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THE MULTI-PARAMETERS OVERRELAXATION METHOD*1)

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Abstract

In this paper, first, we present the comparison theorem and the (generalized) Stein-Rosenberg theorem for the GMPOR method, which improves some recent results[9,11,13]. Second, we also give the convergent theorem of the GMPOR method, which generalizes the corresponding result of [9]. Finally, we provide the real interval such that the generalized extrapolated Jacobi iterative method and the generalized SOR methods simultaneously converge, one of the main results in [1] is extended.

Key words: GMPOR iterative method, convergence, comparison theorem, Stein-Rosenberg theorem

1. Introduction

Recently, many mathematical literatures have provided some new iterative methods for solving the linear system. Kuang[2] presented a two-parameter iterative method called TOR method, which is effective to give the numerical solution of partial differential equations. Wang[10] extended the TOR method to the GTOR methed and improves some results of [3, 11, 12]. In [5], Li also discussed the GTOR method, and extended the corresponding results of [10, 11]. Recently, Song and Dai[9] presented the multi-parameters overrelaxation (MPOR) method, whose specific cases involve the iterative methods mentioned as above. Now, let us make a generalization of the MPOR method.

Let
$$Ax = u, \tag{1.1}$$
where $A = D - \sum_{i=1}^{k} E_i - F$, and D is a nonsingular matrix. Then the generalized multi-parameters overrelaxation (GMPOR) method can be defined by
$$x^{m+1} = L(a_1, \cdots, a_k; b)x^m + v, \quad m = 0, 1, \cdots, \tag{1.2}$$
where x^0 is an initial approximation,
$$v = \Big(D - \sum_{i=1}^{k} a_i E_i\Big)^{-1} bu$$

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and

$$L(a_1,\cdots,a_k;b)=\Big(D-\sum_{i=1}^{k}a_iE_i\Big)^{-1}\Big[(1-b)D+\sum_{i=1}^{k}(b-a_i)E_i+bF\Big], \tag{1.3}$$

which is called the GMPOR iteration matrix, where a_i $i=1,\cdots,k$ and b are independent parameters, D is nonsingular matrix, E_i, $i=1,\cdots,k$ and F are any matrix (In [9] D, E and F respectively nonsingular block diagonal, strictly lower and upper triangular matrices).

Notice that for specific value of the parameters a_i and b, the GMPOR method reduces to the following well-known methods:

$L(0;1)=L_{GJ}$, the iteration matrix of the GJ method (generalized Jacobi method);

$L(1;1)=L_{GGS}$, the iteration matrix of the GGS method (generalized Gauss-Seidel method);

$L(0;b)=L_{GJOR}$, the iteration matrix of the GJOR method (generalized extrapolated Jacobi method);

$L(b;b)=L_{GSOR}$, the iteration matrix of the GSOR method (generalized SOR method);

$L(a;b)=L_{GAOR}$, the iteration matrix of the GAOR method (generalized AOR method).

$L(a_1,a_2;b)=L_{GTOR}$, the iteration matrix of the GTOR method (generalized TOR method).

From the above statement, one can easily understand that the GMPOR method includes the GJ method, GGS method, GJOR method, GSOR method, GAOR method, GTOR method and MPOR method as its specific cases. This paper is organized as follows. In Section 2, we present a comparison theorem and the (generalized) Stein-Rosenberg theorem for the GMPOR method, which improves some recent results[9,11]. Section 3 contains the convergence theorem of the GMPOR method for solving the nonsingular linear system, which extends the corresponding result of [9]. In the final Section, two theorems are given. The first theorem provides a necessary and sufficient condition such that the GMPOR method for solving the singular linear system is convergent. The second theorem reveals the real interval for which the GJOR method and the GSOR method are simultaneously convergent, one of the main results in [1] is generalized. All definitions and notations here are standard and can be found in [8] or [13].

2. Comparison Theorem and Stein-Rosenberg Theorem

Let n be a natural number. By $<n>$ we denote the set $\{1,\cdots,n\}$

Throughout this section we always assume that the following conditions hold:

$$L_i=D^{-1}E_i\geq 0,\ i=1,\cdots,k,\ U=D^{-1}F\geq 0,\quad \text{and } B=\sum_{i=1}^{k}L_i+U=L_{GJ} \tag{2.1}$$

$$\rho\Big(\sum_{i=1}^{k}L_i\Big)<1 \tag{2.2}$$

Lemma 2.1. $L(a_1,\cdots,a_k;b)=(1-b)I+b\Big(I-\sum_{i=1}^{k}a_iL_i\Big)^{-1}\Big[\sum_{i=1}^{k}(1-a_i)L_i+U\Big].$

Proof. The result follows from the direct computation.

