Journal of Computational Mathematics, Vol.16, No.5, 1998, 417–436.

FOURIER-CHEBYSHEV PSEUDOSPECTRAL METHOD FOR THREE-DIMENSIONAL VORTICITY EQUATION

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Abstract

In this paper, a Fourier-Chebyshev pseudospectral scheme with mixed filtering is proposed for three-dimensional vorticity equation. The generalized stability and convergence are proved. The numerical results show the advantages of this method.

Key words: Pseudospectral method, vorticity equation, error estimates.

1. Introduction

In studying boundary layers, flows past suddenly heated vertical plates and other related problems, we have to consider bilaterally periodic problems. There are several ways to solve them numerically. For instance, Murdok[1], Macaraeg[2] and Ben-yu Guo, Yue-shan Xiong[3] proposed spectral-difference schemes, while Canuto, Maday, Quarteroni[4] and Guo Ben-yu, Cao Wei Ming[5] developed spectral-finite element schemes. But the accuracy of all these schemes is still limited due to finite difference and finite element approximations, even if the genuine solution is very smooth. Therefore some authors provided various mixed spectral approximations, such as Fourier-Chebyshev approximation[6,7].

In this paper, we consider three-dimensional unsteady vorticity equation which is one of representations of incompressible flow. It possesses more unknown variables than Navier-Stokes equation and leads to non-standard boundary conditions. But in computation, this representation avoids the difficult job of constructing trial function space whose elements satisfy the incompressible condition. Thus we still use it often. We shall follow the idea of [8] to propose a mixed method by using Fourier pseudospectral approximation in periodic directions and Chebyshev pseudospectral approximation in remaining direction. This method can be implemented simply. In particular, it is easy to deal with nonlinear terms. But the pseudospectral approximation is not as stable as spectral one usually, due to the aliasing. Thus two kinds of filtering technique have been developed. The first was based on Bochner summation by Kuo Pen-yu[9,10]. The second was given by Woodward, Collela and Vandeven[11,12]. Recently, Guo Ben-yu

* Received June 19, 1995.
1) This work is supported in part by the National Nature Science Foundation of China.

improved the first one and generalized it to Chebyshev approximation[13]. The authors also developed a new mixed filtering technique for mixed approximation[8]. In this paper, we also adopt this technique and so the proposed scheme keeps the spectral accuracy, i.e., the convergence rate of infinite order.

The outline of this paper is as follows. We construct the scheme in Section 2 and present the numerical results in Section 3. The advantages of this method and the efficiency of the new mixed filtering technique are shown numerically. In Section 4, we give the main theoretical results. We list some lemmas in Section 5 and then prove the theorems in Section 6.

2. The Scheme

Let $x = (x_1, x_2, x_3)^T$ and $\Omega = I \times Q$ where $I = \{x_1 / -1 < x_1 < -1\}$, $Q = \{(x_2, x_3)/ -\pi < x_2, x_3 < \pi\}$. Let $\xi(x,t)$ and $\psi(x,t)$ be the vorticity vector and stream vector respectively with the components $\xi^{(q)}(x,t)$ and $\psi^{(q)}(x,t)$, $q = 1,2,3. \nu > 0$ is the kinetic viscosity. f_1, f_2 and ξ_0 are given functions. We consider the following problem

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + J(\xi,\psi) - H(\xi,\psi) - \nu \nabla^2 \xi = f_1, & \text{in } \Omega \times (0,T], \\ -\nabla^2 \psi = \xi + f_2, & \text{in } \Omega \times (0,T], \\ \xi(x,0) = \xi_0(x), & \text{in } \Omega \bigcup \partial\Omega, \end{cases} \tag{2.1}$$

where

$$J(\xi,\psi) = [(\nabla \times \psi) \cdot \nabla]\xi, \quad H(\xi,\psi) = (\xi \cdot \nabla)(\nabla \times \psi).$$

Assume that all functions in (2.1) have the period 2π for the variables x_2 and x_3. For simplicity of the analysis, we also suppose that ξ and ψ satisfy the following boundary-value conditions as in [14],

$$\xi(\pm 1, x_2, x_3, t) = \psi(\pm 1, x_2, x_3, t) = 0. \tag{2.2}$$

The existence and uniqueness of local solution can be studied in the same way as in [14].

The inner products and norms of vector function spaces $L^2(I)$ and $L^2(Q)$ are denoted by $(\cdot,\cdot)_I, (\cdot,\cdot)_Q, \|\cdot\|_I$ and $\|\cdot\|_Q$ respectively. Let $\omega(x_1) = (1 - x_1^2)^{-\frac{1}{2}}$ and define

$$(u,v)_{\omega,I} = \int_{-1}^{1} \omega u v dx_1, \quad \|v\|_{\omega,I} = (v,v)_{\omega,I}^{\frac{1}{2}},$$
$$L^2_\omega(I) = \{v/v \text{ is measurable on } I \text{ and } \|v\|_{\omega,I} < \infty\}.$$

Also define

$$(u,v)_\omega = \frac{1}{4\pi^2} \int_\Omega \omega u v dx, \quad \|v\|_\omega = (v,v)_\omega^{\frac{1}{2}}$$
$$L^2_\omega(I) = \{v/v \text{ is measurable on } \Omega \text{ and } \|v\|_\omega < \infty\}.$$

Let M and N be positive integers. Suppose that there exist positive constants c_1 and c_2 such that

$$c_1 N \le M \le c_2 N. \tag{2.3}$$

Let $\mathcal{P}_M$ be the set of all algebraic polynomials of degree equal or less than M, and define

$$V_M^{(1)} = \{v(x_1) \in \mathcal{P}_M / v(-1) = v(1) = 0\}.$$

Let $y = (x_2, x_3)^T$ and $l = (l_2, l_3), l_q$ being integers. Moreover $|l| = (l_2^2 + l_3^2)^{\frac{1}{2}}$, $ly = l_2 x_2 + l_3 x_3$, and

$$\tilde{V}_N^{(2)} = \text{Span}\ \{e^{ily} / |l_q| \le N, q = 2, 3\}, \qquad \tilde{W}_N^{(2)} = \text{Span}\ \{e^{ily} / |l| \le N\}.$$

Let $V_N^{(2)}$ (or $W_N^{(2)}$) be the subset of $\tilde{V}_N^{(2)}$ (or $\tilde{W}_N^{(2)}$), containing all real-valued functions. Set $S_{M,N} = (V_M^{(1)} \times W_N^{(2)})^3$.

Let $P_M^{(1)} : L^2_\omega(I) \to [V_M^{(1)}]^3$ be the orthogonal projection such that for any $u \in L^2_\omega(I)$,

$$(P_M^{(1)} u - u, v)_{\omega, I} = 0, \quad \forall v \in [V_M^{(1)}]^3,$$

while $P_N^{(2)} : L^2(Q) \to [W_N^{(2)}]^3$ is the orthogonal projection such that for any $u \in L^2(Q)$,

$$(P_N^{(2)} u - u, v)_Q = 0, \quad \forall v \in [W_M^{(2)}]^3.$$

Let $P_{M,N} = P_M^{(1)} \otimes P_N^{(2)}$. Obviously, for any $u \in L^2_\omega(\Omega)$,

$$(P_{M,N} u - u, v)_\omega = 0, \quad \forall v \in S_{M,N}.$$

We denote the nodes and weights of Gauss-Lobatto integration formula by $x_1^{(j)}$ and ω_j, namely

$$x_1^{(j)} = \cos \frac{j\pi}{M}, \quad 0 \le j \le M; \quad \omega_0 = \omega_M = \frac{\pi}{2M}; \quad \omega_j = \frac{\pi}{M}, \text{ for } 1 \le j \le M-1.$$

Let $h = \dfrac{2\pi}{2N+1}$ be the mesh size for the variables x_2 and x_3. Set

$$\Omega_{M,N} = \{(x_1^{(j)}, q_2 h, q_3 h) / 1 \le j \le M-1, -N \le q_2, q_3 \le N\},$$
$$\overline{\Omega}_{M,N} = \{(x_1^{(j)}, q_2 h, q_3 h) / 0 \le j \le M, -N \le q_2, q_3 \le N\}.$$

We also introduce the following discrete inner products and norms

$$\langle u, v \rangle_{M,\omega} = \sum_{j=0}^{M} \omega_j u(x_1^{(j)}) v(x_1^{(j)}),$$
$$\langle u, v \rangle_N = \frac{1}{(2N+1)^2} \sum_{q_2, q_3 = -N}^{N} u(q_2 h, q_3 h) \overline{v}(q_2 h, q_3 h),$$

$$(u,v)_{M,N,\omega} = \frac{1}{(2N+1)^2} \sum_{j=0}^{M} \sum_{q_2,q_3=-N}^{N} \omega_j u(x_1^{(j)}, q_2 h, q_3 h) \overline{v}(x_1^{(j)}, q_2 h, q_3 h),$$

$$\|u\|_{M,N,\omega} = (u,v)_{M,N,\omega}^{\frac{1}{2}}.$$

Let $P_C^{(1)}$ be the interpolation from $C(\overline{I})$ to $[\mathcal{P}_M]^3$, and $P_C^{(2)}$ be the interpolation from $C(\overline{Q})$ to $[V_N^{(2)}]^3$ such that

$$P_C^{(1)} u(x_1^{(j)}) = u(x_1^{(j)}), \quad P_C^{(2)} u(q_2 h, q_3 h) = u(q_2 h, q_3 h), \quad 0 \le j \le M, -N \le q_2, q_3 \le N.$$

Furthermore, let $P_C = P_N^{(2)} \otimes P_C^{(2)} \otimes P_C^{(1)}$.

