Journal of Computational Mathematics, Vol.16, No.5, 1998, 445–456.

PROBABILISTIC ANALYSIS OF GALERKIN-LIKE METHODS FOR THE FREDHOLM EQUATION OF THE SECOND KIND*[1)]

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Abstract

This paper deals with the approximate solution of the Fredholm equation $u - T_K u = f$ of the second kind from a probabilistic point of view. With Wiener type measures on the set of kernels and free terms we determine statistical features of the approximation process, i.e., the most likely rate of convergence and the dominating individual behavior. The analysis carried out for a kind of Galerkin-like method.

Key words: Probabilistic analysis, fredholm equation, galerkin-like method, abstract wiener space

1. Introduction

Quantitative probabilistic analysis was carried out for several numerical problems. For a systematic survey, we refer to Traub et al. (1988) and references therein. Smale (1985) gave the first quantitative analysis for concrete measure. He expected that the approach there might lead to a more systematic way of analysing for the cost of numerical algorithms. Heinrich (1991) continued this line and gave the first quantitative analysis for concrete measures and algorithms for integral equation of the second kind. There, the analysis was carried out for the Galerkin method and the iterated Galerkin method. It is natural to ask whether other numerical problems can be analyzed from this point of view. In this paper we get counterparts for a kind of Galerkin-like method, which was proposed by Schock (1971). For brevity, later on, it was called Q-method (see, e.g., Schock (1982)). For a more precise discussion of relation between Q-method and Galerkin method and iterated Galerkin method we refer to Schock (1982).

Finally, we briefly outline the contents of this paper. Section 2 reviews some basic facts about Gaussian measures. Section 3 deals with the main problem in terms of general Banach spaces and Gaussian measures. Section 4 specifies our main problem and formulates the principal results. Section 5 and 6 are devoted to the proofs of the principal results.

* Received January 8, 1996.

[1)] This work was partially supported by the National Natural Science Fundation of China, the Chinese National Key Project for Basic Researchers, the Chinese National High Technology Program(863), and the UNSW Vice-Chancellor's Post Doctoral Fellowship in Australia.

2. Preliminaries on Gaussian Measures

We consider only Banach spaces over the field of reals throughout this paper. Given Banach spaces X and Y we let $L(X,Y)$ denote the spaces of all bounded linear operators T from X to Y, equipped with the operator norm $\|T\|$. $K(X,Y)$ is the space of compact operators, and we write $L(X)$ and $K(X)$ if $X=Y$. X^* stands for the dual space of $X, \mathcal{B}(X)$ is the σ-algebra of all Borel subsets of X. The symbol $\langle , \rangle$ is used for the duality between X and X^*, while $(,)$ always denotes inner products. If $X=H$ is a Hilbert space, we identify X^* with H in the usual way, so that $\langle , \rangle$ and $(,)$ coincide. For $x^* \in X^*$, $y \in Y$, $x^* \otimes y \in L(X,Y)$ denotes the operator defined by $(x^* \otimes y)(x) = \langle x, x^* \rangle y$.

Now we list some basic notions and facts about Gassian measures, the emphasis laid on the operator theoretic aspect. A Gaussian measure on a Banach space X is a Radon probability measure μ such that each $x^* \in X^*$ is a symmetric Gaussian random variable on (X,μ) (which may be degenerate, that is, $=0$ almost everywhere). We shall consider only symmetric, i.e., mean zero Gaussian measures. For a Hilbert space H we let γ_H denote the standard Gaussian cylindrical probability (see [Kuo (1975)], [Pietsch (1980)]). For $T \in L(H,X)$ let

$$E_\gamma(T) = \sup_{\substack{F \subset H \\ \dim_{F<\infty}}} \int_F \|Th\| d\gamma_F(h), \tag{1}$$

and let $\Pi_\gamma(H,X)$ denote the set of all $T \in L(H,X)$ with $E_\gamma(T) < \infty$. E_γ is a norm on $\Pi_\gamma(H,X)$ turning it into a Banach space. It is easily checked that

$$\|T\| \leq (\pi/2)^{1/2} E_\gamma(T). \tag{2}$$

For a further Hilbert space H_0, a Banach space X_0, $S \in L(H_0,H)$ and $U \in L(X,X_0)$,

