0(\Gamma_2))^3}\times \bar D(W,W)^{\frac{1}{2}} \le Ch^\sigma|M(U)|_{(H^0(\Gamma_2))^3}\times|||W||| \end{aligned} \tag{4.12}$$

By applying the embedding theorem (see Marti(86))

$$||\xi_n||_{H^0(\Gamma_2)} \le C||\xi||_{H^2(\Omega)} \le C|||W||| \tag{4.13}$$

and by noting that functions $\bar b_1$ and $\bar b_2$ are the q order interpolatory polynomials of b_1 and b_2, the second term in (4.11) leads to from the Schwarz inequality

$$\begin{aligned} II =& \Big|\int_{\Gamma_2}\{m(y)(\bar b_1 - b_1)\xi_n + m(z)(\bar b_2 - b_1)\xi_n\}d\Gamma\Big| \\ \le& Ch^{q+1}\{|m(y)|_{H^0(\Gamma_2)}\times|b_1|_{H^{q+1}(\Gamma_2)} \\ &+ |m(z)|_{H^0(\Gamma_2)}\times|b_2|_{H^{q+1}(\Gamma_2)}\}\times||\xi_n||_{H^0(\Gamma_2)} \end{aligned}$$

$$\leq Ch^{q+1}|M(U)|_{(H^0(\Gamma_2))^3} \times |B|_{(H^{q+1}(\Gamma_2))^2} \times |||W||| \tag{4.14}$$

Below we estimate rest of the terms in (4.11), we have again from (4.1)

$$\begin{aligned} III =& \Big| \int_0^1 P(U(r,0)) \cdot (W(r,0) - W(r,1)) d\Gamma \Big| \\ \leq& |P(U)|_{(H^0(\bar{AB}))^3} \times \Big(\int_0^1 ||W(r,0) - W(r,1)||^2 d\Gamma \Big)^{\frac{1}{2}} \\ \leq& Ch^\sigma |P(U)|_{(H^0(\Gamma_1))^3} \times \bar{D}(W,W)^{\frac{1}{2}} \leq Ch^\sigma |P(U)|_{(H^0(\Gamma_1))^3} \times |||W||| \end{aligned} \tag{4.15}$$

and

$$IV = \Big| \int_0^1 M(U(r,0)) \cdot (W_n(r,0) - W_n(r,1)) dr \Big| \leq Ch^\sigma |M(U)|_{(H^0(\Gamma_1))^3} \times |||W||| \tag{4.16}$$

Since $\bar{F}$ is piecewise linear (or bilinear) functions of F,

$$\begin{aligned} V =& \Big| \iint_\Omega (F - \bar{F}) \cdot W d\Omega \Big| \leq Ch^2 |F|_{(H^2(\Omega))^3} \times C\|W\|_{(H^0(\Omega))^3} \\ \leq& Ch^2 |F|_{(H^2(\Omega))^3} \times |||W||| \end{aligned} \tag{4.17}$$

Therefore, combining (4.11)–(4.17) leads to

$$|A_P(U - U_h^*, W)| \leq I + II + III + IV + V \leq CQ|||W||| \tag{4.18}$$

where

$$\begin{aligned} Q =& h^\sigma (|M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3}) \\ &+ h^{q+1} |B|_{(H^{q+1}(\Gamma_2))^2} \times |M(U)|_{(H^0(\Gamma_2))^3} + h^2 |F|_{(H^2(\Omega))^3} \end{aligned} \tag{4.19}$$

Moreover, letting $E = W - U_h^*$ and $E \in V_0$, we obtain from Lemma 4.1 and (4.18)

$$\begin{aligned} C_0 |||E|||^2 \leq& A_P(E,E) \leq 2(|A_P(U - W, E)| + |A_P(U - U_h^*, E)|) \\ \leq& C_1(|||U - W||| + Q) \times |||E||| \end{aligned} \tag{4.20}$$

The desired results (4.8) are obtained by dividing two sides of (4.11) by $|||E|||$ and applying the triangular inequality

$$|||U - U_h^*||| \leq |||U - W||| + |||U_h^* - W||| \tag{4.21}$$

Thus the proof of Lemma 4.2 is completed. □

Now let us prove a main theorem.

Theorem 4.1. *Let $U \in (H^4(\Omega))^3$, $F \in (H^2(\Omega))^3$, $U \in (H^4(\Gamma))^3$, $U_n \in (H^4(\Gamma))^3$, $M(U) \in (H^0(\Gamma))^3$, $P(U) \in (H^0(\Gamma_1))^3$ and $B \in (H^{q+1}(\Gamma_2))^2$ be given. Then there exists a bounded constant C independent of h, U and W such that*

$$|||U - U_h^*||| \leq C\{h^2(|U|_{(H^4(\Omega))^3} + |F|_{(H^2(\Omega))^3}) + h^\sigma(|M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3})$$

$$+ h^{4-\sigma}(\|B\|_{(H^{q+1}(\Gamma_2))^2} \times |U_n|_{(H^4(\Gamma_2))^3} + |U|_{(H^4(\Gamma_1))^3} + |U_n|_{(H^4(\Gamma_1))^3})$$
$$+ h^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2} \times |M(U)|_{(H^0(\Gamma_2))^3}\} \tag{4.22}$$

Proof. Let $\bar{U}_h$ be the piecewise bi-cubic Hermite interpolatory functions of U, them $\bar{U}_h \in V$. If letting $W = \bar{U}_h$ we have from Lemma 4.2,

$$|||U - U_h^*||| \le C\{|||U - \bar{U}_h||| + Q\} \tag{4.23}$$

where Q is given in (4.19). Moreover,

$$|||U - \bar{U}_h||| = \|U - \bar{U}_h\|_{(H^2(\Omega))^3} + \bar{D}(U - \bar{U}_h, U - \bar{U}_h)^{\frac{1}{2}} \tag{4.24}$$