In order to weaken the nonlinear instability in computation and raise the accuracy of numerical solutions, we shall use the mixed filtering technique as in [8]. Let $\gamma_1(M) \ge 1, \gamma_2(N) \ge 1$ and $R = R(M, N, \gamma_1, \gamma_2)$ be the filtering operator. It means that if

$$u = \sum_{(j,l)\in R_{M,N}} a_{j,l} T_j(x_1) e^{ily} + \sum_{(j,l)\notin R_{M,N}} a_{j,l} T_j(x_1) e^{ily}, \ R_{M,N} = \{(j,l)/0 \le j \le M, |l| \le N\},$$

$T_j(x_1)$ being the Chebyshev polynomials of order j, then

$$Ru = \sum_{(j,l)\in R_{M,N}} \Big(1 - \Big|\frac{j}{M}\Big|^{\gamma_1}\Big)\Big(1 - \Big|\frac{l}{N}\Big|^{\gamma_2}\Big) a_{j,l} T_j(x_1) e^{ily} + \sum_{(j,l)\notin R_{M,N}} a_{j,l} T_j(x_1) e^{ily}.$$

Let τ be the step size of time t, and

$$\dot{S}_\tau = \Big\{t/t = k\tau, 1 \le k \le \Big[\frac{T}{\tau}\Big]\Big\}, \quad S_\tau = \dot{S}_\tau \bigcup \{0\}.$$

For simplicity, $u(x,t)$ is denoted by $u(t)$ or u usually. Let

$$u_{\hat{t}}(t) = \frac{1}{2\tau}(u(t+\tau) - u(t-\tau)), \quad \hat{u}(t) = \frac{1}{2}(u(t+\tau) + u(t-\tau)).$$

Now, let η and φ be the approximations to ξ and ψ respectively. Let $\partial_j = \dfrac{\partial}{\partial x_j}$ $(j = 1, 2, 3)$ and define

$$J_{RC}(\eta,\varphi) = \sum_{j=1}^{3} \partial_j R P_C((\nabla \times \varphi)^{(j)} \eta), \quad H_{RC}(\eta,\varphi) = \sum_{j=1}^{3} R P_C(\eta^{(j)} \partial_j (\nabla \times \varphi)).$$

The nonlinear terms $J(\xi,\psi)$ and $H(\xi,\psi)$ are approximated by $J_{RC}(\eta,\varphi)$ and $H_{RC}(\eta,\varphi)$. The Fourier-Chebyshev pseudospectral scheme for solving (2.1) is to find $(\eta,\varphi) \in S_{M,N} \times S_{M,N}$ for all $t \in S_\tau$, such that

$$\begin{cases} \eta_{\hat{t}} + J_{RC}(\eta,\varphi) - H_{RC}(\eta,\varphi) - \nu \nabla^2 \hat{\eta} = P_C f_1, & \text{in} \quad \Omega_{M,N} \times \dot{S}_\tau, \\ -\nabla^2 \varphi = \eta + P_C f_2, & \text{in} \quad \Omega_{M,N} \times S_\tau, \\ \eta(\tau) = P_{M,N}\Big(\xi_0 + \tau \dfrac{\partial}{\partial t}\xi(0)\Big), & \text{in} \quad \Omega_{M,N}, \\ \eta(0) = P_{M,N}\xi_0, & \text{in} \quad \Omega_{M,N} \end{cases} \tag{2.4}$$

where

$$\frac{\partial \xi}{\partial t}(0) = -J(\xi_0, \psi_0) + H(\xi_0, \psi_0) + \nu \nabla^2 \xi_0 + f_1(0),$$
$$-\nabla^2 \psi_0 = \xi_0 + f_2(0).$$

3. The Numerical Results

We take the following test functions,

$$\xi^{(1)} = 0.4e^{At}(x_1^2 - 1)(2x_1^2 - 13)\sin 2x_2 \cos 2x_3,$$
$$\xi^{(2)} = 0.4e^{At}(x_1^2 - 1)(2x_1^2 - 13)\cos 2x_2 \sin 2x_3,$$
$$\xi^{(3)} = 0.4e^{At}(x_1^2 - 1)(2x_1^2 - 13)\cos 2x_2 \cos 2x_3 - 1.2 \times 10^{-4} e^{At}(x_1^2 - 1).$$

$E(\xi(t))$ denotes the relative error of $\xi(t)$.

We use scheme (2.4) to solve (2.1). For comparison, we also consider the Fourier pseudospectral-finite element scheme (FPSFE), by using linear finite element approximation in the direction x_1, in which I is uniformly partitioned with the mesh size $h = \frac{2}{M^*}$. We take $A = 0.1, M = M^* = N = 4$ and $\tau = 0.005$. Scheme (2.4) costs the same computational time as FPSFE scheme. But scheme (2.4) gives much better results, see Table I and Table II. In Table III, we list the numerical results of scheme (2.4) with different choices of γ_1 and γ_2. Obviously the new mixed filtering operator $R(M, N, \gamma_1, \gamma_2)$ improves the stability and raises the accuracy.

Table I. $\nu = 0.01, \gamma_1 = \gamma_2 = 1.$

$E(\xi(t))$	Scheme (2.4)	FPSFE
t=0.5	0.6241E-4	0.1599E-1
t=1.0	0.1205E-3	0.3067E-1
t=1.5	0.1737E-3	0.4415E-1
t=2.0	0.2288E-3	0.5653E-1
t=2.5	0.2845E-3	0.6789E-1

Table II. $\nu = 0.001, \gamma_1 = \gamma_2 = 1.$

$E(\xi(t))$	Scheme (2.4)	FPSFE
t=0.5	0.4689E-4	0.1656E-2
t=1.0	0.8479E-4	0.3222E-2
t=1.5	0.1246E-3	0.4706E-2
t=2.0	0.1779E-3	0.6113E-2
t=2.5	0.2382E-3	0.7450E-2

Table III. The errors of scheme (2.4),$\nu = 0.001$.

$E(\xi(t))$	$\gamma_1 = \gamma_2 = \infty$	$\gamma_1 = 5, \gamma_2 = 3$	$\gamma_1 = \gamma_2 = 1$
t=0.5	0.6196E-4	0.6127E-4	0.4689E-4
t=1.0	0.1537E-3	0.1294E-3	0.8479E-4
t=1.5	0.4037E-3	0.2443E-3	0.1246E-3
t=2.0	0.1133E-2	0.5538E-3	0.1779E-3
t=2.5	0.2828E-2	0.1265E-2	0.2382E-3

4. The Theoretical Results

In order to estimate errors, we need some notations. Let $L^\infty(I), L^\infty(\Omega), W^{q,\infty}(\Omega)$, $\|\cdot\|_{\infty,I}$, $\|\cdot\|_\infty$, and $\|\cdot\|_{q,\infty}$ be the usual spaces and their norms, etc.. We also introduce some Sobolev spaces of functions defined on I, with the weight $\omega(x_1)$. For any integer $r \geq 0$, set

$$|v|_{r,\omega,I} = \Big\|\frac{d^r v}{dx_1^r}\Big\|_{\omega,I}, \quad \|v\|_{r,\omega,I} = \Big(\sum_{k=0}^{r} |v|_{k,\omega,I}^2\Big)^{\frac{1}{2}},$$

$$H^r_\omega(I) = \{v/\|v\|_{r,\omega,I} < \infty\}.$$

Clearly $H^0_\omega(I) = L^2_\omega(I)$ and $\|v\|_{0,\omega,I} = \|v\|_{\omega,I}$. For any real $r > 0$, $H^r_\omega(I)$ is defined by the complex interpolation between the spaces $H^{[r]}_\omega(I)$ and $H^{[r+1]}_\omega(I)$. Furthermore, $H^r_{0,\omega}(I)$ denotes the closure of $C^\infty_0(I)$ in $H^r_\omega(I)$.

Let B be a Banach space with the norm $\|\cdot\|_B$, and Λ be a domain in R^2. Define

$$\begin{aligned} L^2(\Lambda, B) &= \{v(z) : \Lambda \to B/v \text{ is strongly measurable}, \|v\|_{L^2(\Lambda,B)} < \infty\},\\ C(\Lambda, B) &= \{v(z) : \Lambda \to B/v \text{ is strongly measurable}, \|v\|_{C(\Lambda,B)} < \infty\}\end{aligned}$$

where

$$\|v\|_{L^2(\Lambda,B)} = \Big(\int_\Lambda \|v(z)\|^2_B dz\Big)^{\frac{1}{2}}, \quad \|v\|_{C(\Lambda,B)} = \max_{z\in\Lambda} \|v(z)\|_B.$$

Moreover, for all integer $\mu \ge 0$,

$$H^\mu(\Lambda, B) = \{v(z) \in L^2(\Lambda, B)/\|v\|_{H^\mu(\Lambda,B)} < \infty\}$$

equipped with the norm

$$\|v\|_{H^\mu(\Lambda,B)} = \Big(\sum_{k=0}^{\mu} \Big\|\frac{\partial^k v}{\partial z^k}\Big\|^2_B\Big)^{\frac{1}{2}}.$$

We can define $H^\mu(\Lambda, B)$ for real number $\mu > 0$ in the same way as before.

For simplifying the statements, we also introduce some non-isotropic spaces. Let

$$H^{r,s}_\omega(\Omega) = L^2(Q, H^r_\omega(I)) \bigcap H^s(Q, L^2_\omega(I)), \quad r, s \ge 0$$

equipped with the norm

$$\|v\|_{H^{r,s}_\omega(\Omega)} = (\|v\|^2_{L^2(Q,H^r_\omega(I))} + \|v\|^2_{H^s(Q,L^2_\omega(I))})^{\frac{1}{2}}.$$

Also define

$$\begin{aligned} M^{r,s}_\omega(\Omega) =& H^{r,s}_\omega(\Omega) \bigcap H^1(Q, H^{r-1}_\omega(I)) \bigcap H^{s-1}(Q, H^1_\omega(I)), \quad r, s \ge 1,\\ X^{r,s}_\omega(\Omega) =& H^s(Q, H^{r+1}(I)) \bigcap H^{s+1}(Q, H^r(I)), \quad r, s \ge 0,\\ Y^{r,s,\delta}_{1,\omega}(\Omega) =& M^{r+2,s+2}_\omega(\Omega) \bigcap L^2(Q, H^{r+3}_\omega(I)) \bigcap H^{1+\delta}(Q, H^{\frac{r}{2}+\frac{1}{2}+\delta}_\omega(I))\\ & \bigcap H^{\frac{3}{2}+\frac{3}{2}\delta}(Q, H^{r-\frac{1}{4}+\frac{\delta}{4}}_\omega(I)) \bigcap H^2(Q, H^{r+1}_\omega(I)) \bigcap H^{2+2\delta}(Q, H^{\frac{r}{2}+\frac{1}{8}+\frac{5}{8}\delta}_\omega(I))\\ & \bigcap H^{\frac{s}{2}+1+\delta}(Q, H^{\frac{1}{2}+\delta}_\omega(I)), \quad r, s \ge 0, \delta > 0,\\ Y^{r,s,\delta}_{2,\omega}(\Omega) =& M^{r+2,s+2}(\Omega) \bigcap L^2(Q, H^{r+3}_\omega(I)) \bigcap H^{1+\delta}(Q, H^{\frac{r}{2}+\frac{3}{2}+\delta}_\omega(I))\\ & \bigcap H^{\frac{3}{2}+\frac{3}{2}\delta}(Q, H^{r+\frac{3}{4}+\frac{1}{4}\delta}_\omega(I)) \bigcap H^2(Q, H^{r+1}_\omega(I)) \bigcap H^{2+2\delta}(Q, H^{\frac{r}{2}+\frac{9}{8}+\delta}_\omega(I))\\ & \bigcap H^{\frac{5}{2}+\frac{3}{2}\delta}(Q, H^{r-\frac{1}{4}+\frac{\delta}{4}}_\omega(I)) \bigcap H^3(Q, H^{r-1}_\omega(I)) \bigcap H^{3+2\delta}(Q, H^{\frac{r}{2}+\frac{1}{8}+\frac{5}{8}\delta}_\omega(I))\end{aligned}$$

$$\bigcap H^{\frac{s}{2}+1+\delta}(Q,H_\omega^{\frac{3}{2}+\delta}(I))\bigcap H^{\frac{s}{2}+2+\delta}(Q,H_\omega^{\frac{1}{2}+\delta}(I))\bigcap H^{s-1}(Q,H_\omega^2(I)),$$
$$r,s\geq 0,\delta>0,$$

and for non-negative integer k,

$$X_{0,\omega}^{r,s}(\Omega)=H^s(Q,H_\omega^r(I)),\quad X_{k,\omega}^{r,s}(\Omega)=X_{k-1,\omega}^{r+2,s}(\Omega)\bigcap H^{r+k}(Q,H_\omega^s(I)).$$