$$E_\gamma(UTS) \leq \|U\| E_\gamma(T) \|S\|, \tag{3}$$

(it follows from [Linde and Pietsch (1974), Lemma 2]). Let $R_\gamma(H,X)$ be the closure of the finite rank operators in $\Pi_\gamma(H,X)$. For $T \in L(H,X)$, let T_{γ_H} denote the cylindrical probability measure induced on X by T, that is, $T_{\gamma_H} = \gamma_H(T^{-1}(B))$ for cylindrical sets B. Now $T \in R_\gamma(H,X)$ if and only if T_{γ_H} has an extension $\tilde{T}_{\gamma_H}$ to $\mathcal{B}(X)$ which is a radon measure (such an extension is unique). So $T \in R_\gamma(H,X)$ implies that $\tilde{T}_{\gamma_H}$ is Gaussian. Conversely, If μ is a Gaussian measure on X, there is a separable Hilbert space H and an injection $J \in R_\gamma(H,X)$ with $\mu = \tilde{J}_{\gamma_H}$. H and J are essentially unique (up to isometries). Note that (J,H,X) is then an abstract Wiener space (see [Kuo(1975)]). If $\mu = \tilde{T}_{\gamma_H}$, $T \in R_\gamma(H,X)$, then $C_\mu = TT^*$ is the covariance operator of μ, the closure of $\mathrm{Im}T$ is the support of μ, and

$$E_\gamma(T) = \int_X \|x\| d\mu(x). \tag{4}$$

These facts can be found in [Kuo (1975), Linde et al (1974), Traub et al (1988)]. If $X=G$ is a Hilbert space, then $R_\gamma(H,G)$ coincides with the class of Hilbert-Schmidt operators $S_2(H, G)$ and

$$(1 + (\pi/2)^{3/2})^{-1}\sigma_2(T) \le E_\gamma(T) \le \sigma_2(T), \quad (5)$$

where $\sigma_2(T)$ denotes the Hilbert-Schmidt norm. This is a consequence of [Pisier (1986), Corollary 2.5 and inequality (2.7)]. The following result will plays crucial pole in our analysis. It takes from Pisier (1986).

Propsition 2.1. *Let X and Y be Banach space, let μ be a Gaussian measure on X, $\mu = \tilde{J}_\gamma(H, X)$ and H is a Hilbert space. Let $T \in L(X, Y)$. Then, for all $t \ge 0$ and $\tau = \pm 1$,*

$$\mu\{x \in X : \tau(\|Tx\| - E_\gamma(TJ)) > t\} \le \exp(-t^2/(2\|TJ\|^2)). \quad (6)$$

We also need the following result of [Chevet (1878), Lemma 3.1] and [Gordon (1985), Corollary 2.4]

Propsition 2.2. *Let X and Y be Banach spaces, $m, n \in N$, $x_1^*, \cdots, x_m^* \in X^*$, $y_1, \cdots, y_n \in Y$. Define the operator $U \in L(l_2^m, X^*)$, $V \in L(l_2^n, Y)$ and $W \in L(l_2^{mn}, L(X, Y))$ by*

$$U(\xi_j) = \sum_{j=1}^{m} \xi_j x_j^*, \quad V(\eta_i) = \sum_{i=1}^{n} \eta_i y_i, \quad W(\zeta_{ij}) \sum_{i=1}^{n} \sum_{j=1}^{m} \zeta_{ij} x_j^* \otimes y_i.$$

Then $E_\gamma(W) \le \|U\| E_\gamma(V) + E_\gamma(U)\|V\| \le 2E_\gamma(W)$.

Finally, we define some notations which we will need in the sequel. If A is a set and $f, g : A \to [0, +\infty)$ are nonnegative functions, we write

$$f(a) \prec g(a)$$

if there is a constant $c > 0$ such that $f(a) \le cg(a)$ for all $a \in A$. Next,

$$f(a) \asymp g(a)$$

means $f(a) \prec g(a)$ and $g(a) \prec f(a)$. If f and g depend on a further variable (collection of parameters, etc.), say $f(a, b), g(a, b)$, $b \in B$,

$$f(a, b) \prec_a g(a, b)$$

means that, for each $b \in B, f(a, b) \prec g(a, b)$ (consequently, the constant of c may depend on b). Analogously, $\asymp_a$ is defined. If the choice of A is ambiguous, we write $\prec_{a \in A}$ and $\asymp_{a \in A}$.

3. General Estimates

We begin with an abstract formulation of the Galerkin-like method. Let X be a Banach space and I the identity operator on X. Let $T \in K(X)$ and $y \in X$. Assume that the Fredholm equation of the second kind,

$$x - Tx = y, \quad (7)$$

has a unique solution $x = x(T, y)$. We want to approximate this solution. For this purpose let $(P_n)_{n\in N} \subset L(X)$ be a sequence of finite rank projections, N always means $\{1, 2, \cdots\}$. Let $n \in N$ and assume that there exists a unique $z_n = z_n(T, y) \in \mathrm{Im}P_n$ satisfying the Galerkin-like equation

$$z_n - P_nTz_n = P_nTy. \tag{8}$$

We define the approximate solution to be

$$x_n^Q = x_n^Q(T, y) = y + z_n, \tag{9}$$

and the error of the Galerkin-like method by

$$\delta_n^Q(T, y) = \|x(T, y) - x_n^Q(T, y)\|. \tag{10}$$

Lemma 3.1. *Let $T \in K(T)$. If $I - P_nT$ is invertible, then, for each $y \in X$, (8) has a unique solution. Hence*

$$x_n^Q(T, y) = (I - P_nT)^{-1}y. \tag{11}$$

Proof. Equation (9) has a unique solution if and only if equation (8) has a unique solution. If $I - P_nT$ is invertible, then equation (8) has a unique solution. Thus, if $I - P_nT$ is invertible equation (9) has a unique solution.