Lemma 2.2.

(1) *if* $b_1\ge b_2>0$ *and* $1\ge\lambda_2\ge\lambda_1\ge 0$, *then* $(1-b_1)+b_1\lambda_1\le(1-b_2)+b_2\lambda_2$;

(2) *if* $b_1\ge b_2>0$ *and* $\lambda_1\ge\lambda_2>1$, *then* $(1-b_2)+b_2\lambda_2\le(1-b_1)+b_1\lambda_1$.

Proof. Easy.

Lemma 2.3.[4] *Let* $A\in R^{nn}$, *and let* $A=M_1-N_1=M_2-N_2$ *be both M-splittings of* A. *If* $N_1\ge N_2$, *then one and only of the following statements holds:*

(1) $0\le\rho(M_2^{-1}N_2)\le\rho(M_1^{-1}N_1)<1$.

(2) $\rho(M_2^{-1}N_2)=\rho(M_1^{-1}N_1)=1$.

(3) $\rho(M_2^{-1}N_2)\ge\rho(M_1^{-1}N_1)>1$.

The following is a comparison theorem about two different GMPOR iterative methods and it will be showed by the similar proof with [5].

Theorem 2.1. *Let* $A=D-\sum_{i=1}^{k}E_i-F$ *be a splitting of* A *satisfied the conditions* (2.1) *and* (2.2). *If the parameters satisfy* $(1,\cdots,1;1)\ge(a_1^{(1)},\cdots,a_k^{(1)};\,b_1)\ge(a_1^{(2)},\cdots,a_k^{(2)};b_2)\ge(0,\cdots,0;0)$ *and* $b_i\ne 0$, $i=1,2$, *then*

(1) $1-b_1\le\rho(L(a_1^{(1)},\cdots,a_k^{(1)},b_1))\le\rho(L(a_1^{(2)},\cdots,a_k^{(2)};\,b_2))\le 1-b_2+b_2\rho(B)<1$ *if and only if* $D^{-1}A$ *is a nonsingular M-matrix.*

(2) $\rho(L(a_1^{(1)},\cdots,a_k^{(1)};b_1))=\rho(L(a_1^{(2)},\cdots,a_k^{(2)};b_2))=1$ *if and only if* $D^{-1}A$ *is a singular M-matrix;*

(3) $\rho(L(a_1^{(1)},\cdots,a_k^{(1)};b_1))\ge\rho(L(a_1^{(2)},\cdots,a_k^{(2)};b_1))\ge 1-b_2+b_2\rho(B)>1$ *if and only if* $D^{-1}A$ *is not an M-matrix.*

Proof. We always assume that $M_j=I-\sum_{i=1}^{k}a_i^{(j)}L_i$ and $N_j=\sum_{i=1}^{k}(1-a_i^{(j)})L_i+U$ in the proof of this theorem.

From the hypothesis of this theorem and (2.2) it follows that $D^{-1}A=M_1-N_1=M_2-N_2=I-B$ are all M-splittings of $D^{-1}A$ and satisfy the following condition:

$$0\le N_1\le N_2\le B. \tag{2.3}$$

It follows from Lemma 2.3 that one and only one of the following results holds:

$$0\le\rho(M_1^{-1}N_1)\le\rho(M_2^{-1}N_2)\le\rho(B)<1 \tag{2.4}$$

$$\rho(M_1^{-1}N_1)=\rho(M_2^{-1}N_2)=\rho(B)=1 \tag{2.5}$$

$$\rho(M_1^{-1}N_1)\ge\rho(M_2^{-1}N_2)\ge\rho(B)>1 \tag{2.6}$$

(1): By Lemma 2.1 it is readily to see $\rho(L(a_1^{(1)},\cdots,\,a_k^{(1)};b_1))\ge 1-b_1$. Let $D^{-1}A$ be a nonsingular M-matrix. Then $\rho(B)<1$, and thus (2.4) occurs. It follows from Lemma 2.2 that $(1-b_1)+b_1\rho(M_1^{-1}N_1)\le(1-b_2)+b_2\rho(M_2^{-1}N_2)\le 1-b_2+b_2\rho(B)$. This implies $0\le\rho(L(a_1^{(1)},\cdots,a_k^{(1)};b_1))\le\rho(L(a_1^{(2)},\cdots,a_k^{(2)};b_2))\le 1-b_2+b_2\rho(B)<1$ from

Lemma 2.1. Conversely, let $1-b_2+b_2\rho(B)<1$. Then $\rho(B)<1$. Since $D^{-1}A=I-B$ with $B\geq 0$, $D^{-1}A$ is a nonsingular M-matrix, which establishes (1).