The norms of the above spaces are defined in analogy with $\|\cdot\|_{H_\omega^{r,s}(\Omega)}$. Furthermore, let $C_{0,p}^\infty(\Omega)$ be the set of all infinitely differential functions defined on $\overline{\Omega}$, which vanish at $x_1=\pm1$ and have the period 2π for x_2 and x_3. $H_{0,p,\omega}^{r,s}(\Omega)$ and $M_{0,p,\omega}^{r,s}(\Omega)$ denote the closures of $C_{0,p}^\infty$ in $H_\omega^{r,s}(\Omega)$ and $M_\omega^{r,s}(\Omega)$ respectively. If $r=s$, we denote $\|\cdot\|_{H_\omega^{r,s}(\Omega)}$ by $\|\cdot\|_{r,\omega}$ for simplicity, etc..

We now consider the generalized stability of scheme (2.4). Suppose that the initial values $\eta(0),\eta(\tau)$ and the right terms f_1,f_2 have the errors $\tilde\eta(0),\tilde\eta(\tau),\tilde f_1$ and $\tilde f_2$ respectively, which induce the errors of $\eta(t)$ and $\varphi(t)$, denoted by $\tilde\eta(t)$ and $\tilde\varphi(t)$. Then they satisfy the following equation

$$\begin{cases}(\tilde\eta_{\hat t}+J_{RC}(\eta,\tilde\varphi)+J_{RC}(\tilde\eta,\varphi+\tilde\varphi)-H_{RC}(\eta,\tilde\varphi)\\ \qquad -H_{RC}(\tilde\eta,\varphi+\tilde\varphi)-\nu\nabla^2\hat{\tilde\eta},v)_{M,N,\omega}=(P_c\tilde f_1,v)_{M,N,\omega}, & \forall\, v\in S_{M,N},\quad t\in\dot S_\tau,\\ -(\nabla^2\tilde\varphi,v)_{M,N,\omega}=(\tilde\eta+P_C\tilde f_2,v)_{M,N,\omega}, & \forall\, v\in S_{M,N},\quad t\in S_\tau.\end{cases}\tag{4.1}$$

Let $|||y|||_{q,\infty}=\max\limits_{t\in S_\tau}\|y(t)\|_{q,\infty}$. For describing the errors, we introduce

$$E(\tilde\eta,t)=\|\tilde\eta(t)\|_\omega^2+\frac{\nu\tau}{2}\sum_{t'=\tau}^{t-\tau}\|\hat{\tilde\eta}(t')\|_{1,\omega}^2,$$
$$\rho(t)=2\|\tilde\eta(0)\|_\omega^2+2\|\tilde\eta(\tau)\|_\omega^2+4\tau\sum_{t'=\tau}^{t-\tau}G_1(t')$$

where

$$G_1(t)=8\|P_C\tilde f_1(t)\|_\omega^2+\frac{c}{\nu}\|\eta\|_{1,\infty}^2\|P_C\tilde f_2\|_\omega^2+\frac{cM^2\ln N}{\nu}\|P_C\tilde f_2\|_\omega^4.$$

Hereafter c is a positive constant independent of M,N,ν and any function, which could be different in different cases. We have the following result.

Theorem 1. *Let* (2.3) *hold. There exist positive constants* d_1 *and* d_2 *depending only on* $|||\eta|||_{1,\infty}$, $|||\varphi|||_{2,\infty}$ *and* ν *such that if for some* $t_1\in S_\tau$,

$$\rho(t_1)e^{d_1t}\leq\frac{d_2}{M^2\ln N},$$

then for all $t\in S_\tau,t\leq t_1$, *we have*

$$E(\tilde\eta,t)\leq\rho e^{d_1t}.$$

We next turn to consider the convergence. For analyzing the errors, let $P_{M,N}^1$: $H_{0,p,\omega}^{1,1}(\Omega)\to S_{M,N}$ be the projection operator such that for any $u\in H_{0,p,\omega}^{1,1}(\Omega)$,

$$(\nabla(u-P_{M,N}^1u),\nabla(\omega v))=0,\quad\forall v\in S_{M,N}.$$

Set

$$\xi^* = P^1_{M,N}\xi, \quad \psi^* = P^1_{M,N}\psi, \quad \tilde{\xi} = \eta - \xi^*, \quad \tilde{\psi} = \varphi - \psi^*.$$

By (2.1) and (2.4), we get

$$\begin{cases} (\tilde{\xi}_{\hat{t}} + J_{RC}(\xi^*, \tilde{\psi}) + J_{RC}(\tilde{\xi}, \psi^* + \tilde{\psi}) - H_{RC}(\xi^*, \tilde{\psi}) - H_{RC}(\tilde{\xi}, \psi^* + \tilde{\psi}) - \nu\nabla^2\hat{\tilde{\xi}}, v)_{M,N,\omega} \\ \qquad = \sum_{j=1}^{6} A_j, \qquad\qquad \forall v \in S_{M,N}, \\ -(\nabla^2\tilde{\psi}, v)_{M,N,\omega} = (\tilde{\xi}, v)_{M,N,\omega} + \sum_{j=7}^{9} A_j, \qquad \forall v \in S_{M,N}, \\ \tilde{\xi}(\tau) = P_{M,N}\Big(\xi_0 + \tau\dfrac{\partial\xi}{\partial t}(0)\Big) - P^1_{M,N}\xi(\tau), \\ \tilde{\xi}(0) = P_{M,N}\xi_0 - P^1_{M,N}\xi_0, \end{cases} \tag{4.2}$$

where $A_j = A_j(t)$, and

$$\begin{aligned} &A_1 = \Big(\frac{\partial}{\partial t}\xi, v\Big)_\omega - (\xi^*_{\hat{t}}, v)_{M,N,\omega}, && A_2 = (J(\xi,\psi), v)_\omega - (J_{RC}(\xi^*, \psi^*), v)_{M,N,\omega}, \\ &A_3 = -(H(\xi,\psi), v)_\omega + (H_{RC}(\xi^*, \psi^*), v)_{M,N,\omega}, && A_4 = -\nu(\nabla^2\hat{\xi}^*, v)_\omega + \nu(\nabla^2\hat{\xi}^*, v)_{M,N,\omega}, \\ &A_5 = -\nu(\nabla^2\xi, v)_\omega + \nu(\nabla^2\hat{\xi}, v)_\omega, && A_6 = -(f_1, v)_\omega + (P_C f_1, v)_{M,N,\omega}, \\ &A_7 = -(\nabla^2\psi^*, v)_\omega + (\nabla^2\psi^*, v)_{M,N,\omega}, && A_8 = -(\xi, v)_\omega + (\xi^*, v)_{M,N,\omega}, \\ &A_9 = -(f_2, v)_\omega + (P_C f_2, v)_{M,N,\omega}. \end{aligned}$$

Theorem 2. *Let $\tau = O((M^2 \ln N)^{-\frac{1}{4}})$ and (2.3) hold. $\gamma_1(M)$ and $\gamma_2(N)$ are suitably big. Also assume that for $r > 5/4, s > 1, \alpha > 1/2, \beta > 1$ and $\delta > 0$,*

$$\begin{aligned} &\xi \in C(0,T; Y^{r,s,\delta}_{1,\omega}(\Omega) \textstyle\bigcap X^{\alpha,\beta}_\omega(\Omega) \bigcap W^{3,\infty}(\Omega)) \bigcap C^1(0,T; M^{r,s}(\Omega)) \\ &\quad \textstyle\bigcap H^2(0,T; M^{1,1}(\Omega)) \bigcap H^3(0,T; L^2_\omega(\Omega))), \\ &\psi \in C(0,T; Y^{r,s,\delta}_{2,\omega}(\Omega) \textstyle\bigcap X^{\alpha,\beta}_{2,\omega}(\Omega)), \;\; f_j \in C(0,T; H^{r,s}_\omega(\Omega) \bigcap H^s(Q, H_\omega^{\frac{1}{2}+\delta}(I))), \;\; j = 1, 2. \end{aligned}$$

Then for some $t_1 \in S_\tau, t \le t_1$,

$$\|\xi(t) - \eta(t)\|^2_\omega \le d^*_1(\tau^4 + M^{-2r} + N^{-2s}),$$

where d^*_1 and d^*_2 are positive constants depending only on ν and the norms of ξ, ψ, f_1 and f_2 in the spaces mentioned in the above. If $\tau = o((M^2 \ln N)^{-\frac{1}{4}})$, then $t_1 = T$.

5. Some Lemmas

In order to prove the theorems, we need some lemmas.

Lemma 1. *If $u \in C(\overline{\Omega})$ and $v \in \mathcal{P}_M \times V^{(2)}_N$, then*

$$\begin{aligned} &\|v\|_\omega \le \|v\|_{M,N,\omega} \le \sqrt{2}\|v\|_\omega, \\ &|(u,v)_{M,N,\omega} - (u,v)_\omega| \le c(\|u - P_{M-1,N}u\|_\omega + \|u - P_C u\|_\omega)\|v\|_\omega. \end{aligned}$$

Furthermore, if $u \in \mathcal{P}_M \times V_N^{(2)}$, *then*

$$|(u,v)_{M,N,\omega} - (u,v)_\omega| \le cM^{-r}\|u\|_{H_\omega^{r,0}(\Omega)}\|v\|_\omega.$$

By using some results in [15-17], we can prove Lemma 1 in the same way as in proof of Lemma 1 of [8].