and

$$\|U - \bar{U}_h\|_{(H^2(\Omega))^2} \le Ch^2|U|_{(H^4(\Omega))^3} \tag{4.25}$$

Also

$$\begin{aligned}\bar{D}(U - \bar{U}_h, U - \bar{U}_h) =& \frac{P_c}{h^{2\sigma}}\Big\{\int_{\Gamma_2}(y_n - (\bar{y_n})_h - \bar{b}_1(x_n - (\bar{x_n})_h))^2 d\Gamma \\ &+ \int_{\Gamma_3}(z_n - (\bar{z_n})_h - \bar{b}_2(x_n - (\bar{x_n})_h))^2 d\Gamma + \|\delta(U - \bar{U}_h)\|^2_{[0,1]} \\ &+ \|\delta(U_n - (\bar{U_n})_h)\|^2_{[0,1]}\Big\} = \frac{P_c}{h^{2\sigma}}\{I^* + II^* + III^* + IV^*\}\end{aligned} \tag{4.26}$$

where the notation

$$\|\delta U\|^2_{[0,1]} = \int_0^1 \|U(r,0) - U(r,1)\|^2 dr \tag{4.27}$$

Since Γ is parallel to axis r or t, derivatives $(\bar{U}_n)_h$ on Γ are also the piecewise cubic Hermite interpolatory polynomials of U_n, to yield interpolation errors of $O(h^4)$. Hence we obtain

$$\begin{aligned}I^* =& \int_{\Gamma_2}(y_n - (\bar{y_n})_h - \bar{b}_1(x_n - (\bar{x_n})_h))^2 d\Gamma \le C\int_{\Gamma_2}(|y_n - (\bar{y_n})_h|^2 + \bar{b}_1^2|x_n - (\bar{x}_n)_h|^2)d\Gamma \\ \le& Ch^8(|y_n|^2_{H^4(\Gamma_2)} + \|\bar{b}_1\|_{H^0(\Gamma_2)}|x_n|^2_{H^4(\Gamma_2)}) \\ \le& Ch^8(|U_n|^2_{(H^4(\Gamma_2))^3} + |\bar{B}|^2_{(H^0(\Gamma_2))^2} \times |U_n|^2_{(H^4(\Gamma_2))^3}) \\ \le& Ch^8\|B\|^2_{(H^{q+1}(\Gamma_2))^2} \times |U_n|^2_{(H^4(\Gamma_2))^3}\end{aligned} \tag{4.28}$$

where we have used the following inequality

$$\begin{aligned}|\bar{B}|_{(H^0(\Gamma_2))^2} \le& |B|_{(H^0(\Gamma_2))^2} + |B - \bar{B}|_{(H^0(\Gamma_2))^2} \\ \le& |B|_{(H^0(\Gamma_2))^2} + Ch^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2} \le C\|B\|_{(H^{q+1}(\Gamma_2))^2}\end{aligned} \tag{4.29}$$

Similarly

$$II^* \le \int_{\Gamma_3}(z_n - (\bar{z_n})_h - \bar{b}_2(x_n - (\bar{x_n})_h))^2 d\Gamma \le Ch^8\|B\|^2_{(H^{q+1}(\Gamma_2))^2} \times |U_n|^2_{(H^4(\Gamma_2))^3} \tag{4.30}$$

Also we have

$$III^* = \|\delta(\bar{U} - U_h)\|^2_{[0,1]} \le 2\int_0^1 (\|U(r,0) - \bar{U}_h(r,0)\|^2 + \|U(r,1) - \bar{U}_h(r,1)\|^2)dr$$
$$\le Ch^8(|U|^2_{(H^4(\bar{AB})^3} + |U|^2_{(H^4(\bar{CD}))^3}) \le Ch^8|U|^2_{(H^4(\Gamma_1))^3} \quad (4.31)$$

By noting that $(\bar{U_n})_h$ on Γ_1 is also piecewise cubic Hermite interpolatory polynomials of U_n, we have similarly

$$IV^* = \|\delta(U_n - (\bar{U_n})_h)\|^2_{[0,1]} \le Ch^8|U_n|^2_{(H^4(\Gamma_1))^3} \quad (4.32)$$

Therefore, combining (4.26)–(4.32) leads to

$$\{\bar{D}(U - \bar{U}_h, U - \bar{U}_h)\}^{\frac{1}{2}} = Ch^{-\sigma}\{(I^*)^{\frac{1}{2}} + (II^*)^{\frac{1}{2}} + (III^*)^{\frac{1}{2}} + (IV^*)^{\frac{1}{2}}\}$$
$$\le Ch^{4-\sigma}(||B||_{(H^{q+1}(\Gamma_2))^2} \times |U_n|_{(H^4(\Gamma_2))^3} + |U|_{(H^4(\Gamma_1))^3} + |U_n|_{(H^4(\Gamma_1))^3}) \quad (4.33)$$

Finally the desired bounds (4.22) are obtained from (4.23)–(4.25) and (4.33). This completes the proof of Theorem 4.1. □

Corollary 4.1. *Let all the conditions in Theorem* 4.1 *and* $q \ge 1$ *hold. Then* $\sigma = 2$ *is the best choice, leading to the following bounds as* $h \to 0$,

$$|||U - U_h^*||| \le Ch^2\{|U|_{(H^4(\Omega))^3} + |F|_{(H^2(\Omega))^3} + |M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3}$$
$$+ \|B\|_{(H^{q+1}(\Gamma_2))^2} \times |U_n|_{(H^4(\Gamma_2))^3} + |U|_{(H^4(\Gamma_1))^3} + |U_n|_{(H^4(\Gamma_1))^3}\}$$
$$+ Ch^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2} \times |M(U)|_{(H^0(\Gamma_2))^3} \quad (4.34)$$

Proof. Based on Theorem 4.1, when $q \ge 1$ and $h \to 0$, the best choice of σ is given by $\sigma = 4 - \sigma$, leading to $\sigma = 2$, then to obtain (4.34). This completes the proof of Corollary 4.1. □

It is worthy noting that $O(h^2)$ of second derivatives implies a high accuracy of the solutions from Method I of BP-FEMs. The optimal convergence rate $O(h^2)$ is also given for plate problems by other finite element methods (see Ciarlet (90)). Now we have the following corollary.