From

$$I + (I - P_nT)^{-1}P_nT = (I - P_nT)^{-1}$$

we can obtain (11). This completes the proof of Lemma 3.1.

Now we specify our measures. Let G and H be Hilbert spaces, let $\Phi \in R_\gamma(G, K(X))$, $J \in R_\gamma(H, X)$ and assume that J is an injection. Put $\mu = \tilde{\Phi}_{\gamma_G}$ and $\nu = \tilde{J}_{\gamma_H}$. To handle stability, we introduce the following sets. Let $\alpha_1, \alpha_2,\ \beta_1, \beta_2 > 0$, $n_0 \in N$, and let $W^Q(\alpha_1, \alpha_2,\ \beta_1, \beta_2, n_0)$ be the set of all $T \in K(X)$ satisfying

(i) $T(J(H)) \subset J(H)$ and $\|J^{-1}(I - T)J\| \le \alpha_1$,

(ii) $I - T$ is invertible, $(I - T)^{-1}(J(H)) \subset J(H)$, and $\|J^{-1}(I - T)^{-1}J\| \le \beta_1$,

(iii) for all $n > n_0$, $\|I - P_nT\| \le \alpha_2$,

(iv) for all $n > n_0$, $I - P_nT$ is invertible and $\|(I - P_nT)\| \le \beta_2$.

Since H is a Hilbert space and J is compact, the image of the unit ball, $J(B_H)$, is closed. From this, it is easily derived that W^Q is a Borel set.

For the quantitative analysis, we have to introduce certain operators related to the approximation process. For $n \in N$ define $\Pi_n^Q \in L(K(X))$ by $\Pi_n^Q T = (I - P_n)T$, for $T \in K(X)$. For the sake of brevity, we put

$$E(n) = E_\gamma((I - P_n)TJ). \tag{12}$$

$$L(n) = \|(I - P_n)TJ)\|. \tag{13}$$

$$E_1(n) = E_\gamma(\Pi_n^Q\Phi). \tag{14}$$

$$L_1(n) = \|(\Pi_n^Q\Phi)\|. \tag{15}$$

Now we come to the probabilistic estimate of the stability sets W^Q. More precisely, we shall reduce it to the estimate of the set $U(\beta)$, defined for $\beta > 0$ by

$$U(\beta) = \{T \in K(X) : \|(I-T)^{-1}\| \le \beta\}. \tag{16}$$

Later on, we shall use the results of Heinrich (1990a) where the probability of this set was estimated. Define the operator $\Psi_{X,H} : \mathrm{Dom}\Psi_{X,H} \to L(X,H)$ as follows. Let $\mathrm{Dom}\Psi_{X,H}$ be the set of those $T \in K(X)$ such that $T(X) \subset J(H)$. By the closed-graph theorem, $J^{-1}T \in L(X,H)$. Now set $\Psi_{X,H}T = J^{-1}T$.

Lemma 3.2. *Suppose that* $Im\Phi \subset Dom\Psi_{X,H}$ *and that* $\Psi_{X,H}\Phi \in R_\gamma(G, L(X,H))$. *Let* $\alpha > 0$, $\beta > 0$, $n_0 \in N$ *and define*

$$\alpha_1 = \|J\|(\alpha + E_\gamma(\Psi_{X,H}\Phi)) + 1, \quad \alpha_2 = \alpha + E_\gamma(\Phi)) + 1/(2\beta) + 1,$$
$$\beta_1 = \|J\|(\alpha + E_\gamma(\Psi_{X,H}\Phi))\beta + 1, \quad \beta_2 = 2\beta.$$

If $E_1(n) \le 1/(4\beta)$ *for all* $n > n_0$, *then*

$$\begin{aligned}\mu(W^Q(\alpha_1,\alpha_2,\beta_1,\beta_2,n_0)) \ge &\mu(U_\beta) - \exp(-\alpha^2/(2\|\Psi_{X,H}\Phi\|^2)) \\ &- \exp(-\alpha^2/(2\|\Phi\|^2)) - \sum_{n>n_0} \exp(-1/(32\beta^2 l_1(n)^2)).\end{aligned}$$

Proof. It follows from [Heinrich (1991), Lemma 2.2].

Now we are ready for the convergence analysis of the Galerkin-like method.