Lemma 2. (Lemma 6 of [7]). *If* $v \in S_{M,N}$, *then*

$$\|v\|_\infty \le cM^{\frac{1}{2}}(\ln N)^{\frac{1}{2}}(\|v\|_\omega + \|\partial_2 v\|_\omega + \|\partial_3 v\|_\omega).$$

Lemma 3. *If* $v \in H_{0,p,\omega}^{r,s}(\Omega)$ *and* $r, s \ge 0$, *then*

$$\|v - P_{M,N}v\|_\omega \le c(M^{-r} + N^{-s})\|v\|_{H_\omega^{r,s}(\Omega)}.$$

If in addition $v \in H^\beta(Q, H_\omega^r(I)) \bigcap H^s(Q, H_\omega^\alpha(I)) \bigcap H^{s'}(Q, H_\omega^{r'}(I))$, $0 \le \alpha \le \min(r, r')$, $0 \le \beta \le \min(s, s')$, $r, r' > 1/2$ *and* $s, s, > 1$, *then*

$$\begin{aligned}\|v - P_C v\|_{H^\beta(Q,H_\omega^\alpha(I))} \le & cM^{2\alpha-r}\|v\|_{H^\beta(Q,H_\omega^r(I))} + cN^{\beta-s}\|v\|_{H^s(Q,H_\omega^\alpha(I))} \\ & + cq(\beta)M^{2\alpha-r'}N^{\beta-s'}\|v\|_{H^{s'}(Q,H_\omega^{r'}(I))}\end{aligned}$$

where $q(\beta) = 0$ *for* $\beta > 1$ *and* $q(\beta) = 1$ *for* $\beta \le 1$.

Proof. The first conclusion comes from Lemma 2 of [7]. We only prove the second one. Let $\mathcal{I}$ be the identity operator. Then

$$\begin{aligned}&\|v - P_C v\|_{H^\beta(Q,H_\omega^\alpha(I))} \le D_1 + D_2, \\ &D_1 = \|v - P_C^{(1)}v\|_{H^\beta(Q,H_\omega^\alpha(I))} + \|v - P_C^{(2)}v\|_{H^\beta(Q,H_\omega^\alpha(I))}, \\ &D_2 = \|(P_C^{(2)} - \mathcal{I})(\mathcal{I} - P_C^{(1)})v\|_{H^\beta(Q,H_\omega^\alpha(I))}.\end{aligned}$$

By (9.7.7) and (9.7.26) in [16],

$$D_1 \le cM^{2\alpha-r}\|v\|_{H^\beta(Q,H_\omega^r(I))} + cN^{\beta-s}\|v\|_{H^s(Q,H_\omega^\alpha(I))}.$$

If $\beta > 1$, then by (9.7.26) of [16],

$$D_2 \le c\|(\mathcal{I} - P_C^{(1)})v\|_{H^\beta(Q,H_\omega^\alpha(I))} \le cM^{2\alpha-r}\|v\|_{H^\beta(Q,H_\omega^r(I))}.$$

If $\beta \le 1$, then

$$D_2 \le cN^{\beta-s'}\|(\mathcal{I} - P_C^{(1)})v\|_{H^{s'}(Q,H_\omega^\alpha(I))} \le cM^{2\alpha-r'}N^{\beta-s'}\|v\|_{H^{s'}(Q,H_\omega^{r'}(I))}.$$

Lemma 4. (Lemma 3 of [7]). *Let* (2.3) *hold. If* $v \in H_{0,p,\omega}^{1,1}(\Omega) \bigcap M_\omega^{r,s}(\Omega)$ *and* $r, s \ge 1$, *then*

$$\begin{aligned}&\|v - P_{M,N}^1 v\|_{1,\omega} \le c(M^{1-r} + N^{1-s})|v|_{M_\omega^{r,s}(\Omega)}. \\ &\|v - P_{M,N}^1 v\|_\omega \le c(M^{-r} + N^{-s})|v|_{M_\omega^{r,s}(\Omega)}.\end{aligned}$$

Lemma 5. *Let* (2.3) *hold and* $v \in H^{1,1}_{0,p,\omega}(\Omega)\bigcap H^{s+q_2}(Q, H^{r+2q_1}(I))$ *with* $r > 1/2, s > 1$. *Then there exists a positive constant* c *independent of* M, N *and* v *such that*

$$\|\partial_1^{q_1}\partial_2^{q_2-\lambda}\partial_3^{\lambda}P^1_{M,N}v\|_\infty \leq c\|v\|_{H^{s+q_2}(Q,H^{r+2q_1}_\omega(I))}, \quad 0 \leq \lambda \leq q_2.$$

Proof. Let

$$v = \sum_{|l|=0}^{\infty} v_l(x_1)e^{ily}, \quad P^1_{M,N}v = \sum_{|l|\leq N} v_l^*(x_1)e^{ily}$$

and

$$a_l(w,u) = (\partial_1 w, \partial_1(\omega u))_{L^2(I)} + |l|^2(w,u)_{\omega,I}, \quad |l| \leq N.$$

Then $v_l^* \in V_M^{(1)}$ and $a_l(v_l - v_l^*, u) = 0$ for all $u \in V_M^{(1)}$. Therefore,

$$a_l(v_l - v_l^*, v_l - v_l^*) = a_l(v_l - v_l^*, v_l - u), \quad \forall u \in V_M^{(1)}.$$

By Lemma 2 and Lemma 3 of [17], we know that if $u \in H^1_\omega(I)$ and $u(-1) = u(1) = 0$, then

$$\begin{aligned} &a_l(u,u) \geq \frac{1}{4}\|u\|^2_{1,\omega,I} + |l|^2\|u\|^2_{\omega,I}, \\ &|a_l(w,u)| \leq c(\|w\|_{1,\omega,I} + |l|\|w\|_{\omega,I})(\|u\|_{1,\omega,I} + |l|\|u\|_{\omega,I}). \end{aligned}$$

Denote by $v_{l,*}$ the $H^1_\omega(I)$ projection of v_l onto $V_M^{(1)}$. Then the combination of the above statements with (2.3) and (9.5.17)in [16], leads to

$$\begin{aligned} &\frac{1}{4}\|v_l - v_l^*\|^2_{1,\omega,I} + |l|^2\|v_l - v_l^*\|^2_{\omega,I} \leq a_l(v_l - v_l^*, v_l - v_l^*) \\ &\leq c \inf_{u\in V_M^{(1)}} \Big(\frac{1}{4}\|v_l - u\|^2_{1,\omega,I} + |l|^2\|v_l - u\|^2_{\omega,I}\Big) \\ &\leq c(\|v_l - v_{l,*}\|^2_{1,\omega,I} + |l|^2\|v_l - v_{l,*}\|^2_{\omega,I}) \leq cM^{2-2r}\|v_l\|^2_{r,\omega,I}. \end{aligned}$$

Moreover by means of the duality as in [6],

$$\|v_l - v_l^*\|_{\mu,\omega,I} \leq cM^{\mu-r}\|v_l\|_{r,\omega,I}, \quad \mu = 0,1. \tag{5.1}$$

Obviously

$$\|\partial_1^{q_1}\partial_2^{q_2-\lambda}\partial_3^{\lambda}P^1_{M,N}v\|_\infty \leq \sum_{|l|\leq N}|l|^{q_2}(|v_l|_{q_1,\infty,I} + |v_l - P_C^{(1)}v_l|_{q_1,\infty,I} + |P_C^{(1)}v_l - v_l^*|_{q_1,\infty,I}). \tag{5.2}$$

We now estimate the terms in the right side of (5.2). First of all, by embedding theory,

$$|v_l|_{q_1,\infty,I} \leq c\|v_l\|_{r+q_1,I} \leq c\|v_l\|_{r+q_1,\omega,I}, \quad r > 1/2.$$

Next, we estimate the term $|v_l - P_C^{(1)}v_l|_{q_1,\infty,I}$. Let $x_1 = \cos\theta, I_\theta = (0, 2\pi)$ and

$$v_l(x_1) = \sum_{j=0}^{\infty} v_l^{(j)}T_j(x_1), \quad |l| \leq N.$$

Then

$$v_l(x_1) = \hat{v}_l(\theta) = \sum_{j=0}^{\infty} v_l^{(j)} \cos j\theta, \quad \theta \in I_\theta.$$

Let $\hat{P}_C$ be the trigonometric interpolation on I_θ. Then $\widehat{P_C^{(1)} v_l} = \widehat{P_C}\hat{v}_l$. Let $\hat{v}_l^{(j)}$ be the coefficient of the Fourier expansion of $\widehat{P_C}\hat{v}_l$. By (2.1.29) of [16],

$$\hat{v}_l^{(j)} = v_l^{(j)} + \sum_{\sigma=1}^{\infty} (v_l^{(j+2\sigma M)} + v_l^{(-j+2\sigma M)}), \quad 0 \le j \le M.$$

Hence

$$\begin{aligned}\|\hat{v}_l - \widehat{P_C}\hat{v}_l\|_{\infty,I_\theta} &\le c \sum_{j>M} |v_l^{(j)}| \\ &\le c\Big(\sum_{j>M} (1+j^2)^r |v_l^{(j)}|^2\Big)^{1/2} \Big(\sum_{j>M} (1+j^2)^{-r}\Big)^{1/2} \le cM^{1/2-r} \|v_l\|_{r,I_\theta}, \quad r > 1/2.\end{aligned}$$

Since the mapping $v_l \to \hat{v}_l$ is continuous from $H^r_\omega(I)$ to $H^r(I_\theta)$,

$$\|v_l - P_C^{(1)} v_l\|_{\infty,I} \le cM^{1/2-r} \|v_l\|_{r,\omega,I}, \quad r > 1/2. \tag{5.3}$$

Furthermore,

$$|v_l - P_C^{(1)} v_l|_{q_1,\infty,I} \le \|\partial_1^{q_1} v_l - P_C^{(1)}(\partial_1^{q_1} v_l)\|_{\infty,I} + \|P_C^{(1)}(\partial_1^{q_1} v_l) - \partial_1^{q_1}(P_C^{(1)} v_l)\|_{\infty,I}.$$

By (5.3),

$$\|\partial_1^{q_1} v_l - P_C^{(1)}(\partial_1^{q_1} v_l)\|_{\infty,I_1} \le cM^{1/2-r} \|v_l\|_{r+q_1,\omega,I}.$$