Corollary 4.2. *Let all the conditions in Theorem* 4.1 *and* $\sigma = 2$ *and* $q \ge 1$ *hold, When* $h \to 0$ *the solutions from Method I of BP-FEMs have the following asymptotes.*

$$|||U - U_h^*||| = O(h^2), \quad ||U - U_h^*||_{(H^2(\Omega))^3} = O(h^2) \quad (4.35)$$
$$\|\delta U_h^*\|_{[0,1]} = O(h^4), \quad \|\delta(U_n^*)_h\|_{[0,1]} = O(h^4)$$
$$||(y_n^*)_h - \bar{b}_1(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^4), \quad ||(z_n^*)_h - \bar{b}_2(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^4) \quad (4.36)$$
$$||(y_n^*)_h - b_1(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^{q+1}), \quad ||(z_n^*)_h - b_2(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^{q+1}) \quad (4.37)$$

In addition, when $q = 3$, *i.e., when using cubic Lagrange* (*or Hermite*) *interpolation,* $\bar{b}_l$ *and* $\bar{b}_2$, *then the solutions from Method I of BP-FEMs satisfy the tangent boundary conditions* (2.6) *with the following high convergence rate*

$$||(y_n^*)_h - b_1(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^4), \quad ||(z_n^*)_h - b_2(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^4) \quad (4.38)$$

Proof. Based on Corollary 4.1 we can see

$$\|\delta U_h^*\|_{[0,1]} = \|\delta(U - U_h^*)\|_{[0,1]} \leq Ch^2 \bar{D}(U - U_h^*, U - U_h^*)^{1/2} \leq Ch^2 |||U - U_h^*||| \leq Ch^4 \tag{4.39}$$

Similarly

$$||(y_n^*)_h - \bar{b}_1(x_n^*)_h||_{H^0(\Gamma_2)} = Ch^4 \tag{4.40}$$

Also we have

$$\begin{aligned} ||(y_n^*)_h - b_1(x_n^*)_h||_{H^0(\Gamma_2)} &\leq ||(y_n^*)_h - \bar{b}_1(x_n^*)_h||_{H^0(\Gamma_2)} + ||b_1 - \bar{b}_1||_{H^0(\Gamma_2)} \times ||(x_n^*)_h||_{H^0(\Gamma_2)} \\ &\leq C\{h^4 + h^{q+1}||b_1||_{H^{q+1}(\Gamma_2)} \times ||(x_n^*)_h||_{H^0(\Gamma_2)}\} \\ &\leq C\{h^4 + h^{q+1}||B||_{(H^{q+1}(\Gamma_2))^2}\} \end{aligned} \tag{4.41}$$

In the above inequality, we have used the boundedness of $||(x_n^*)_h||_{H^0(\Gamma_2)}$,

$$\begin{aligned} ||(x_n^*)_h||_{H^0(\Gamma_2)} &\leq C||(x_n^*)_h||_{H^2(\Omega)} \leq C||U_h^*||_{(H^2(\Omega))^3} \\ &\leq C\{||U||_{(H^2(\Omega))^3} + ||U - U_h^*||_{(H^2(\Omega))^3}\} \leq C\{||U||_{(H^2(\Omega))^3} + |||U - U_h^*|||\} \\ &\leq C\{||U||_{(H^2(\Omega))^3} + h^2\} \leq C \end{aligned} \tag{4.42}$$

The first inequality in (4.37) is obtained, then leading to the left inequality in (4.38) when $q = 3$. The proof of other bounds in Corollary 4.3 are similar, thus to complete the proof. □

4.2. Error Bounds of Solutions from Method II of BP-FEMs Involving Approximate Integration

In this subsection, we also derive error bounds of the solutions from Method II, (3.18). Our concern is which order of accuracy of integration rules should be used to maintain $O(h^2)$ of errors in Ω, as in Corollary 4.1, and $O(h^4)$ in Corollary 4.2 for the tangent boundary conditions. Denote another norm

$$|||U|||_H = \{||U||^2_{(H^2(\Omega))^3} + D(U,U)\}^{1/2} \tag{4.43}$$

where the true functions b_1 and b_1 are chosen in $D(U,U)$ of (3.20), instead of $\bar{b}_1$ and $\bar{b}_2$ in (3.17). Then we have the following lemma (See Ciarlet(91)).

Lemma 4.3. *Let $M(U) \in (H^0(\Gamma))^3$ and $P(U) \in (H^0(\Gamma_1))^3$ and the following inequalities be given*

$$|\hat{A}_P(U,W)| \leq C_0 |||U|||_H \times |||W|||_H, \quad U \in H_0 \text{ and } W \in V_0 \tag{4.44}$$

and

$$C_1 |||U|||_H{}^2 \leq \hat{A}_P(U,U), \quad U \in V_0 \tag{4.45}$$

There exists a bounded constant C independent of h, U, W and E such that

$$|||U - \hat{U}_h^*|||_H \leq C\Big\{ \inf_{W\in V} |||U - W|||_H + h^\sigma(|M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3})$$

$$+\sup_{E\in V_0}\frac{|A(W,E)-\hat{A}(U,E)|}{|||E|||_H}+\sup_{E\in V_0}\frac{|F(E)-\hat{F}(E)|}{|||E|||_H}$$
$$+\sup_{E\in V_0}\frac{|D(W,E)-\hat{D}(W,E)|}{|||E|||_H}\Big\} \tag{4.46}$$

where $\hat{U_h^*}(\in V)$ are the solution from Method II of BP-FEMs involving integration approximation.

Comparing Lemma 4.3 with Lemma 4.2, there disappear the terms

$$h^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2}\times|M(U)|_{(H^0(\Gamma))^3}+h^2|F|_{(H^2(\Omega))^3}$$

because of the true functions b_1 and b_2 and F are chosen; but there appear more terms resulting from integration approximation. Below let us prove a useful lemma.

Lemma 4.4. *Let $U\in(H^4(\Gamma_2))^3$, $U_n\in(H^4(\Gamma_2))^3$ and $B\in(H^p(\Gamma_2))^2$ be given. Assume that the integral rule of $\hat{\int}_{\Gamma_2}$ has $2p-2$ order of accuracy with $p\geq\sigma$. Then there exist the following bounds*

$$\begin{aligned}|D(\bar{U}_h,E)-\hat{D}(\bar{U}_h,E)|\leq& Ch^{p-\sigma}(||B||_{(H^p(\Gamma_2))^2}\times||U_n||_{(H^4(\Gamma_2))^3}\\&+||U_n||_{(H^4(\Gamma_1))^3}+||U||_{(H^4(\Gamma_1))^3})\times|||E|||_H,\quad E\in V_0\end{aligned} \tag{4.47}$$

where $\bar{U}_h$ are the piecewise Hermite bi-cubic interpolatory polynomials of the true solution U.