Propsition 3.1. *Let* $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$, $n_0 \in N$, *and assume that* $T \in W^Q(\alpha_1,\alpha_2,\beta_1,\beta_2,n_0)$. *Then, for each* $n > n_0$,

$$\begin{aligned}\nu\{y \in X : \quad &(2\alpha_1\alpha_2)^{-1}E(n) \le \delta_n^Q(T,y) \le (3/2)\beta_1\beta_2 E(n)\} \\ &\ge 1 - 2\exp(-E(n)^2/(8(\alpha_1\alpha_2\beta_1\beta_2 L(n))^2)).\end{aligned}$$

Proof. By the assumption on T, we can define, for $n > n_0$,

$$\Delta_n^Q(T) = (I-T)^{-1} - (I-P_nT)^{-1}. \tag{17}$$

By (7) and (11) we have

$$\delta_n^Q(T,y) = \|\Delta_n^Q(T)y\|. \tag{18}$$

Furthermore, one verifies directly that

$$\begin{aligned}\Delta_n^Q(T)J &= (I-P_nT)^{-1}(I-P_nT-I+T)(I-T)^{-1}J \\ &= (I-P_nT)^{-1}(I-P_n)T(I-T)^{-1}J \\ &= (I-P_nT)^{-1}(I-P_n)TJ(J^{-1}(I-T)J)^{-1}\end{aligned} \tag{19}$$

By the definition of W^Q, (3),(12) and (19) we get

$$\begin{aligned}(\alpha_1\alpha_2)^{-1}E(n) &= (\alpha_1\alpha_2)^{-1}E_\gamma((I-P_n)TJ) \le E_\gamma(\Delta_n^Q(T)J) \\ &\le \beta_1\beta_2 E_\gamma((I-P_n)TJ) \le \beta_1\beta_2 E(n).\end{aligned} \tag{20}$$

With the operator norm in place of E_γ it follows analogously that

$$(\alpha_1\alpha_2)^{-1}L(n) \le \|\Delta_n^Q(T)J\| \le \beta_1\beta_2 L(n). \tag{21}$$

Now we apply Proposition 1.1 to get

$$\begin{aligned}\nu\{y \in X :&(1/2)E_\gamma(\Delta_n^Q(T)J) \le \|\delta_n^Q(T,y)\| \le (3/2)E_\gamma(\Delta_n^Q(T)J)\}\\ &\ge 1 - 2\exp(-E_\gamma(\Delta_n^Q(T)J)^2/(8\|\delta_n^Q(T)J\|^2)).\end{aligned} \tag{22}$$

Inserting (20)-(22) arrives at the desired result.

For the main results we need the following simple consequence on the global, i.e., $\mu \times \nu$ probability.

Corollary 3.1. *Let α_1, α_2, $\beta_1, \beta_2 > 0$, $n_0 \in N$. Then*

$$\begin{aligned}\mu \times \nu\{(T,y) :&(2\alpha_1\alpha_2)^{-1}E(n) \le \delta_n^Q(T,y) \le (3/2)\beta_1\beta_2 E(n), \quad \text{for all} \quad n > n_0\}\\ &\ge \mu(W^Q(\alpha_1,\alpha_2,\beta_1,\beta_2,n_0))2\sum_{n>n_0}\exp(-E(n)^2/(8(\alpha_1\alpha_2\beta_1\beta_2 L(n))^2)).\end{aligned} \tag{23}$$

4. Principal Results

Let $\Gamma = \{e^{it} : 0 \le t \le 2\pi\}$ denote the unit circle, and $\{e_n\}_{n=-\infty}^{\infty}$ be normalized in $L_2(\Gamma)$ trigonometric basis, i.e.,

$$e_0(t) = (2\pi)^{-1/2}, \quad e_n(t) = (\pi)^{-1/2}\cos nt, \quad e_{-n}(t) = (\pi)^{-1/2}\sin nt, \quad n \in N. \tag{24}$$

$n \in N$, and let $P_n, n \in N$, be the orthogonal projection onto span$\{e_j : |j| \le n\}$. With the choice of X and $P_n, n \in N$ the error functions δ_n^Q is defined well. Let $L_2(\Gamma^2) = L_2(\Gamma^2, \lambda^2)$. For $K \in L_2(\Gamma^2)$ let $T_k \in K(L_2(\Gamma))$ be the integral operator with kernel k defined by

$$(T_k x)(u) = \int_\Gamma k(u,v)x(v)dv. \tag{25}$$

The error analysis will be carried out for such operator only, so it is convenient to write $\delta_n^Q(k,y)$ instead of $\delta_n^Q(T_k,y)$.

We define the periodic Sobolev space $H^s(\Gamma)$ for any real $s \ge 0$ as

$$H^s(\Gamma) = \Big\{f \in L_2(\Gamma) : \|f\|_{H^s}^2(\Gamma) = \sum_{j=-\infty}^{\infty}(1+j^2)^s(f,e_j)^2 < \infty\Big\} \tag{26}$$

where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Gamma)$.