(2): Follows immediately from (2.5).

(3): Let $D^{-1}A$ be not an M-matrix. Since $D^{-1}A=I-B$. we have $\rho(B)>1$, and thus (2.6) occurs. Hence inequality (3) follows from Lemmas 2.1 and 2.2. Conversely, let inequality (3) hold. By the last inequality that $(1-b_2)+b_2\rho(B)>1$, we obtain $\rho(B)>1$. This implies that $D^{-1}A$ is not an M-matrix.

Applying Theorem 2.1, the following generalized Stein-Rosenberg theorem is derived.

Theorem 2.2. *Let* $(0,\cdots,0;0)\leq(a_1,\cdots,a_k;b)\leq(1,\cdots,1;1)$ *and* $b\neq 0$. *Then*

(1) $1-b\leq\rho(L(a_1,\cdots,a_k;b))<1$ *if and only if* $\rho(B)<1$, *in this case, we have* $\rho(L(a_1,\cdots,a_k;b))\leq 1-b+b\rho(B)$. *Forthermore, if* $\rho(B)=0$, *then* $\rho(L(a_1,\cdots,a_k;b))=1-b$.

(2) $\rho(L(a_1,\cdots,a_k;b))=1$ *if and only if* $\rho(B)=1$;

(3) $\rho(L(a_1,\cdots,a_k;b))>1$ *if and only if* $\rho(B)>1$, *in this case, we have* $\rho(L(a_1,\cdots,a_k;b))\geq 1-b+b\rho(B)$.

Proof. Let $(a_1^{(1)},\cdots,a_k^{(1)};b_1)=(a_1^{(2)},\cdots,a_k^{(2)};b_2)=(a_1,\cdots,a_k;b)$. The results follow immediately from Theorem 2.1 and Lemma 2.1.

Remark: From $\rho(L(a_1,\cdots,a_k;b))=1-b$ one can not deduce that $\rho(B)=0$. For example, let

$$A=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}0&1\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\0&0\end{pmatrix}=I-L-U.$$

Then $\rho(L(1,1))=\rho((I-L)^{-1}U)=0$, but $\rho(B)=\rho(L+U)=1$. It is easy to see that Theorem 2.2 improves Theorem 2.6[13] and Theorem 3.4[9].

Corollary 2.1.[11,Theorem 3.1'] *Suppose that matrices* L *and* U *satisfy* $U\geq 0$ *and* $L\geq 0$ *with* $\rho(L)<1$. *Then one and only one of the following mutually exclusive relations is valid:*

(1) $0\leq\rho((I-L)^{-1}U)\leq\rho(B)<1$.

(2) $\rho((I-L)^{-1}U)=\rho(B)=1$.

(3) $\rho((I-L)^{-1}U)\geq\rho(B)>1$.

Proof. We set that $A\doteq I-L-U(=I-B)$, this is a splitting of A and satisfies the conditions (2.1) and (2.2). Applying Theorem 2.2 to $L(1;1)$ we know that this corollary holds.

3. Convergence of the GMPOR Method: the Nonsingular Case

In this section we consider the nonsingular linear system and give a sufficent condition such that the generalized multi-parameters overrelaxation (GMPOR) method is convergent, which extends theorem 3.5[9].

Lemma 3.1.[6] *Let* $A\in R^{nn}$ *be a nonsingular matrix, and* $A=M-N$ *be an* M*-splitting of* A. *Then* $\rho(M^{-1}N)<1$ *if and only if* A *is a nonsingular M-matrix.*

Theorem 3.1 *Let* $A=D-\sum_{i=1}^{k}E_i-F$, D *be nonsingular and* $L_i=D^{-1}E_i\geq 0$ *and*

$U = D^{-1}F \geq 0$, $i \in < k >$, and let $D^{-1}A$ be a nonsingular M-matrix. If

$$0 \leq a_i, b < \frac{2}{1+\rho(B)}, \quad b \neq 0,\ i \in < k > \tag{3.1}$$

where $B = \sum_{i=1}^{k} D^{-1}E_i + D^{-1}F = \sum_{i=1}^{k} L_i + U$, then the GMPOR method converges.