Proof. We have

$$\begin{aligned}|D(\bar{U}_h,E)-\hat{D}(\bar{U}_h,E)|\leq&\frac{P_c}{h^{2\sigma}}\Big\{\Big|\Big(\int_{\Gamma_2}-\hat{\int}_{\Gamma_2}\Big)(\bar{y}_n-b_1\bar{x}_n)(\eta_n-b_1\xi_n)d\Gamma\Big|\\&+\Big|\Big(\int_{\Gamma_2}-\hat{\int}_{\Gamma_2}\Big)(\bar{z}_n-b_2\bar{x}_n)(\zeta_n-b_2\xi_n)d\Gamma\Big|\\&+\Big|\Big(\int_0^1-\hat{\int}_0^1\Big)(\bar{U}_h(r,0)-\bar{U}_h(r,1))\cdot(W_h(r,0)-W_h(r,1))dr\Big|\\&+\Big|\Big(\int_0^1-\hat{\int}_0^1\Big)((\bar{U_n})_h(r,0)-(\bar{U_n})_h(r,1))\\&\cdot((W_n)_h(r,0)-(W_n)_h(r,1))dr\Big\}\\\leq& C\,h^{-2\sigma}\{I+II+III+IV\}\end{aligned} \tag{4.48}$$

The bounds of all the four terms in (4.48) will be provided below. First let us show

$$\begin{aligned}I=&\Big|\Big(\int_{\Gamma_2}-\hat{\int}_{\Gamma_2}\Big)(\bar{y}_n-b_1\bar{x}_n)(\eta_n-b_1\xi_n)d\Gamma\Big|\\\leq& Ch^{p+\sigma}||B||_{H^p(\Gamma_2)^2}\times||\bar{U}_n||_{H^p(\Gamma_2)^3}\times||E|||_H\end{aligned} \tag{4.49}$$

where $\bar{x_n}$ and $\bar{y}_n$ are the piecewise Hermite interpolatory polynomials of the true solution x_n and y_n. Denote

$$f=(\bar{y}_n-b_1\bar{x}_n),\quad g=(\eta_n-b_1\xi_n). \tag{4.50}$$

Since the integration rule has $2p-2$ order of accuracy, we have from Ciarlet (90)

$$I = \Big|\Big(\int_{\Gamma_2} - \hat{\int}_{\Gamma_2}\Big) fg d\Gamma| \le Ch^p ||f||_{H^p(\Gamma_2)} \times ||g||_{H^0(\Gamma_2)} \tag{4.51}$$

By applying the binomial formula

$$(bx)^{(p)} = \sum_{k=0}^{p} C_p^k b^{(k)} x^{(p-k)} \tag{4.52}$$

we obtain

$$\begin{aligned} ||f||_{H^p(\Gamma_2)} \le& ||\bar{y}_n||_{H^p(\Gamma_2)} + C||b_1||_{H^p(\Gamma_2)} \times ||\bar{x}_n||_{H^p(\Gamma_2)} \\ \le& ||\bar{U}_n||_{(H^p(\Gamma_2))^3} + C||B||_{(H^p(\Gamma_2))^2} \times ||\bar{U}_n||_{(H^p(\Gamma_2))^3} \\ \le& C(||U_n||_{(H^4(\Gamma_2))^3} + ||B||_{(H^p(\Gamma_2))^2} \times ||U_n||_{(H^4(\Gamma_2))^3}) \\ \le& C||B||_{(H^p(\Gamma_2))^2} \times ||U_n||_{(H^4(\Gamma_2))^3} \end{aligned} \tag{4.53}$$

In (4.53) we have used the following bounds

$$\begin{aligned} ||\bar{U}_n||_{(H^p(\Gamma_2))^3} \le& ||U_n||_{(H^p(\Gamma_2))^3} + ||U_n - \bar{U}_n||_{(H^p(\Gamma_2))^3} \\ \le& ||U_n||_{(H^p(\Gamma_2))^3} + Ch^{(4-p)}|U_n|_{(H^4(\Gamma_2))^3} \le C||U_n||_{(H^4(\Gamma_2))^3}, \text{ as } p \le 3 \end{aligned} \tag{4.54}$$

and by noting that $\bar{U}_n$ are only piecewise cubic Hermite polynomials on Γ_2,

$$||\bar{U}_n||_{(H^p(\Gamma_2))^3} = ||\bar{U}_n||_{(H^3(\Gamma_2))^3} \le C||U_n||_{(H^4(\Gamma_2))^3}, \text{ as } p \ge 4 \tag{4.55}$$

Also since

$$||g||_{H^0(\Gamma_2)} \le C\, h^\sigma D(E,E)^{1/2} \le C\, h^\sigma |||E|||_H \tag{4.56}$$

we have from (4.51), (4.53) and (4.56)

$$I \le Ch^{p+\sigma} ||B||_{(H^p(\Gamma_2))^2} \times ||U_n||_{(H^4(\Gamma_2))^3} \times |||E|||_H \tag{4.57}$$

Similarly,

$$\begin{aligned} II =& \Big|\Big(\int_{\Gamma_2} - \hat{\int}_{\Gamma_2}\Big)(\bar{z}_n - b_2\bar{x}_n)(\zeta_n - b_2\xi_n)d\Gamma\Big| \\ \le& Ch^{p+\sigma}||B||_{(H^p(\Gamma_2))^2} \times ||U_n||_{(H^4(\Gamma_2))^3} \times |||E|||_H \end{aligned} \tag{4.58}$$

By further manipulation, we can see from (4.51)

$$\begin{aligned} III =& \Big|\Big(\int_0^1 - \hat{\int}_0^1\Big)(\bar{U}_h(r,0) - \bar{U}_h(r,1)) \cdot (W_h(r,0) - W_h(r,1))dr\Big| \\ \le& Ch^p ||\bar{U}_h||_{(H^p(\Gamma_1))^3} \times \Big\{\int_0^1 ||W_h(r,0) - W_h(r,1)||^2 dr\Big\}^{\frac{1}{2}} \\ \le& Ch^{p+\sigma}||U||_{(H^4(\Gamma_1))^3} \times |||E|||_H \end{aligned} \tag{4.59}$$

Similarly,

$$
\begin{aligned}
IV =&\Big|\Big(\int_0^1 - \hat{\int_0^1}\Big)((\bar{U_n})_h(r,0) - (\bar{U_n})_h(r,1)) \cdot ((W_n)_h(r,0) - (W_n)_h(r,1))dr\Big| \\
\le& Ch^{p+\sigma}||U_n||_{(H^4(\Gamma_1))^3} \times |||E|||_H
\end{aligned} \tag{4.60}
$$

By applying (4.48), the desired results (4.47) are obtained from (4.57)–(4.60). This completes the proof of Lemmas 4.4. □

We also have the following lemma.