Lemma 4.1. [Kress (1989), Theorem 8.2] *The Sobolev space $H^r(\Gamma)$ is a Hilbert space with the scalar product defined by*

$$(f,g)_{H^s(\Gamma)} := \sum_{j=-\infty}^{\infty}(1+j^2)^s(f,e_j)(g,e_j). \tag{27}$$

Note that the norm on $H^s(\Gamma)$ is given by

$$\|f\|_{H^s(\Gamma)} = \Big\{ \sum_{j=-\infty}^{\infty} (1+j^2)^s (f, e_j)^2 \Big\}^{1/2}. \tag{28}$$

The trigonometric polynomials are dense in $H^s(\Gamma)$.

It is clear that $H^s(\Gamma)$ is isomorphism to the space which consists of the sequence $\{x_j\}_{j=-\infty}^{\infty}$ satisfying

$$\sum_{j=-\infty}^{\infty} (1+j^2)^s)|x_j|^2 < \infty \tag{29}$$

The functions

$$e_{mn}(s,t) = e_m(s)e_n(t), \quad m,n \in Z, \quad s,t \in \Gamma \tag{30}$$

form an orthogonal basis of $L_2(\Gamma^2)$. We define the periodic Sobolev space $H^r(\Gamma^2)$ for any real $r \geq 0$ as

$$H^r(\Gamma^2) = \{g \in L_2(\Gamma^2) : \|g\|^2_{H^r}(\Gamma^2) = \sum_{m,n\in Z} (1+m^2+n^2)^r (g, e_{mn})^2 < \infty\} \tag{31}$$

By Φ_r we denote the identical embedding $H^r(\Gamma^2) \to L_2(\Gamma^2)$. We assume $r > 1$ and $s > 1/2$. Then we have, by (5), $\Phi_r \in R_\gamma(H^r(\Gamma^2), L_2(\Gamma^2))$ and $J_s \in R_\gamma(H^s(\Gamma), L_2(\Gamma))$. Hence, we can define the Gaussian measures μ_r on $L_2(\Gamma^2)$, by

$$\mu_r = \tilde{\Phi}_{r\gamma_{H^r(\Gamma^2)}}, \tag{32}$$

and ν_s on $L_2(\Gamma)$, by

$$\nu_s = \tilde{J}_{s\gamma_{H^s(\Gamma)}}. \tag{33}$$

These measures are of Wiener type in the following sense. As the classical Wiener measures they are generated by the identical embedding of a Hilbert space of smooth functions into some function space (see [Kuo (1975)]). Concequently, they represent a certain degree of smoothness. To make this precise, let $\sigma \geq 0$ and let us consider $H^\sigma(\Gamma)$ as a subset of $L_2(\Gamma)$. Clearly, this is a Borel set, so $\nu_s(H^\sigma(\Gamma)$ is defined. Then the following holds:

$$\nu_s(H^\sigma(\Gamma)) = \begin{cases} 1 & \text{for} \quad \sigma < s - 1/2 \\ 0 & \text{for} \quad \sigma \geq s - 1/2. \end{cases} \tag{34}$$

Roughly speaking, (34) means that ν_s corresponds to the smoothness $H^{s-1/2}$. Similarly,

$$\mu_r(H^\rho(\Gamma)) = \begin{cases} 1 & \text{for} \quad \rho < s - 1/2 \\ 0 & \text{for} \quad \rho \geq s - 1/2. \end{cases} \tag{35}$$

Now we can formulate the principal results. First we provide estimates for an individual, fixed operator T_K and the probability on the set of free terms only.

Theorem 4.1. *$\rho \geq s > 1/2$, $k \in H^\rho(\Gamma^2)$, and assume that $I - T_k$ is invertible. Then there exist constants $c_i(k) > 0$, $i = 1,2,3$, and $n_0(k) \in N$ such that, for each $n > n_0(k)$,*

$$\nu_s\{y \in L_2(\Gamma) : c_1(k)n^{-\rho+1/2} \leq \delta_n^Q(k,y) \leq c_2(k)n^{-\rho+1/2}\} \geq 1 - \exp(-c_3(k)n). \tag{36}$$

Remark 4.1. *It is wll-known that if the right-hand sides y of the Fredholm integral equation $x - T_K x = y$ is in $H^{s-1/2}(\Gamma)$, then the worst-case rate of Q-Method is $n^{s-1/2}$. Theorem* 4.1 *shows that this rate occurs for most of the right-hand sides. Moreover, the exceptional set is of exponentially small probability.*

Theorem 4.2. *Let $r - 1/2 > s > 1/2$. For each $\varepsilon > 0$ there exist constants $c_i(\varepsilon) > 0$, $i = 1, 2, 3$, and $n_1(\varepsilon) \in N$ such that*

$$\mu_r \times \nu_s\{(k, y) : c_1(\varepsilon)n^{-r+1/2} \leq \delta_n^Q(k, y) \leq c_2(\varepsilon)n^{-r+1/2} \text{ for all } \quad n > n_1(\varepsilon)\} \geq 1 - \varepsilon. \tag{37}$$