According to (9.5.3) and (9.5.20) in [16],

$$\begin{aligned}\|P_C^{(1)}(\partial_1^{q_1} v_l) - \partial_1^{q_1}(P_C^{(1)} v_l)\|_{\infty,I} &\le cM^{1/2}(\|P_C^{(1)}(\partial_1^{q_1} v_l) - \partial_1^{q_1} v_l\|_{\omega,I} + \|v_l - P_C^{(1)} v_l\|_{q_1,\omega,I}) \\ &\le cM^{1/2-r} \|v_l\|_{r+2q_1,\omega,I}.\end{aligned}$$

Consequently

$$|v_l - P_C^{(1)} v_l|_{q_1,\infty,I} \le cM^{1/2-r} \|v_l\|_{r+2q_1,\omega,I}.$$

Finally we estimate the term $|P_C^{(1)} v_l - v_l^*|_{q_1,\infty,I}$. From (9.5.3) and (9.5.4) in [16],

$$|P_C^{(1)} v_l - v_l^*|_{q_1,\infty,I} \le cM^{1/2} \|P_C^{(1)} v_l - v_l^*\|_{q_1,\omega,I} \le cM^{1/2+2q_1} \|P_C^{(1)} v_l - v_l^*\|_{\omega,I}.$$

Moreover

$$\|P_C^{(1)} v_l - v_l^*\|_{\omega,I} \le \|P_C^{(1)} v_l - v_l\|_{\omega,I} + \|v_l - v_l^*\|_{\omega,I} \le cM^{-r-2q_1} \|v_l\|_{r+2q_1,\omega,I}.$$

The previous statements and (5.2) lead to that

$$\|\partial_1^{q_1} \partial_2^{q_2-\lambda} \partial_3^\lambda P^1_{M,N} v\|_\infty \le c\Big(\sum_{|l|\le N} (1+|l|^2)^{s+q_2} \|v_l\|^2_{r+2q_1,\omega,I}\Big)^{1/2} \Big(\sum_{|l|\le N} (1+|l|^2)^{-s}\Big)^{1/2}$$

$$\leq c\|v\|_{H^{s+q_2}(Q,H_\omega^{r+2q_1}(I))}.$$

Remark 1. Clearly, if the conditions of Lemma 5 hold, then for all non-negative integer k,

$$\|P_{M,N}^1 v\|_{k,\infty} \leq c\|v\|_{X_{k,\omega}^{r,s}(\Omega)}, \quad k \geq 1.$$

Remark 2. The bound of $\|P_{M,N}^1 v\|_{1,\infty}$ can be improved as[7]

$$\|P_{M,N}^1 v\|_{1,\infty} \leq c\|v\|_{X_\omega^{r,s}(\Omega)}.$$

Lemma 6. *If $v \in H^\beta(Q, H_\omega^r(I)) \bigcap H^s(Q, L_\omega^2(I))$, $0 \leq r \leq \gamma_1(M)$ and $0 \leq s-\beta \leq \gamma_2(N)$, then*

$$\|Rv - v\|_{H^\beta(Q,L_\omega^2(I))} \leq cM^{-r}\|v\|_{H^\beta(Q,H_\omega^r(I))} + cN^{\beta-s}\|v\|_{H^s(Q,L_\omega^2(I))}.$$

This Lemma can be proved in the same way as in [8].

Lemma 7. *For any $v \in S_{M,N}$,*

$$-(\nabla^2 v, v)_{M,N,\omega} \geq \frac{1}{4}\|v\|_{1,\omega}^2.$$

Proof. Let

$$v = \sum_{|l|\leq N} v_l(x_1)e^{ily}.$$

Notice that if $u, z \in \mathcal{P}_M \times V_N^{(2)}$ and $uz \in \mathcal{P}_{2M-1} \times V_{2N}^{(2)}$, then

$$(u,z)_{M,N,\omega} = (u,z)_\omega. \tag{5.4}$$

Therefore by Lemma 1 and Lemma 2 in [17],

$$\begin{aligned}-(\nabla^2 v, v)_{M,N,\omega} &= \sum_{|l|\leq N}(\partial_1 v_l, \partial_1(\omega v_l))_{L^2(I)} + \|\partial_2 v\|_{M,N,\omega}^2 + \|\partial_3 v\|_{M,N,\omega}^2,\\ &\geq \frac{1}{4}\sum_{|l|\leq N}\|v_l\|_{1,\omega,I}^2 + \|\partial_2 v\|_\omega^2 + \|\partial_3 v\|_\omega^2 \geq \frac{1}{4}\|v\|_{1,\omega}^2.\end{aligned}$$

Lemma 8. *If $u \in L_\omega^2(\Omega)$ and $v \in H_{0,p,\omega}^{1,0}(\Omega)$, then*

$$|(u, \partial_1(\omega v))_{L^2(\Omega)}| \leq 2\|u\|_\omega\|\partial_1 v\|_\omega.$$

Proof. We know from Lemma 1 of [17] that if $z \in H_\omega^1(I)$ and $z(-1) = z(1) = 0$, then

$$\|\omega^2 z\|_{\omega,I} \leq |z|_{1,\omega,I}. \tag{5.5}$$

Thus

$$\begin{aligned}|(u, \partial_1(\omega v))_{L^2(\Omega)}| &\leq |(u, \partial_1 v)_\omega| + |(u, x_1\omega^2 v)_\omega|\\ &\leq \|u\|_\omega(\|\partial_1 v\|_\omega + \|\omega^2 v\|_\omega) \leq 2\|u\|_\omega\|\partial_1 v\|_\omega.\end{aligned}$$

Lemma 9. *If* (2.3) *holds and* $u, v, z \in S_{M,N}$*, then*

$$|(J_{RC}(u,z),v)_{M,N,\omega}| \le c\|u\|_\infty |z|_{1,\omega}|v|_{1,\omega},$$
$$|(J_{RC}(u,z),v)_{M,N,\omega}| \le c\|u\|_\omega |z|_{1,\infty}|v|_{1,\omega}.$$

Proof. By (5.4),

$$(J_{RC}(u,z),v)_{M,N,\omega} = D_1 + D_2 + D_3,$$
$$D_1 = -(RP_C((\nabla \times z)^{(1)}u), \partial_1(\omega v))_{L^2(\Omega)}, \quad D_2 = -(RP_C((\nabla \times z)^{(2)}u), \partial_2 v)_{M,N,\omega},$$
$$D_3 = -(RP_C((\nabla \times z)^{(3)}u), \partial_3 v)_{M,N,\omega}.$$

By Lemma 1, Lemma 6 and Lemma 8,

$$\begin{aligned}|D_1| &\le 2\|RP_C((\nabla \times z)^{(1)}u)\|_\omega \|\partial_1 v\|_\omega \le c\|P_C((\nabla \times z)^{(1)}u)\|_{M,N,\omega}|v|_{1,\omega}\\ &= c\|(\nabla \times z)^{(1)}u\|_{M,N,\omega}|v|_{1,\omega} \le c\|u\|_\infty |z|_{1,\omega}|v|_{1,\omega}.\end{aligned}$$

Next, by Lemma 1 and Lemma 6,

$$\begin{aligned}|D_2| &\le \|RP_C((\nabla \times z)^{(2)}u)\|_{M,N,\omega}\|\partial_2 v\|_{M,N,\omega}\\ &\le c\|RP_C((\nabla \times z)^{(2)}u)\|_\omega |v|_{1,\omega} \le c\|u\|_\infty |z|_{1,\omega}|v|_{1,\omega}.\end{aligned}$$

We can estimate $|D_3|$ similarly and get the first conclusion. The second one follows similarly.

Lemma 10. *Let* (2.3) *hold and* $v \in S_{M,N}$*,* $g \in (\mathcal{P}_M \times W_N^{(2)})^3$ *satisfy*

$$-(\nabla^2 v, u)_{M,N,\omega} = (g,u)_{M,N,\omega}, \quad \forall\ u \in S_{M,N}. \tag{5.6}$$

Then

$$\|v\|_{1,\omega}^2 + \|\partial_2 v\|_{1,\omega}^2 + \|\partial_3 v\|_{1,\omega}^2 \le c\|g\|_\omega^2,$$
$$\|\partial_1^2 v\|_\omega^2 \le cM\|g\|_\omega^2.$$

Proof. Let $u = v$ in (5.6). We have from Lemma 1 and Lemma 7 that

$$\frac{1}{4}\|v\|_{1,\omega}^2 \le \|v\|_{M,N,\omega}\|g\|_{M,N,\omega} \le c\|v\|_\omega \|g\|_\omega$$

and so $\|u\|_{1,\omega}^2 \le c\|g\|_\omega^2$. Now, let

$$v = \sum_{|l|\le N} v_l(x_1)e^{ily}, \quad g = \sum_{|l|\le N} g_l(x_1)e^{ily}.$$

By putting $u = v_l e^{ily}$ in (5.6), we obtain from (5.4) that

$$(\partial_1 v_l, \partial_1(\omega v_l))_{L^2(I)} + |l|^2\langle v_l, v_l\rangle_{M,\omega} = \langle g_l, v_l\rangle_{M,\omega}.$$

By Lemma 6 of [17], the above equality reads

$$\frac{1}{4}\|v_l\|_{1,\omega,I}^2 + |l|^2\|v_l\|_{\omega,I}^2 \le |\langle g_l, v_l\rangle_{M,\omega}| \le 2\|g_l\|_{\omega,I}\|v_l\|_{\omega,I}$$

$$\leq \frac{3}{4}|l|^2\|v_l\|^2_{\omega,I} + \frac{4}{3|l|^2}\|g_l\|^2_{\omega,I}.$$

Thus

$$\begin{aligned}\|\partial_2 v\|^2_{1,\omega} + \|\partial_3 v\|^2_{1,\omega} &= \sum_{|l|\leq N} |l|^2(\|v_l\|^2_{1,\omega,I} + |l|^2\|v_l\|^2_{\omega,I}) \\ &\leq \frac{16}{3}\sum_{|l|\leq N}\|g_l\|^2_{\omega,I} = \frac{16}{3}\|g\|^2_{\omega}\end{aligned}$$

which completes the proof of the first conclusion.