Lemma 4.5. *Let $U \in (H^4(\Omega))^3$ and $F \in (H^2(\Omega))^3$ be given. Assume that Simpson's rule is used for the 2D integrals in $A_h(U_h, W_h)$ and $F(W_h)$. There exist the following bounds*

$$\sup_{E \in V_0} |A(U_h, E) - \hat{A}(U_h, E)| \le Ch^2 ||U||_{(H^4(\Omega))^3} \times |||E|||_H, \tag{4.61}$$

and

$$\sup_{E \in V_0} |F(E) - \hat{F}(E)| \le Ch^2 ||F||_{(H^2(\Omega))^3} \times |||E|||_H \tag{4.62}$$

Proof. We have from (4.51)

$$|A(\bar{U}_h, E) - \hat{A}(\bar{U}_h, E)| \le Ch^2 \|D^2(\bar{U}_h)\|_{(H^2(\Omega))^3} \times |||E|||_H \tag{4.63}$$

where

$$
\begin{aligned}
&\|D^2(\bar{U}_h)\|_{(H^2(\Omega))^3} \le C\|\bar{U}_h\|_{(H^4(\Omega))^3} \le C\{\|U\|_{(H^4(\Omega))^3} + \|U - \bar{U}_h\|_{(H^4(\Omega))^3}\} \\
&C\{\|U\|_{(H^4(\Omega))^3} + h^\epsilon \|U\|_{(H^{4+\epsilon}(\Omega))^3}\} \le C\|U\|_{(H^{4+\epsilon}(\Omega))^3}
\end{aligned} \tag{4.64}
$$

where $0 < \epsilon < 1$ and $\|U\|_{(H^{4+\epsilon}(\Omega))^3}$ are the Sobolev norms with nonintegrals. The bounds (4.61) are obtained from (4.64) and (4.63) as $\epsilon \to 0$; the other bounds (4.62) can be proved similarly. This completes the proof of Lemma 4.5. □

Finally, by following the proof of Theorem 4.1, we obtain the following theorem from Lemmas 4.3 – 4.5.

Theorem 4.2. *Let $U \in (H^4(\Omega))^3$, $F \in (H^2(\Omega))^3$, $U \in (H^4(\Gamma))^3$, $U_n \in (H^4(\Gamma))^3$, $M(U) \in (H^0(\Gamma))^3$, $P(U) \in (H^0(\Gamma_1))^3$ and $B \in (H^p(\Gamma_2))^2$ be given. Then by using Simpson's rules for $A(U, W)$ and $F(U)$, and the integration rules with order $2p-2$ for the penalty integrals in $D(U, W)$, there exist the following bounds of the solutions from Method II of BP-FEMs*

$$
\begin{aligned}
|||U - \hat{U}_h^*|||_H \le& C\{h^2(\|U\|_{(H^4(\Omega))^3} + ||F||_{(H^2(\Omega))^3}) + h^\sigma(|M(U)|_{(H^0(\Gamma))^3} + |P(U)|_{(H^0(\Gamma_1))^3}) \\
&+ h^{4-\sigma}(\|B\|_{(H^0(\Gamma_2))^2} \times |U_n|_{(H^4(\Gamma_2))^3} + |U_n|_{(H^4(\Gamma_1))^2} + |U|_{(H^4(\Gamma_1))^3}) \\
&+ Ch^{p-\sigma}(||B||_{(H^p(\Gamma_2))^2} \times ||U_n||_{(H^4(\Gamma_2))^3} + ||U_n||_{(H^4(\Gamma_1))^2} + ||U||_{(H^4(\Gamma_1))^3})\}
\end{aligned} \tag{4.65}
$$

We have the following corollary.

Corollary 4.3. *Let all the conditions in Theorem 4.2 be given. Choose $\sigma = 2$ and $p = 4$, i.e., the integration rules of order six are used for the penalty integrals. The solutions from Method II of BP-FEMs have with the asymptotes.*

$$\begin{aligned} &|||U - \hat{U}_h^*|||_H = O(h^2), \quad ||U - \hat{U}_h^*||_{(H^2(\Omega))^3} = O(h^2) \\ &\|\delta \hat{U}_h^*\|_{[0,1]} = O(h^4), \quad \|\delta (\hat{U}_n^*)_h\|_{[0,1]} = O(h^4) \\ &||(\hat{y}_n^*)_h - b_1(\hat{x}_n^*)_h||_{H^0(\Gamma_2)} = O(h^4), \quad ||(\hat{z}_n^*)_h - b_2(\hat{x}_n^*)_h||_{H^0(\Gamma_2)} = O(h^4) \end{aligned} \tag{4.66}$$

Proof. When $\sigma = 2$, the value of p should be chosen such that

$$p - \sigma = 2, \text{ then } p = 2 + \sigma = 4, \text{ and } 2p - 2 = 6$$

in order to achieve the optimal convergence rates $O(h^2)$. This indicates order six of accuracy of integration rules. The proof of Corollary 4.3 is completed. □

It is also noted that the inequalities (4.44) and (4.45) hold, since the following bounds can be proven by the norm equivalence of finite dimensions.

$$C_0 D(E, E) \le \hat{D}(E, E) \le C_1 D(E, E), \quad \forall E \in V_0 \tag{4.67}$$

where C_0 and C_1 are two bounded constants independent of h and E.

4.3. Comparisons

Now let us compare different requirements of data of functions b_1 and b_2 and F in Method I and II of BP-FEMs.