Remark 4.2. *Theorem* 4.2 *gives global estimate, i.e., independent of kernels k, with probabilities on the set of kernels and right-hand sides. More precisely, Theorem* 4.2 *shows that the worst-case rate of Q-Method occurs with large probability, in fact almost surely.*

The proof of Theorem 4.2 also provides estimates for the dependence on ε for $\varepsilon \to 0$, namely, the functions of ε occurring there can be chosen in such way that the following hold (here $\asymp$ stands for $\asymp_{\varepsilon\in(0,1/2)}$):

$$c_1(\varepsilon) \asymp (\log(1/\varepsilon))^{-1}, \tag{38}$$

$$c_2(\varepsilon) \asymp (\log(1/\varepsilon))^{3/2+3/(2r)}\varepsilon^{-2}, \tag{39}$$

$$n_1(\varepsilon) \asymp (\log(1/\varepsilon))^{6+3/(r)}\varepsilon^{-6}. \tag{40}$$

We will prove Theorem 4.1 and 4.2 in the following two sections. This will be accomplished by estimating the needed approximation quantities and applying the results of Section 3. For this purpose we have to establish a correspondence to the notation of Section 3. We have already fixed $X = L_2(\Gamma)$ and P_n. Now we put $H = H^s(\Gamma)$, $J = J_s$ and get

$$\nu = \tilde{J}_{\gamma_H} = \nu_s. \tag{41}$$

Our main results are formulated in terms of μ_r, which is a measure on the set of kernels $L_2(\Gamma^2)$. In order to use Section 3, we need a measure μ on the set of compact operators. For this let $\Lambda \in L(L_2(\Gamma^2), K(L_2(\Gamma))$ be the mapping assigning to each $K \in L_2(\Gamma^2)$, the integral operator T_K defined by (25). Then μ will be the measure induced on $K(L_2(\Gamma))$ by μ_r under the action of Λ. This means

$$\mu(B) = \mu_r(\Lambda^{-1}(B)) \tag{42}$$

for every Borel subset B of $K(L_2(\Gamma))$. Now we put $G = H^r(\Gamma^2)$, $\Phi = \Lambda\Phi_r$, and it follows readily that $\mu = \tilde{\Phi}_{\gamma_G}$. With this, $E(n), L(n), E_1(n)$, and $L_1(n)$ are defined well. The following section is devoted to them.

5. Approximate Rates

This section is devoted to determine the order of those quantities which are related to the approximation process. We start with two general results from which the concrete estimates will follow. Let $\sigma, \tau \geq 0$ be reals and the operator

$$\Psi_{\sigma,\tau} : \text{Dom } \Psi_{\sigma,\tau} \to L(H^\sigma(\Gamma), H^\tau(\Gamma)) \tag{43}$$

as follows. The domain Dom $\Psi_{\sigma,\tau}$ is the set of all $T \in K(L_2(\Gamma))$ such that Im $(TJ_\sigma) \subset H^r(\Gamma)$. Now we set, for $T \in$ Dom $\Psi_{\sigma,\tau}$

$$\Psi_{\sigma,\tau}T = J_\tau^{-1}TJ_\sigma. \tag{44}$$

Propsition 5.1. *Let $\sigma, \tau \geq 0$, $r > \tau + 1/2$. Then*
(i) $Im\Phi, Im(\Pi_n^Q\Phi) \subset Dom\Psi_{\sigma,\tau}$,
(ii) $\Psi_{\sigma,\tau}\Phi, \Psi_{\sigma,\tau}\Pi_n^Q\Phi \in R_\gamma(H^r(\Gamma^2), L(H^\sigma(\Gamma), H^\tau(\Gamma))$,
(iii) $\|\Psi_{\sigma,\tau}\Pi_n^Q\Phi\| \asymp_n n^{-r-\sigma+\tau}$,
(iv) $E_\gamma(\Psi_{\sigma,\tau}\Pi_n^Q\Phi) \asymp_n n^{-r-\sigma+\tau+1/2}$.
Proof. This follows from [Heinrich (1991), Proposition 4.2].

Now we define, for $\sigma, \tau \geq 0$, a further operator $\Phi_{\sigma,\tau}^E$: Dom$\Phi_{\sigma,\tau}^E \to R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma))$. We let Dom$\Phi_{\sigma,\tau}^E$ be the set of those $T \in K(L_2(\Gamma))$ for which Im $(TJ_\sigma) \subset H^\tau(\Gamma)$ and $J_\tau^{-1}TJ_\sigma \in R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma))$. The operator is defined for $T \in$ Dom $\Phi_{\sigma,\tau}^E$ by

$$\Phi_{\sigma,\tau}^E T = J_\tau^{-1}TJ_\sigma.$$

Hence,

$$\text{Dom } \Phi_{\sigma,\tau}^E \subset \text{ Dom } \Phi_{\sigma,\tau},$$

and, for $T \in$ Dom $\Phi_{\sigma,\tau}^E$, $\Phi_{\sigma,\tau}^E T$ and $\Phi_{\sigma,\tau}T$ are the same operators.