We next turn to prove the second conclusion. By (5.4) and (5.6),

$$-(\partial_1^2 v, u)_\omega = (g + \partial_1^2 v + \partial_3^2 v, u)_{M,N,\omega}, \quad \forall u \in S_{M,N}. \tag{5.7}$$

For simplicity, we consider the following auxiliary problem. Let $\tilde{v}, \tilde{g} \in V_M^{(1)}$ and

$$-(\partial_1^2 \tilde{v}, \tilde{u})_{\omega,I} = \langle \tilde{g}, \tilde{u}\rangle_{M,\omega}, \quad \forall \tilde{u} \in V_M^{(1)}. \tag{5.8}$$

Assume that

$$\partial_1^2\tilde{v} = \sum_{k=0}^{M-2} a_k T_k(x_1), \quad \tilde{g} = \sum_{k=0}^{M} b_k T_k(x_1).$$

Also, let

$$\tilde{T}_k(x_1) = T_k(x_1) - T_{\alpha(k)}(x_1), \quad 0 \leq k \leq M \tag{5.9}$$

with

$$\alpha(k) = \begin{cases} M, & \text{if} \quad k+M \quad \text{is even}, \\ M-1, & \text{if} \quad k+M \quad \text{is odd}. \end{cases}$$

It can be verified that $\{\tilde{T}_k(x_1)\}$ are the basis in $V_M^{(1)}$. We put $\tilde{u}(x_1) = \tilde{T}_k(x_1)$ in (5.8). The calculation tells us that

$$(\partial_1^2\tilde{v}, \tilde{T}_k)_{\omega,I} = \frac{\pi}{2}c_k a_k, \quad (\tilde{g}, \tilde{T}_k)_{\omega,I} = \frac{\pi}{2}c_k b_k - \frac{\pi}{2}b_{\alpha(k)}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$. Moreover by (5.4) and Lemma 1,

$$|\langle \tilde{g}, \tilde{T}_k\rangle_{M,\omega} - (\tilde{g}, \tilde{T}_k)_{\omega,I}| = \frac{\pi}{2}|b_M|.$$

Thus (5.8) leads to

$$|c_k a_k| \leq |c_k b_k| + |b_{\alpha(k)}| + |b_M|, \quad 0 \leq k \leq M-2.$$

Hence

$$\sum_{k=0}^{M-2} a_k^2 \leq c\Big(Mb_M^2 + Mb_{M-1}^2 + \sum_{k=0}^{M-2} b_k^2\Big)$$

and so $\|\partial_1^2\tilde{u}\|^2_{\omega,I} \leq cM\|\tilde{g}\|^2_{\omega,I}$. Then the second conclusion follows.

Lemma 11. (Lemma 4.16 of [18]). *Suppose that the following conditions are fulfilled*

(i) ρ, b_1, b_2 *are non-negative constants and* $q > 1$*;*

(ii) $E(t)$ *is a non-negative function defined on* S_τ*;*

(iii) $E(0) \leq \rho$ *and for* $t \in S_\tau, t > \tau$,

$$E(t) \leq \rho + b_1 \tau \sum_{t'=0}^{t-\tau} (E(t') + b_2 E^q(t'));$$

(iv) for some $t_1 \in S_\tau, \rho e^{2b_1 t_1} \leq b_2^{1/1-q}$.

Then for all $t \in S_\tau, t \leq t_1$,

$$E(t) \leq \rho e^{2b_1 t}.$$

6. The Proof of Theorems

We first prove Theorem 1. Let $v = 2\hat{\tilde{\eta}}$ in the first formula of (4.1). By using Lemma 7 and the fact that

$$2(\tilde{\eta}_{\hat{t}}, \hat{\tilde{\eta}})_{M,N,\omega} = (\|\tilde{\eta}\|^2_{M,N,\omega})_{\hat{t}},$$

we obtain

$$(\|\tilde{\eta}\|^2_{M,N,\omega})_{\hat{t}} + \frac{\nu}{2}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \sum_{j=1}^{6} F_j \leq \frac{1}{4}\|\hat{\tilde{\eta}}\|^2_{M,N,\omega} + 4\|P_C \tilde{f}_1\|^2_{M,N,\omega} \tag{6.1}$$

where $F_j = F_j(t)$ and

$$\begin{aligned} F_1 &= 2(J_{RC}(\eta, \tilde{\varphi}), \hat{\tilde{\eta}})_{M,N,\omega}, & F_2 &= 2(J_{RC}(\tilde{\eta}, \varphi), \hat{\tilde{\eta}})_{M,N,\omega}, \\ F_3 &= 2(J_{RC}(\tilde{\eta}, \tilde{\varphi}), \hat{\tilde{\eta}})_{M,N,\omega}, & F_4 &= -2(H_{RC}(\eta, \tilde{\varphi}), \hat{\tilde{\eta}})_{M,N,\omega}, \\ F_5 &= -2(H_{RC}(\tilde{\eta}, \varphi), \hat{\tilde{\eta}})_{M,N,\omega}, & F_6 &= -2(H_{RC}(\tilde{\eta}, \tilde{\varphi}), \hat{\tilde{\eta}})_{M,N,\omega}. \end{aligned}$$

We apply Lemma 10 to the second formula of (4.1), and get

$$\|\tilde{\varphi}\|^2_{1,\omega} + \|\partial_2\tilde{\varphi}\|^2_{1,\omega} + \|\partial_3\tilde{\varphi}\|^2_{1,\omega} \leq c(\|\tilde{\eta}\|^2_\omega + \|P_C \tilde{f}_2\|^2_\omega), \tag{6.2}$$

$$\|\tilde{\varphi}\|^2_{2,\omega} \leq cM(\|\tilde{\eta}\|^2_\omega + \|P_C \tilde{f}_2\|^2_\omega). \tag{6.3}$$

We now estimate $|F_j| (j = 1, \cdots, 6)$. By Lemma 9 and (6.2),

$$\begin{aligned} |F_1| &\leq c\|\eta\|_\infty \|\tilde{\varphi}\|_{1,\omega} \|\hat{\tilde{\eta}}\|_{1,\omega} \leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \frac{c}{\nu}\|\eta\|^2_\infty (\|\tilde{\eta}\|^2_\omega + \|P_C \tilde{f}_2\|^2_\omega), \\ |F_2| &\leq c\|\varphi\|_{1,\infty} \|\tilde{\eta}\|_\omega \|\hat{\tilde{\eta}}\|_{1,\omega} \leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \frac{c}{\nu}\|\varphi\|^2_{1,\infty} \|\tilde{\eta}\|^2_\omega. \end{aligned}$$

By Lemma 2, Lemma 9 and (6.2),

$$\begin{aligned} |F_3| &\leq c\|\tilde{\varphi}\|_{1,\infty} \|\tilde{\eta}\|_\omega \|\hat{\tilde{\eta}}\|_{1,\omega} \\ &\leq cM^{1/2}(\ln N)^{1/2}(\|\tilde{\varphi}\|_{1,\omega} + \|\partial_2\tilde{\varphi}\|_{1,\omega} + \|\partial_3\tilde{\varphi}\|_{1,\omega})\|\tilde{\eta}\|_\omega \|\hat{\tilde{\eta}}\|_{1,\omega} \\ &\leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \frac{cM\ln N}{\nu}(\|\tilde{\eta}\|^4_\omega + \|P_C \tilde{f}_1\|^4_\omega). \end{aligned}$$

By (2.2.27) of [16], we know that $(Ru,v)_{M,N,\omega} = (u,Rv)_{M,N,\omega}$ for all $u,v \in S_{M,N}$. On the other hand, we know from (5.4) that $(\partial_1 z, uv)_{M,N,\omega} = (\partial_1 z, P_C(uv))_{M,N,\omega} = (\partial_1 z, P_C(uv))_\omega$ for all $u,v,z \in S_{M,N}$. Thus

$$\begin{aligned}
|F_4| =& 2|\sum_{j=1}^{3}(\eta^{(j)}\partial_j(\nabla\times\tilde{\varphi}), R\hat{\tilde{\eta}})_{M,N,\omega}| \le D_1 + D_2,\\
D_1 =& 2|(\eta^{(1)}\partial_1(\nabla\times\tilde{\varphi}), R\hat{\tilde{\eta}} - \hat{\tilde{\eta}})_{M,N,\omega}| + 2|(\partial_1(\nabla\times\tilde{\varphi}), \eta^{(1)}\hat{\tilde{\eta}})_\omega|\\
&+ 2|(\partial_1(\nabla\times\tilde{\varphi}), (P_C - \mathcal{I})(\eta^{(1)}\hat{\tilde{\eta}}))_\omega|,\\
D_2 =& 2\sum_{j=2}^{3}\|\eta^{(j)}\partial_j(\nabla\times\tilde{\varphi})\|_{M,N,\omega}\|\hat{\tilde{\eta}}\|_{M,N,\omega}.
\end{aligned}$$

Furthermore, by Lemma 3, (6.2) and (6.3),

$$\begin{aligned}
D_1 \le& c\|\eta^{(1)}\partial_1(\nabla\times\tilde{\varphi})\|_{M,N,\omega}\|R\hat{\tilde{\eta}} - \hat{\tilde{\eta}}\|_{M,N,\omega} + c|\tilde{\varphi}|_{1,\omega}|\eta^{(1)}\hat{\tilde{\eta}}|_{1,\omega}\\
&+ c|\tilde{\varphi}|_{2,\omega}\|(P_C - \mathcal{I})(\eta^{(1)}\hat{\tilde{\eta}})\|_\omega \le c(M^{-1} + N^{-1})\|\eta^{(1)}\|_\infty|\tilde{\varphi}|_{2,\omega}\|\hat{\tilde{\eta}}\|_{1,\omega}\\
&+ c|\tilde{\varphi}|_{1,\omega}\|\eta^{(1)}\|_{1,\infty}|\hat{\tilde{\eta}}|_{1,\omega} + c|\tilde{\varphi}|_{2,\omega}(M^{-1}\|\eta^{(1)}\hat{\tilde{\eta}}\|_{L^2(Q,H^1_\omega(I))}\\
&+ N^{-1}\|\eta^{(1)}\hat{\tilde{\eta}}\|_{H^1(Q,L^2_\omega(I))} + M^{-1}N^{-2}\|\eta^{(1)}\hat{\tilde{\eta}}\|_{H^2(Q,H^1_\omega(I))})\\
\le& c|\tilde{\varphi}|_{1,\omega}\|\eta^{(1)}\|_{1,\infty}|\hat{\tilde{\eta}}|_{1,\omega} + c(M^{-1} + N^{-1})|\tilde{\varphi}|_{2,\omega}\|\eta^{(1)}\|_{1,\infty}\|\hat{\tilde{\eta}}\|_{1,\omega}\\
\le& c\|\eta\|_{1,\infty}\|\hat{\tilde{\eta}}\|_{1,\omega}(\|\tilde{\eta}\|^2_\omega + \|P_C\tilde{f}_2\|^2_\omega),\\
D_2 \le& c\|\eta\|_\infty\|\hat{\tilde{\eta}}\|_\omega\sum_{j=2}^{3}\|\partial_j\tilde{\varphi}\|_{1,\omega} \le c\|\eta\|_\infty\|\hat{\tilde{\eta}}\|_{1,\omega}(\|\tilde{\eta}\|^2_\omega + \|P_C\tilde{f}_2\|^2_\omega).
\end{aligned}$$