(1). Based on Corollaries 4.2 and 4.3, if for both

$$||U - U_h||_{(H^2(\Omega))^3} = O(h^2) \tag{4.68}$$

and

$$||(y_n)_h - b_1(x_n)_h||_{H^0(\Gamma_2)} = O(h^4), \quad ||(z_n)_h - b_2(x_n)_h||_{H^0(\Gamma_2)} = O(h^4) \tag{4.69}$$

where $U_h = U_h^*$ or $U_h = \hat{U}_h^*$, the same requirements of $B \in (H^4(\Gamma_2))^2$ are needed by methods I as $q = 3$ and Method II as $p = 4$. Hence, we may employ the values of $B_1(0, t_j + k\delta t_j/3)$, $k = 0, 1, 2, 3$ in Method I to the Lagrange interpolation nodes, but the values $b_1(0, \frac{1+\theta_k}{2})$, $k = 1, 2, 3, 4$ to Gaussian integration nodes. As to F, although the assumption $F \in (H^2(\Omega))^3$ is the same, only the values of $F(r_i, t_j)$ are used in Method I; but both $F(r_i, t_j)$ and $F(r_i + \delta r_j/2,\, t_j + \delta t_j/2))$ in Method II to Simpson's rule. This shows an advantage of Method I using fewer and simpler data of b_1 and b_1 and F.

(2) Let us consider the case of less smoothness as $B \in (H^m(\Gamma_2))^2$, $m = 1, 2, 3$, which may occur in different applications. For instance, when functions b_1 and b_2 are supplied by discrete, experimental data. Hence $U_n \in (H^m(\Gamma_2))^3$ and $U \in (H^\alpha(\Omega))^3$ with $\alpha = \min(4, m + \frac{3}{2})$ should be also assumed by the solution regularity. We only give corollaries involving choices of σ, p and q, and convergence rates; detailed proofs may follow those in Theorems 4.1 and 4.2.

First consider Method I, different bounds of the terms in Theorem 4.1 may be provided below by some manipulation

$$C\{h^{\alpha-2}|U|_{(H^\alpha(\Omega))^3} + h^{m-\sigma}(\|B\|_{(H^{q+1}(\Gamma_2))^2} \times |U_n|_{(H^m(\Gamma_2))^3}) + h^{q+1}|B|_{(H^{q+1}(\Gamma_2))^2} \times |M(U)|_{(H^0(\Gamma_2))^3}\} \quad (4.70)$$

where $q = m - 1$. the best σ should be chosen such that

$$m - \sigma = \sigma, \text{ then } \sigma = \frac{m}{2}$$

We then give the following corollary.

Corollary 4.4. *Let $B \in (H^m(\Gamma_2))^2$, $U_n \in (H^m(\Gamma_2))^3$, $U \in (H^\alpha(\Omega))^3$ with $\alpha = \min(4, m + \frac{3}{2})$, and other conditions in Theorem 4.1 be given. Suppose $m = 1, 2, 3$ and choose $q = m - 1$ and $\sigma = \frac{m}{2}$. The solutions from Method I of BP-FEMs lead to the reduced convergence rates*

$$|||U - U_h^*|||_H = O(h^{\frac{m}{2}}), \qquad ||U - U_h^*||_{(H^2(\Omega))^3} = O(h^{\frac{m}{2}}) \quad (4.71)$$

$$\|\delta U_h^*\|_{[0,1]} = O(h^m), \quad \|\delta(U_n^*)_h\|_{[0,1]} = O(h^m)$$
$$||(y_n^*)_h - b_1(x_n^*)_h||_{H^0(\Gamma_2)} = O(h^m), \quad ||(z_n^*)_h - b_2(\hat{x}_n^*)_h||_{H^0(\Gamma_2)} = O(h^m) \quad (4.72)$$

Similarly, we obtain the following corollary for Method II.

Corollary 4.5. *Let $B \in (H^m(\Gamma_2))^2$, $U_n \in (H^m(\Gamma_2))^3$, $U \in (H^\alpha(\Omega))^3$ with $\alpha = \min(4, m + \frac{3}{2})$ and other conditions in Theorem 4.2 be given. Suppose $m = 1, 2, 3$ and choose $\sigma = \frac{m}{2}$ and $p = m$, i.e., the integration rules of order $2m - 2$ are used for the penalty integrals. The solutions from Method II of BP-FEMs satisfy the reduced convergence rates*

$$|||U - \hat{U}_h^*|||_H = O(h^{\frac{m}{2}}), \quad ||U - \hat{U}_h^*||_{(H^2(\Omega))^3} = O(h^{\frac{m}{2}}) \quad (4.73)$$

$$\|\delta\hat{U}_h^*\|_{[0,1]} = O(h^m), \quad \|\delta(\hat{U}_n^*)_h\|_{[0,1]} = O(h^m)$$
$$||(\hat{y}_n^*)_h - b_1(\hat{x}_n^*)_h||_{H^0(\Gamma_2)} = O(h^m), \quad ||(\hat{z}_n^*)_h - b_2(\hat{x}_n^*)_h||_{H^0(\Gamma_2)} = O(h^m) \quad (4.74)$$

It is interesting to note that the best choice of σ and the reduced convergence rates are the same in Corollaries 4.4 and 4.5. Table 1 provides a clear view on the important theoretical results of Corollaries 4.2–4.5.

Table 1 The best parameters and convergence rates of solutions from Methods I and II by the assumptions of $B \in (H^m(\Gamma_2))^2$, $U_n \in (H^m(\Gamma_2))^3$, and $U \in (H^\alpha(\Omega))^3$, where $\alpha = \min(4, m + \frac{3}{2})$, $1 \le m \le 4$

m	σ	Convergence Rates		Method I		Method II	
		$\|U - U_h\|_{(H^2(\Omega))^3}$	$\|(y_n)_h - b_1(x_n)_h\|_{(H^2(\Gamma_2))^3}$	$q = m - 1$	order $6 + 2q$	p	order $2p - 2$
4	2	$O(h^2)$	$O(h^4)$	3	12	4	6
3	$\frac{3}{2}$	$O(h^{\frac{3}{2}})$	$O(h^3)$	2	10	3	4
2	1	$O(h)$	$O(h^2)$	1	8	2	4
1	$\frac{1}{2}$	$O(h^{\frac{1}{2}})$	$O(h)$	0	6	1	2

(3) However, Method II seems to be simpler since integration rules with lower order are employed. To combining both advantages we may adopt integral rules in Method II to approximate all the integrals in Method I, thus leading to Method III, (3.21), of BP-FEMs. Error analysis related can be carried out similarly.