Propsition 5.2. Let $r > \tau + 1$. Then
(i) Im Φ, Im $(\Pi_n^Q\Phi) \subset$ Dom $\Phi_{\sigma,\tau}^E$,
(ii) $\Phi_{\sigma,\tau}^E$, $\Phi_{\sigma,\tau}^E\Pi_n^Q \in R_\gamma(H^r(\Gamma^2), R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma)))$,
(iii) $\|\Phi_{\sigma,\tau}^E\Pi_n^Q\Phi\| \asymp_n n^{-r-\sigma+\tau}$,
(iv) $E_\gamma(\Phi_{\sigma,\tau}^E\Pi_n^Q\Phi) \asymp_n n^{-r-\sigma+\tau+1}$.
Proof. This follows from [Heinrich (1991), proposition 4.3].

Now we can easily derive the desired estimates for our concrete situation.

Corollary 5.1. *Let $r - 1/2 > s > 1/2$. Then*

$$E(n) \asymp n^{-r+1/2}, \quad L(n) \asymp n^{-r},$$
$$E_1(n) \asymp n^{-r+1/2}, \quad L_1(n) \asymp n^{-r},$$

We have to separate another immediate consequence of Proposition 5.1 (ii), which will be needed for the application of Lemma 3.2.

Corollary 5.2. *Im $\Phi \subset$ Dom $\Psi_{0,s}$ and*

$$\Psi_{0,s}\Phi \in R_\gamma(H^r(\gamma^2), L(L_2(\Gamma), H^s(\Gamma))).$$

6. Proofs of the Principal Results

Proof of Theorem 4.1. Let $k \in H^\rho(\Gamma^2)$ and $\rho \geq s \geq 1/2$, then

$$T_k(L_2(\Gamma)) \subset H^\rho(\Gamma) \subset H^s(\Gamma). \tag{45}$$

This implies

$$\|T_k - P_nT_k\| \prec n^{-\rho}. \tag{46}$$

Since $\rho > 1/2 > 0$, we have $n^{-\rho} \to 0$ as $n \to \infty$. Hence there exists an $n = n_0(k)$ such that, for $n > n_0$,

$$\|T_k - P_n T_k\| \leq 1/(2\|I - T_k)^{-1}\|). \tag{47}$$

With these relations it is readily cheeked that

$$T_k \in W^Q(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$$

for certain constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ depending on k.

It remains to prove the inequality (36). It is easily followed from Proposition 3.1 and Corollary 5.1. This completes the proof of Theorem 4.1.

Now we come to prove Theorem 4.2. For this purpose, we need an elementary technical lemma which is from Heinrich (1991).

Lemma 6.1. *Let $a > 0, b \geq 1, \varepsilon > 0$ be reals, $n_0 \in N$. If*

$$n_0 \geq ((\log a + \log(1/\varepsilon))a)^{1/b},$$

then

$$\sum_{n > n_0} \exp(-n^b/a) \leq \varepsilon.$$

Proof of Theorem 4.2. We start with estimating the probability of the sets W^Q with the help of the Lemma 3.2. By Theorem 3.3 of Heinrich (1990*a*) and (42), there exists a function $\beta : (0, 1/2) \to (0, +\infty)$ such that

$$\mu(U(\beta(\varepsilon))) = \mu_r\{k \in L_2(\Gamma^2) : \|(I - T_k)^{-1}\| \leq \beta(\varepsilon))\} \geq 1 - \varepsilon/6 \tag{48}$$

and

$$\beta(\varepsilon) \asymp (\log(1/\varepsilon))^{1/2+3/(4r)}\varepsilon^{-1} \tag{49}$$

(in this section $\asymp$ and $\prec$ always refer to $\varepsilon \in 0, 1/2)$). Furthermore, it is clearly possible to choose a function $\alpha : (0, 1/2) \to (0, +\infty)$ such that

$$\exp(-\alpha(\varepsilon)^2/(2\|\Psi_{0,s}\Phi\|^2)) + \exp(-\alpha(\varepsilon)^2/(2\|\Phi\|^2)) \leq \varepsilon/6 \tag{50}$$

and

$$\alpha(\varepsilon) \asymp \log(1/\varepsilon))^{1/2}. \tag{51}$$

Corollary 5.2 says that the assumptions of Lemma 3.2 are satisfied. Then let $\alpha_1(\varepsilon)$, $\alpha_2(\varepsilon)$, $\beta_1(\varepsilon)$, $\beta_2(\varepsilon)$ be as defined in Lemma 3.2 when we replace α and β by $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. It is clear that

$$\alpha_1(\varepsilon) \asymp \alpha_2(\varepsilon) \asymp \log(1/\varepsilon))^{1/2}. \tag{52}$$

$$\beta_1(\varepsilon) \asymp \log(1/\varepsilon))^{1+3/(4r)}\varepsilon^{-1}. \tag{53}$$

$$\beta_2(\varepsilon) \asymp \log(1/\varepsilon))^{1/2+3/(4r)}\varepsilon^{-1}. \tag{55}$$