Proof. Let $D^{-1}A$ be a nonsingular M-matrix. Since $D^{-1}A = I - \sum_{i=1}^{k} L_i - U = I - B$ is a nonsingular M-matrix and $B \geq 0$, we have $\rho(B) < 1$. Without loss of generality we may assume that $a_1 \geq a_2 \geq \cdots \geq a_k$. If $a_1 \leq b$, then the result follows from Theorem 2.1 and the same proof as Theorem 3.5[9]. Hence it need only show that the result holds in the case where $1 < b < \dfrac{2}{1+\rho(B)}$ and there is an integer $t \in < k >$ such that $a_1 \geq \cdots \geq a_t > b \geq a_{t+1} \geq \cdots \geq a_k$ or $a_k > b$. The proof of the last case is similar to the first case. Hence it need only show that the result holds in the first case.

Now let $M = I - \sum_{i=1}^{k} a_i L_i$ and $N = (b-1)I + \sum_{i=1}^{t}(a_i - b)L_i + \sum_{i=t+1}^{k}(b - a_i)L_i + bU$, then $N \geq 0$ and

$$|L(a_1, \cdots, a_k; b)| \leq M^{-1}N \tag{3.2}$$

Since $\rho\Big(\sum_{i=1}^{k} a_i L_i\Big) \leq \rho\Big(a_1 \sum_{i=1}^{k} L_i\Big) \leq a_1 \rho\Big(\sum_{i=1}^{k} L_i\Big) \leq a_1\rho(B) < \dfrac{2}{1+\rho(B)}\rho(B) < 1$, M is a nonsingular M-matrix. Hence $M - N = (2-b)I - \sum_{i=1}^{t}(2a_i - b)L_i - \sum_{i=t+1}^{k} bL_i - bU \doteq A'$ is an M-splitting of A'. Let $B' = \sum_{i=1}^{t}(2a_i - b)L_i + \sum_{i=t+1}^{k} bL_i + bU$. Then $B' \geq 0$ and $A' = (2-b)I - B'$. Notice that $b \leq 2a_1 - b$ and $2a_i - b \leq 2a_1 - b$, $i \in < k >$, then we obtain $B' \leq (2a_1 - b)B$. Therefore,

$$\rho(B') \leq \rho((2a_1 - b)B) \leq (2a_1 - b)\rho(B) \tag{3.3}$$

Since $a_1, b \in \Big(1, \dfrac{2}{1+\rho(B)}\Big)$ and $a_1 > b$, $2a_1\rho(B) + b(1-\rho(B)) \leq 2a_1\rho(B) + a_1(1-\rho(B)) \leq a_1(1+\rho(B)) < 2$. Hence $(2a_1 - b)\rho(B) < 2 - b$, which together with (3.3) gives $\rho(B') < 2 - b$. This proves that A' is a nonsingular M-matrix. Hence $A' = M - N$ is an M-splitting of a nonsingular M-matrix. It follows from Lemma 3.1 that $\rho(M^{-1}N) < 1$. From (3.2) one can deduce $\rho(L(a_1, \cdots, a_k; b)) < 1$, which proves that the GMPOR method converges.

Remark: Let $A = D - \Big(\sum_{i=1}^{k} E_i + F\Big)$ be an M-splitting. Then A is a nonsingular M-matrix if and only if $D^{-1}A$ is also a nonsingular M-matrix from Lemma 3.1. Therefore, it is easy to see that the Theorem 3.1 is a generalization of Theorem 3.5[9].

4. Convergence of the GMPOR Method: the Singular Case

In this section we always assume that the linear system (1.1) is singular, i.e., A is a singular matrix. Now we discuss convergence of the GMPOR method and the common convergence inteval of the GJOR method and the GSOR iterative method.

Recall that an M-matrix with property c[13] is the M- matrix with $ind_0(A) \leq 1$, by $ind_\lambda(A)$ we mean the size of the largest Jordan block corresponding to the eigenvalue λ of A[8]. Now we give two lemmas.