(4). For the purpose of blending surfaces, small division numbers m and n may be chosen. For example, let $n = m = 10$ with uniform rectangles, then $h = 0.1$ and $h^4 = 0.0001$. Based on Corollaries 4.2 and 4.3, the boundary conditions of solutions have the optimal convergence rate $O(h^4) = O(10^{-4})$. Suppose that such an accuracy is even lower than that required by engineering designs, we may suitably modify the numerical solutions. For example, since the property GC^1 is our main concern, we may use the average values as the final solutions on Γ_1

$$U_h^{=}(r,0) = U_h^{=}(r,1) = (U_h(r,0) + U_h(r,1))/2,\ 0 < r < 1 \tag{4.75}$$

$$(U_n^{=})_h(r,0) = (U_n^{=})_h(r,1) = ((U_n)_h(r,0) + (U_n)_h(r,1))/2,\ 0 < r < 1 \tag{4.76}$$

where $U_h = U_h^*$, $\hat{U}_h^*$ or $\hat{\hat{U}}_h^*$. Similarly, a suitable modification can also be made on the derivatives on the boundary Γ_2.

5. Numerical Experiments

In this section, the numerical experiments are designed to verify the basic optimal convergence rates (4.68) and (4.69). A simple model of $u(r,t)$ is chosen; the real blending examples with three solutions, $x(r,t)$, $y(r,t)$ and $z(r,t)$, will appear elsewhere. Consider

$$\Delta^2 u(r,t) = 0, (r,t) \text{ in } \Omega, \tag{5.1}$$

$$u = g \text{on } \Gamma_2, \frac{\partial u}{\partial n} = g_1 \text{ on } \Gamma_2, \tag{5.2}$$

$$u(r,0) = u(r,1), u_t(r,0) = u_t(r,1) \text{ on } \Gamma_1, \tag{5.3}$$

where Ω, Γ_1 and Γ_2 are given in Fig.2. Choose the particular solution

$$u(r,t) = \sinh 2\pi r \cos 2\pi t \tag{5.4}$$

satisfying (5.1) and (5.3). So the functions g and g_1 in (5.2) are found from (5.4).

We employ the following boundary penalty method

$$I(u_h) = \min_{v \in V_h^0} I(v) \tag{5.5}$$

where the energy

$$I(v) = \iint_\Omega (\Delta v)^2 d\Omega + \frac{P_c}{h^{2\sigma}} \int_{\Gamma_2} \Big(\frac{\partial v}{\partial n} - \bar{g}_1\Big)^2 d\Gamma + \frac{P_c}{h^{2\sigma}} \Big[\int_{\Gamma_1} (v(r,1) - v(r,0))^2 d\Gamma + \int_{\Gamma_1} (v_t(r,1) - v_t(r,0))^2 d\Gamma\Big]. \tag{5.6}$$

Here the optimal parameter $\sigma = 2$, based on Corollaries 4.2 and 4.3. V_h^0 denotes the space of the piecewise cubic Hermite functions satisfying the boundary condition

$$u = \bar{g} \text{ on } \Gamma_1. \tag{5.7}$$

The functions $\bar{g}$ and $\bar{g}_1$ in (5.6) and (5.7) are the piecewise cubic Hermite interpolants of g and g_1 respectively.

The boundary penalty method (5.5) is, indeed, analogous to Method I of BP-FEMs in (3.14)–(3.17). Since the integrands in (5.6) are piecewise polynomials of order up to 12, we may adopt the composite Gaussian rule of order six, to evaluate all the integrals in (5.6) exactly. The Gaussian rule of sixth order is given in Davis and Rabinowitz(84) as

$$\int_{-1}^{1} f(\theta)d\theta \approx \widehat{\int_{-1}^{1}} f(x)dx = \sum_{k=1}^{4} w_k f(\theta_k),$$

with the following four nodes and weights

$$\theta_k = \pm 0.8611363116, \quad w_k = 0.3478546451, \quad k = 1,\ 2$$
$$\theta_k = \pm 0.3399810436, \quad w_k = 0.6521451549, \quad k = 3,\ 4.$$

Eq. (5.5) leads to a linear system of algebraic equations

$$AW = b \tag{5.8}$$

where A is positive definite and symmetric, and W is a unknown vector. We use the Gaussian elimination method to obtain W, i.e., u_h in (5.5). For simplicity, the uniform square elements, $\square_{ij}$, are chosen with the boundary length $h = 1/N$, where N is the division number along $\overline{AB}$ in Fig.2.

After trial computation, it is found that $P_c = 1 \sim 10$ is a good choice due to rather smaller errors of u_h and condition number of matrix A. So we choose $P_c = 5$ in our computation. The calculated results are provided in Table 2, where $\varepsilon = u_h - u$, u is the true solution (5.4). In Table 2, $\|\varepsilon\|_{k,\Omega}$ are the Sobolev norm over $H^k(\Omega)$, and other notations are

$$|\varepsilon|_{0,\infty} = \max_{\Omega} |\varepsilon|,\ |\varepsilon|_{1,\infty} = \max\{|\varepsilon_r|, |\varepsilon_t|\}$$
$$|\varepsilon|_{2,\infty} = \max_{\Omega}\{|\varepsilon_{rr}|, |\varepsilon_{rt}|, |\varepsilon_{tt}|\},$$
$$\|(u_h)_n - g_1\|_{\Gamma_2} = \|(u_h)_n - g_1\|_{H^0(\Gamma_2)} = \Big(\int_{\Gamma_1} ((u_h)_n - g_1)^2 d\Gamma\Big)^{\frac{1}{2}}$$
$$\|\varepsilon\|_{\Gamma_1} = \Big\{\int_0^1 [(u_h(r,1) - u(r,1))^2 + ((u_h)_t(r,1) - u_t(r,1))^2]\Big\}^{\frac{1}{2}}$$
$$\|\delta u_h\|_{\Gamma_1} = \Big(\|\delta u_h\|^2_{[0,1]} + \|\delta (u_h)_t\|^2_{[0,1]}\Big)^{\frac{1}{2}}$$
$$= \Big\{\int_0^1 [(u_h(r,1) - u_h(r,0))^2 + ((u_h)_t(r,1) - (u_h)_t(r,0))^2] d\ell\Big\}^{\frac{1}{2}}$$

Table 2 The error norms for different N as $\sigma = 2$ and $P_c = 5$.