By Corollary 5.1 there exists a constant c_1 such that, for all n,

$$E_1(n) \leq c_1 n^{-r+1/2}.$$

Hence, if $n > N_1(\varepsilon) = [(4c_1\beta(\varepsilon))^{1/(r-1/2)}]$, then

$$E_1(n) \leq 1/(4\beta(\varepsilon)), \tag{56}$$

where $[a]$ stands for the smallest integer $m \geq a$. By (48),

$$N_1(\varepsilon) \asymp \log(1/\varepsilon))^{(2r+3)/(4r^2-2r)}\varepsilon^{-1/(r-1/2)}. \tag{57}$$

Corollary 5.1 gives that there is a constant $c_2 > 0$ such that

$$L_1(n) \leq c_2 n^{-r}.$$

Let $N_2(\varepsilon)$ be the smallest $n_0 \in N$ such that

$$\sum_{n>n_0} \exp(-n^{2r}/(32c_2^2\beta(\varepsilon)^2)) \leq \varepsilon/6. \tag{58}$$

It follows that

$$\sum_{n>N_2(\varepsilon)} \exp(-1/(32\beta(\varepsilon)^2)L_1(n))^2)) \leq \varepsilon/6. \tag{59}$$

From (57) and Lemma 6.1 with $a = 32c_2^2\beta(\varepsilon)^2$ and $b = 2r$ we get

$$N_2(\varepsilon) \leq (\log(32c_2^2\beta(\varepsilon)^2) + (\log(1/\varepsilon)32c_2^2\beta(\varepsilon)^2))^{1/(2r)} + 1,$$

hence, by (49),

$$N_2(\varepsilon) \prec (\log(1/\varepsilon))^{(4r+3)/(4r^2)}\varepsilon^{-1/r}. \tag{60}$$

Lemma 3.2, together with (48), (50), (55),and (58), gives

$$\mu(W^Q(\alpha_1(\varepsilon), \alpha_2(\varepsilon), \beta_1(\varepsilon), \beta_2(\varepsilon), n_0)) \geq 1 - \varepsilon/2 \tag{61}$$

for all $n_0 \geq \max(N_1(\varepsilon),\, N_1(\varepsilon))$. By Corollary 5.1 there is a constant $c_3 > 0$ such that

$$E(n)/L(n) \geq c_3 n^{1/2}.$$

Let $N_3(\varepsilon)$ be the smallest $n_0 \in N$ such that

$$\sum_{n>n_0} \exp(-c_3^2 n/(8\gamma(\varepsilon))) \leq \varepsilon/4, \tag{61}$$

where $\gamma(\varepsilon) = (\alpha_1(\varepsilon)\alpha_2(\varepsilon)\beta_1(\varepsilon)\beta_2(\varepsilon))^2$. Thus

$$\gamma(\varepsilon) \asymp (\log(1/\varepsilon))^{(5+3/r)}\varepsilon^{-4}. \tag{62}$$

We have

$$\sum_{n>N_3(\varepsilon)} \exp(-E(n)^2/(8\gamma(\varepsilon)L(n))) \leq \varepsilon/4, \tag{63}$$

By Lemma 6.1, (62) and (63),

$$N_3(\varepsilon) \leq (\log(8\gamma(\varepsilon)/c_3^2) + (\log(4/\varepsilon)8\gamma(\varepsilon)/c_3^2 + 1 \prec \log(1/\varepsilon))^{6+3/r}\varepsilon^{-4}. \tag{64}$$

We put

$$n_1(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon)),$$

apply Corollary 3.1 and get, by (60) and (63),

$$\mu \times \nu\{(T,y) : (2\alpha_1\alpha_2)^{-1}E(n) \leq \delta_n^Q(T,y) \leq (3/2)\beta_1\beta_2 E(n), \text{for all} \quad n > n_0\} \geq 1 - \varepsilon \tag{65}$$

for all $\varepsilon \in (0, 1/2)$. Now, Theorem 4.2 follows from (40), (41) and Corollary 5.1, while the corresponding parts of (34), (38), (39) are a consequence of (52)–(54), (56), (59) and (64). This completes the proof of Theorem 4.2.

Acknowledgement: The author is highly grateful to the anonymous referees for their comments and suggestions, which improved the presentation of the paper.

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