Hence

$$|F_4| \le \frac{\nu}{16}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \frac{c}{\nu}\|\eta\|^2_{1,\infty}(\|\tilde{\eta}\|^2_\omega + \|P_C\tilde{f}_2\|^2_\omega).$$

Obviously by Lemma 1 and Lemma 6,

$$|F_5| \le c\|\varphi\|_{2,\infty}\|\tilde{\eta}\|_\omega\|\hat{\tilde{\eta}}\|_\omega \le \frac{\nu}{16}\|\hat{\tilde{\eta}}\|^2_{1,\omega} + \frac{c}{\nu}\|\varphi\|^2_{1,\infty}\|\tilde{\eta}\|^2_\omega.$$

By Lemma 1, Lemma 2, Lemma 6 and (6.3),

$$\begin{aligned}
|F_6| &\le c\|\tilde{\varphi}\|_{2,\omega}\|\tilde{\eta}\|_\infty\|\hat{\tilde{\eta}}\|_\omega \le c(M\ln N)^{1/2}\|\tilde{\varphi}\|_{2,\omega}\|\hat{\tilde{\eta}}\|_{1,\omega}\|\tilde{\eta}\|_\omega\\
&\le \frac{\nu}{16}\|\hat{\tilde{\eta}}\|_{1,\omega} + \frac{cM^2\ln N}{\nu}(\|\tilde{\eta}\|^4_\omega + \|P_C\tilde{f}_2\|^4_\omega).
\end{aligned}$$

By substituting the above estimations into (6.1), we get

$$(\|\tilde{\eta}\|^2_{M,N,\omega})_{\hat{t}} + \frac{\nu}{8}\|\hat{\tilde{\eta}}\|^2_{1,\omega} \le \frac{1}{2}\|\hat{\tilde{\eta}}\|^2_\omega + d_3\|\tilde{\eta}\|^2_\omega + d_4\|\tilde{\eta}\|^4_\omega + G_1 \tag{6.4}$$

where $G_1(t)$ is given in Section 4, and

$$d_3 = \frac{c}{\nu}(|||\eta|||^2_{1,\infty} + |||\varphi|||^2_{1,\infty}), \quad d_4 = \frac{cM^2\ln N}{\nu}.$$

By summing (6.4) for $t \in \dot{S}_\tau$, we have

$$\begin{aligned}\|\tilde{\eta}(t)\|^2_{M,N,\omega} + \|\tilde{\eta}(t-\tau)\|^2_{M,N,\omega} + \frac{\nu\tau}{4}\sum_{t'=\tau}^{t-\tau}\|\hat{\tilde{\eta}}(t')\|^2_{1,\omega} \\ \le \|\tilde{\eta}(0)\|^2_{M,N,\omega} + \|\tilde{\eta}(\tau)\|^2_{M,N,\omega} + \tau\sum_{t'=\tau}^{t-\tau}(\|\hat{\tilde{\eta}}(t')\|^2_\omega \\ + 2d_3\|\tilde{\eta}(t')\|^2_\omega + 2d_4\|\tilde{\eta}(t')\|^4_\omega + 2G_1(t')).\end{aligned}$$

Let $\tau < 1$. By Lemma 1 and the fact that

$$\|\hat{\tilde{\eta}}(t)\|^2_\omega \le \frac{1}{2}\|\tilde{\eta}(t+\tau)\|^2_\omega + \frac{1}{2}\|\tilde{\eta}(t-\tau)\|^2_\omega,$$

we obtain

$$E(\tilde{\eta}, t) \le \rho(t) + 4\tau\sum_{t'=\tau}^{t-\tau}[(d_3+1)E(\tilde{\eta}, t') + d_4E^2(\tilde{\eta}, t')]$$

where $E(\tilde{\eta}, t)$ and $\rho(t)$ are as shown in Section 4. Finally we use Lemma 11 to complete the proof of Theorem 1.

Next, we prove Theorem 2. The key point is to estimate $|A_j| (j = 1, \cdots, 9)$. By Lemma 1 and Lemma 4, we have that for any $r, s \ge 1$,

$$\begin{aligned}|A_1| \le& |(\xi^*_{\hat{t}} - \xi_{\hat{t}}, v)_{M,N,\omega}| + |(\xi_{\hat{t}}, v)_{M,N,\omega} - (\xi_{\hat{t}}, v)_\omega| + \left|\left(\xi_{\hat{t}} - \frac{\partial}{\partial t}\xi, v\right)_\omega\right| \\ \le& c\|v\|_\omega(\|\xi_{\hat{t}} - P_{M-1,N}\xi_{\hat{t}}\|_\omega + \|\xi_{\hat{t}} - P_C\xi_{\hat{t}}\|_\omega + \|\xi^*_{\hat{t}} - \xi_{\hat{t}}\|_\omega + \tau^{3/2}\|\xi\|_{H^3(t-\tau,t+\tau;L^2_\omega(\Omega))}) \\ \le& \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r} + N^{-2s})\|\xi\|^2_{C^1(0,T;M^{r,s}_\omega(\Omega))} + c\tau^3\|\xi\|^2_{H^3(t-\tau,t+\tau;L^2_\omega(\Omega))}.\end{aligned}$$

It is complicated to estimate $|A_2|$. Let $A_2 = B_1 + B_2$ where

$$B_1 = (J_{RC}(\xi^*, \psi^*), v)_\omega - (J_{RC}(\xi^*, \psi^*), v)_{M,N,\omega}, \quad B_2 = (J(\xi, \psi) - J_{RC}(\xi^*, \psi^*), v)_\omega.$$

By Lemma 1 and Lemma 6, we have

$$\begin{aligned}|B_1| =& |\sum_{j=2}^{3}[(P_C((\nabla\times\psi^*)^{(j)}\xi^*), R\partial_j v)_\omega - (P_C((\nabla\times\psi^*)^{(j)}\xi^*), R\partial_j v)_{M,N,\omega}]| \\ \le& c\|v\|_{1,\omega}\sum_{j=2}^{3}\|(\mathcal{I} - P_{M-1,N})P_C((\nabla\times\psi^*)^{(j)}\xi^*)\|_\omega \\ \le& c\|v\|_{1,\omega}\sum_{j=2}^{3}\|(\mathcal{I} - P_{M-1,N})(P_C - \mathcal{I})(\nabla\times\psi^*)^{(j)}\xi^*\|_\omega \\ &+ \|(\mathcal{I} - P_{M-1,N})(\nabla\times\psi^*)^{(j)}\xi^*\|_\omega).\end{aligned}$$

Moreover, Lemma 3, Lemma 4, and Lemma 5 imply that for $r > 5/4$ and $s > 1$,

$$\|(\mathcal{I} - P_{M-1,N})(P_C - \mathcal{I})(\nabla\times\psi^*)^{(j)}\xi^*\|_\omega \le c\|(P_C - \mathcal{I})(\nabla\times\psi^*)^{(j)}\xi^*\|_\omega$$

$$
\begin{aligned}
&\le c\|(P_C-\mathcal{I})(\nabla\times\psi^*)^{(j)}(\xi^*-\xi)\|_\omega + c\|(P_C-\mathcal{I})(\nabla\times(\psi^*-\psi))^{(j)}\xi\|_\omega\\
&\quad + c\|(P_C-\mathcal{I})(\nabla\times\psi)^{(j)}\xi\|_\omega \le cM^{-1}\|(\nabla\times\psi^*)^{(j)}(\xi^*-\xi)\|_{1,\omega}\\
&\quad + cN^{-2}\|(\nabla\times\psi^*)^{(j)}(\xi^*-\xi)\|_{H^2(Q,L^2_\omega(I))}\\
&\quad + cM^{-1}N^{-2}\|(\nabla\times\psi^*)^{(j)}(\xi^*-\xi)\|_{H^2(Q,H^1_\omega(I))} + cM^{-1}\|\xi(\nabla\times(\psi^*-\psi))^{(j)}\|_{1,\omega}\\
&\quad + cN^{-2}\|\xi(\nabla\times(\psi^*-\psi))^{(j)}\|_{H^2(Q,L^2_\omega(I))} + cM^{-1}N^{-2}\|\xi(\nabla\times(\psi^*-\psi))^{(j)}\|_{H^2(Q,H^1_\omega(I))}\\
&\quad + c(M^{-r}+N^{-s})\|(\nabla\times\psi)^{(j)}\xi\|_{H^{r,s}_\omega(\Omega)\bigcap H^{1+\delta}(Q,H^{r-3/4-3/4\delta}_\omega(I))}\\
&\le c(M^{-r}+N^{-s})\|\psi^*\|_{2,\infty}\|\xi\|_{M^{r,s}_\omega(\Omega)} + cN^{-1}\|\psi^*\|_{2,\infty}\|\xi^*-\xi\|_{H^2(Q,L^2_\omega(I))}\\
&\quad + cM^{-1}\|\psi^*\|_{2,\infty}\|\xi^*-\xi\|_{H^2(Q,H^1_\omega(I))} + cM^{-1}\|\xi\|_{1,\infty}\|\psi^*-\psi\|_{2,\omega}\\
&\quad + cN^{-2}\|\xi\|_{2,\infty}\|\nabla\times(\psi^*-\psi)\|_{H^2(Q,L^2_\omega(I))}\\
&\quad + cM^{-1}N^{-2}\|\xi\|_{3,\infty}\|\nabla\times(\psi^*-\psi)\|_{H^2(Q,H^1_\omega(I))}\\
&\quad + c(M^{-r}+N^{-s})\|(\nabla\times\psi)^{(j)}\xi\|_{H^{r,s}_\omega(\Omega)\bigcap H^{1+\delta}(Q,H^{r-3/4-3/4\delta}_\omega(I))}.
\end{aligned}
$$