N	2	4	6	8	12	16
$\lvert\varepsilon\rvert_{0,\infty}$	12.9	2.95	0.0855	0.0318	0.727×10^{-2}	0.246×10^{-2}
$\lvert\varepsilon\rvert_{1,\infty}$	95.4	34.7	12.6	5.70	1.78	0.765
$\lvert\varepsilon\rvert_{2,\infty}$	0.214×10^{4}	0.125×10^{4}	664	398	185	106
$\lVert\varepsilon\rVert_{0,\Omega}$	4.45	0.878	0.191	0.0626	0.0127	0.406×10^{-2}
$\lVert\varepsilon\rVert_{1,\Omega}$	44.1	10.6	3.14	1.33	0.392	0.165
$\lVert\varepsilon\rVert_{2,\Omega}$	624	260	119	67.5	30.3	17.1
$\lVert(u_h)_n - g_1\rVert_{\Gamma_2}$	30.2	11.6	2.34	0.745	0.148	0.0468
$\lVert\varepsilon\rVert_{\Gamma_1}$	6.50	0.650	0.148	0.0495	0.0101	0.0325
$\lVert\delta u_h\rVert_{\Gamma_1}$	5.96	0.146	0.0141	0.253×10^{-2}	0.211×10^{-3}	0.350×10^{-4}

Note that the values of errors in Table 2 are in an absolute sense. Since the true solution of (5.4) are large, with the norms, $\max_{\Omega} |u| = 264$ and $\|u\|_{0,\Omega} = 53.4$, the relative errors from Table 2 are small. The error curved are depicted in Figs 3 and 4, based on the data in Table 2. It can be discovered that the following asymptotic relations hold.

$$\|\varepsilon\|_{k,\Omega} = O(h^{4-k}),\ k = 0, 1, 2, \tag{5.9}$$

$$\|(u_h)_n - g_1\|_{\Gamma_2} = O(h^4), \tag{5.10}$$

$$\|\varepsilon\|_{\Gamma_1} = O(h^4), \tag{5.11}$$

$$\|\delta u_h\|_{\Gamma_1} = O(h^6). \tag{5.12}$$

The computed results $\|\varepsilon\|_{2,\Omega} = O(h^2)$ and (5.10) have verified perfectly the important optimal convergence rates, (4.68) and (4.69).

Fig.3 The error curves of $\|\varepsilon\|_{k,\Omega}$ versus N with $\sigma = 2$ and $P_c = 5$

Fig.4 The error curves of $\|(u_h)_n - g\|_{\Gamma_2}$ versus N with $\sigma = 2$ and $P_c = 5$ and $\|\delta(u_h)_t\|_{\gamma_1}$

To our surprise, $O(h^6)$ in Eqs. (5.12) is superconvergence, compared to $O(h^4)$ given in Corollaries 4.2 and 4.3. Some numerical solutions at the boundary nodes are listed in

Table 3. The numerical solutions $u_h(r,0)$ (or $(u_h)_r(r,0)$) are almost identical to $u_h(r,1)$ (or $(u_h)_r(r,1)$). Interestingly, the numerical digits of $(u_h)_t(r,0)$ (or $(u_h)_{rt}(r,0)$) are also almost identical to $(u_h)_t(r,1)$ (or $(u_h)_{rt}(r,1)$), but with opposite signs; these solutions are just the errors due to the null true solutions. Other numerical experiments display that by using the boundary penalty techniques in this paper, the periodical conditions (5.3) of numerical solutions are satisfied with an extreme accuracy if the true solutions such as (5.4) also satisfy the conditions, $\frac{\partial}{\partial n}\Delta u|_{\Gamma_1} = \Delta u|_{\Gamma_1} = 0$. The better performance of the penalty techniques on the periodical boundary conditions can be explained by the following arguments. The penalty techniques in (5.6) play a role of enforcing the constraints (5.3); the error estimates in (4.31) by means of the norm triangular inequality and the true solution interpolant $\bar{U}_h$ may be overestimated. In fact, the true errors $\|\varepsilon\|_{\Gamma_1}$ on Γ_1 can have the optimal convergence rates $O(h^4)$, see (5.11).

Table 3 The approximate and true solutions at the periodical boundary on $t = 0, 1$ and $r = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ as $N = 16$, $\sigma = 2$ and $P_c = 5$.

Solutions		$u(r,t)$	$u_r(r,t)$	$u_t(r,t)$	$u_{rt}(r,t)$
$r = \frac{1}{4}$	$t = 0$	2.300990	15.763687	0.700862×10^{-7}	0.442241×10^{-4}
	$t = 1$	2.300990	15.763687	-0.700874×10^{-7}	-0.442241×10^{-4}
	True	2.301299	15.765633	0	0
$r = \frac{1}{2}$	$t = 0$	11.547870	72.832207	0.337905×10^{-6}	0.202781×10^{-3}
	$t = 1$	11.547870	72.832207	-0.337911×10^{-6}	-0.202781×10^{-3}
	True	11.548379	72.834391	0	0
$r = \frac{3}{4}$	$t = 0$	55.653345	349.745630	0.490877×10^{-6}	0.782088×10^{-3}
	$t = 1$	55.653345	349.745630	-0.490902×10^{-6}	-0.782088×10^{-3}
	True	55.654398	349.743337	0	0

Concluding Remarks. In summary, the new approaches using PDEs in this paper are proposed to construct general and complicated blending surfaces. The solutions of blending surfaces are sought to minimize the global curvature of the entire surface, and the additional boundary conditions are then found to lead to unique PDE solutions. The boundary penalty FEMs are significant to treat complicated boundary conditions since the algorithms can be carried out simply and easily, and since theoretical analysis is also provided. New analysis is devoted to error bounds of the boundary constraints so that useful guidance in approximation of functions b_1 and b_2 are discovered. Not only may the optimal convergence rate $O(h^2)$ of second order generalized derivatives be maintained, but also the high convergence rate $O(h^4)$ of the tangent boundary conditions of solutions can be achieved. High accuracy of solutions implies that small division numbers m and n can be chosen in computation, thus to save CPU time significantly. A numerical example is given to support the basic error analysis made. On the whole, the merits of the algorithms in this paper lie in flexibility and generality to produce the blending surfaces subjected to the complicated boundary conditions,

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