By embedding theorem, we have

$$
\begin{aligned}
\|uv\|_{H^{r,s}_\omega(\Omega)} \le & c\|u\|_{H^{r,s}_\omega(\Omega)\bigcap H^{1+\delta}(Q,H^{1/2+1/2+\delta}_\omega(I))\bigcap H^{1/2+1+\delta}(Q,H^{1/2+\delta}_\omega(I))}\\
&\times\|v\|_{H^{r,s}_\omega(\Omega)\bigcap H^{1+\delta}(Q,H^{1/2+1/2+\delta}_\omega(I))\bigcap H^{1/2+1+\delta}(Q,H^{1/2+\delta}_\omega(I))}, \qquad (6.5)\\
\|uv\|_{H^s(Q,H^r_\omega(I))} \le & c\|u\|_{H^s(Q,H^r_\omega(I))\bigcap H^{1/2+1+\delta}(Q,H^{r+1/2+\delta}_\omega(I))\bigcap H^{s+1+\delta}(Q,H^{1/2+1/2+\delta}_\omega(I))}\\
&\times\|v\|_{H^s(Q,H^r_\omega(I))\bigcap H^{1/2+1+\delta}(Q,H^{r+1/2+\delta}_\omega(I))\bigcap H^{s+1+\delta}(Q,H^{1/2+1/2+\delta}_\omega(I))}. \qquad (6.6)
\end{aligned}
$$

By Lemma 3 and Lemma 4,

$$
\begin{aligned}
\|\xi^*-\xi\|_{H^2(Q,H^1_\omega(I))} &\le \|\xi^*-P_C\xi\|_{H^2(Q,H^1_\omega(I))} + \|P_C\xi-\xi\|_{H^2(Q,H^1_\omega(I))}\\
&\le cN^{-2}(\|\xi^*-\xi\|_{1,\omega} + \|\xi-P_C\xi\|_{L^2(Q,H^1_\omega(I))}) + \|P_C\xi-\xi\|_{H^2(Q,H^1_\omega(I))}\\
&\le c(M^{1-r}+N^{1-s})\|\xi\|_{M^{r+2,s+2}_\omega(\Omega)\bigcap L^2(Q,H^{r+3}_\omega(I))\bigcap H^2(Q,H^{r+1}_\omega(I))}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|\psi^*-\psi\|_{2,\omega} \le \|\psi^*-P_C\psi\|_{2,\omega} + \|P_C\psi-\psi\|_{2,\omega}\\
&\le c(M^2+N)(\|\psi^*-\psi\|_{1,\omega} + \|P_C\psi-\psi\|_{1,\omega}) + \|P_C\psi-\psi\|_{2,\omega}\\
&\le c(M^{1-r}+N^{1-s})\|\psi\|_{M^{r+2,s+2}_\omega(\Omega)\cap L^2(Q,H^{r+3}_\omega(I))\cap H^2(Q,H^{r+1}_\omega(I))\cap H^{s-1}(Q,H^2_\omega(I))},\\
&\|\nabla\times(\psi^*-\psi)\|_{H^2(Q,L^2_\omega(I))} \le \|\psi^*-\psi\|_{H^3(Q,L^2_\omega(I))} + \|\psi^*-\psi\|_{H^2(Q,H^1_\omega(I))}\\
&\le c(M^{2-r}+N^{2-s})\|\psi\|_{M^{r+1,s+1}_\omega(\Omega)\cap L^2(Q,H^{r+2}_\omega(I))\cap H^2(Q,H^r_\omega(I))\cap H^3(Q,H^{r-2}_\omega(I))},\\
&\|\nabla\times(\psi^*-\psi)\|_{H^2(Q,H^1_\omega(I))} \le \|\psi^*-\psi\|_{H^3(Q,H^1_\omega(I))} + \|\psi^*-\psi\|_{H^2(Q,H^2_\omega(I))}\\
&\le c(M^{3-r}+N^{3-s})\|\psi\|_{M^{r+2,s+2}_\omega(\Omega)\cap L^2(Q,H^{r+3}_\omega(I))\cap H^2(Q,H^{r+1}_\omega(I))\cap H^3(Q,H^{r-1}_\omega(I))\cap H^{s-1}(Q,H^2_\omega(I))}.
\end{aligned}
$$

We can estimate the term $\|(\nabla\times\psi)^{(j)}\xi\|_{H^{r,s}_\omega(\Omega)\bigcap H^{1+\delta}(Q,H^{r-3/4-3/4\delta}_\omega(I))}$ by (6.5) and (6.6). Finally, for $r>5/4$, $s>1, \alpha>1/2$ and $\beta>1$,

$$
|B_1| \le \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r}+N^{-2s})\|\xi\|^2_{Y^{r,s,\delta}_{1,\omega}(\Omega)\bigcap W^{3,\infty}(\Omega)}\|\psi\|^2_{Y^{r,s,\delta}_{2,\omega}(\Omega)\bigcap X^{\alpha,\beta}_{2,\omega}(\Omega)}.
$$

By Lemma 8,

$$
\begin{aligned}
|B_2| \le & 2\|v\|_{1,\omega} \sum_{j=1}^{3} \|\xi(\nabla\times\psi)^{(j)} - RP_C(\xi^*(\nabla\times\psi^*)^{(j)})\|_\omega \\
\le & 2\|v\|_{1,\omega} \sum_{j=1}^{3} \{\|\xi(\nabla\times\psi)^{(j)} - \xi^*(\nabla\times\psi^*)^{(j)}\|_\omega + \|(\mathcal{I}-P_C)\xi^*(\nabla\times\psi^*)^{(j)}\|_\omega \\
& + \|(\mathcal{I}-R)(P_C-\mathcal{I})\xi^*(\nabla\times\psi^*)^{(j)}\|_\omega + \|(\mathcal{I}-R)(\xi^*(\nabla\times\psi^*)^{(j)})\|_\omega\}.
\end{aligned}
$$

We have from Lemma 4 and Lemma 6 that

$$
\begin{aligned}
\|(\mathcal{I}-R)\xi^*(\nabla\times\psi^*)^{(j)}\|_\omega \le & \|(\mathcal{I}-R)(\xi^*-\xi)(\nabla\times\psi^*)^{(j)}\|_\omega \\
& + \|(\mathcal{I}-R)\xi(\nabla\times(\psi^*-\psi))^{(j)}\|_\omega + \|(\mathcal{I}-R)\xi(\nabla\times\psi)^{(j)})\|_\omega \\
\le & c\|\psi^*\|_{1,\infty}\|\xi^*-\xi\|_\omega + c\|\xi\|_\infty\|\psi^*-\psi\|_{1,\omega} + c(M^{-r}+N^{-s})\|\xi(\nabla\times\psi)^{(j)}\|_{H^{r,s}_\omega(\Omega)} \\
\le & c(M^{-r}+N^{-s})\|\psi^*\|_{1,\infty}\|\xi\|_{M^{r,s}_\omega(\Omega)} + c(M^{-r}+N^{-s})\|\xi\|_\infty\|\psi\|_{M^{r+1,s+1}_\omega(\Omega)} \\
& + c(M^{-r}+N^{-s})\|\xi(\nabla\times\psi)^{(j)}\|_{H^{r,s}_\omega(\Omega)}.
\end{aligned}
$$

By an argument similar to those in the estimation for $|B_1|$, we get

$$
|B_2| \le \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r}+N^{-2s})\|\xi\|^2_{Y^{r,s,\delta}_{1,\omega}(\Omega)\bigcap W^{3,\infty}(\Omega)}\|\psi\|^2_{Y^{r,s,\delta}_{2,\omega}(\Omega)\cap X^{\alpha,\beta}_{2,\omega}(\Omega)}.
$$

Similarly, we know that for $r>5/4, s>1, \alpha>1/2$ and $\beta>1$,

$$
|A_3| \le \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r}+N^{-2s})\|\xi\|^2_{Y^{r,s,\delta}_{1,\omega}(\Omega)\cap W^{3,\infty}(\Omega)}\|\psi\|^2_{Y^{r,s,\delta}_{2,\omega}(\Omega)\cap X^{\alpha,\beta}_{2,\omega}(\Omega)}.
$$

It is easy to verify that for $r\ge 1, s>1$ and $\varepsilon>0$,

$$
\begin{aligned}
|A_4| &\le \nu c(M^{-r}+N^{-s})\|v\|_{1,\omega}\|\xi\|_{C(0,T;M^{r+1,s+1}_\omega(\Omega))} \\
&\le \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r}+N^{-2s})\|\xi\|^2_{C(0,T;M^{r+1,s+1}_\omega(\Omega))} \\
|A_5| &\le \frac{\nu}{128}\|v\|^2_{1,\omega} + c\tau^3\|\xi\|^2_{H^2(t-\tau,t+\tau;H^{1,1}_\omega(\Omega))}, \\
|A_6| &\le \frac{\nu}{128}\|v\|^2_{1,\omega} + c(M^{-2r}+N^{-2s})\|f_1\|^2_{H^{r,s}_\omega(\Omega)\cap H^s(Q;H^{1/2+\delta}_\omega(I))}, \\
|A_7| &\le \varepsilon\|v\|^2_{1,\omega} + \frac{c}{\varepsilon}(M^{-2r}+N^{-2s})\|\psi\|^2_{M^{r+1,s+1}_\omega(\Omega)}, \\
|A_8| &\le \varepsilon\|v\|^2_{1,\omega} + \frac{c}{\varepsilon}(M^{-2r}+N^{-2s})\|\xi\|^2_{M^{r,s}_\omega(\Omega)}, \\
|A_9| &\le \varepsilon\|v\|^2_{1,\omega} + \frac{c}{\varepsilon}(M^{-2r}+N^{-2s})\|f_2\|^2_{H^{r,s}_\omega(\Omega)\bigcap H^s(Q,H^{1/2+\delta}_\omega(I))}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\|\tilde{\xi}(0)\|^2_\omega \le c(M^{-2r}+N^{-2s})\|\xi_0\|^2_{M^{r,s}_\omega(\Omega)}, \\
&\|\tilde{\xi}(\tau)\|^2_\omega \le c(M^{-2r}+N^{-2s})\|\xi(\tau)\|^2_{M^{r,s}_\omega(\Omega)} + c\tau^4\|\xi\|^2_{H^2(0,T;L^2_\omega(\Omega))}, \\
&\|\xi^*\|_{1,\infty} \le c\|\xi\|_{X^{\alpha,\beta}_\omega(\Omega)}, \quad \|\psi^*\|_{2,\infty} \le c\|\psi\|_{X^{\alpha,\beta}_{2,\omega}(\Omega)}, \quad \alpha>\frac{1}{2}, \beta>1.
\end{aligned}
$$

By an argument as in the proof of Theorem 1, we complete the proof of Theorem 2.

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