Lemma 4.1.[7] *Let H be a nonnegative matrix with $\rho(H) = 1$, and let $T = I - H$ and $H_b = (1 - b)I + bH$. Then the following statments are equivalent:*

(1) *T is an M-matrix with property c.*

(2) *For some $b \in (0, 1)$, H_b is a convergent matrix.*

(3) *For each $b \in (0, 1)$, H_b is a convergent matrix.*

Lemma 4.2. *Let $A \in R^{nn}$, and $A = M - N$ be an M-splitting. Then $M^{-1}A$ is an M-matrix if and only if A is an M-matrix.*

Proof. "$\Leftarrow$": From Theorem 4.5[8] it follows that $\rho(M^{-1}N) = 1$, and thus $M^{-1}A$ is an M-matrix..

"$\Rightarrow$": Let $M^{-1}A$ be an M-matrix. Since $M^{-1}A = I - M^{-1}N$, we obtain $\rho(M^{-1}N) \leq 1$. In order to show the assertion, we consider two cases as follows:

Case 1. If A is irreducible, then $A = M - N$ is an M-splitting of an irreducible matrix A. By Lemma 2.4[8] A is a Z-matrix. Let $A = sI - B$, $s > 0$ and $B \geq 0$ be irreducible. If A is not an M-matrix, then $\rho(B) > s$. By Perron-Frobenius theorem of nonnegative matrix there is a positive vector x such that $Ax = (s - \rho(B))x \ll 0$. Since $M^{-1} > 0$, $M^{-1}Ax \ll 0$. This implies that $M^{-1}Nx \gg x$. Applying Perron-Frobenius theorem to the nonnegative matrix $(M^{-1}N)^T$, it is readily to show that $\rho(M^{-1}N) > 1$, which contradicts the hypothesis of the lemma. Thus A is an M-matrix.

Case 2. If A is reducible, then there exists a permutation matrix P such that

$$PAP^T = (A_{ij}), \tag{4.1}$$

where A_{ii} is irreducible, $i \in < s >$ and $A_{ij} = 0$ $(i > j)$. Since an M-splitting is graph compatible (see [8, Lemma 2.4]), $PMP^T = (M_{ij})$ and $PNP^T = (N_{ij})$ are both block upper triangular matrices partitioned in the same way as (4.1). Hence $A_{ii} = M_{ii} - N_{ii}$ is an M-splitting of an irreducible matrix with $\rho(M_{ii}^{-1}N_{ii}) \leq \rho(M^{-1}N) \leq 1$. By Case 1 and Lemma 3.1, A_{ii} is an M-matrix, $i \in < s >$. Since $PAP^T = (A_{ij})$ is a block upper triangular Z-matrix whose all diagonal blocks are M-matrices, PAP^T (and hence A) is an M-matrix.

Theorem 4.1. *Let $A = D - \sum_{i=1}^{k} E_i - F$ be a splitting satisfied the conditions* (2.1) *and* (2.2). *If a_i and b satisfy* (3.1), *$i \in < k >$, then the GMPOR iterative method* (1.2) *converges for any initial approximation vector x^0 if and only if $D^{-1}A$ is an M-matrix with property c.*

Proof. Let $D^{-1}A$ be a singular M-matrix with property c, i.e., $ind_0(D^{-1}A) = 1$. Hence $\rho(B) = 1$. From (3.1) it follows that $0 \leq a_i$, $b < 1$ and $b \neq 0$, $i \in < k >$. This implies that $D^{-1}A = \Big(I - \sum_{i=1}^{k} a_i L_i\Big) - \Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)$ is an M-splitting of a singular M-matrix. It follows from Theorem 4.5[8] that $\rho\Big(\Big(I - \sum_{i=1}^{k} a_i L_i\Big)^{-1}\Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)\Big) = 1$ and $ind_1\Big(\Big(I - \sum_{i=1}^{k} a_i L_i\Big)^{-1}\Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)\Big) = ind_0(D^{-1}A) = 1$. Hence $I - \Big(I - \sum_{i=1}^{k} a_i L_i\Big)^{-1}\Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)$ is an M-matrix with property c. From Lemma 2.1 and Lemma 4.1 we conclude that the GMPOR iteration matrix $L(a_1, \cdots, a_k; b)$ is convergent.

Conversely, assume that the GMPOR iteration matrix $L(a_1, \cdots, a_k; b)$ is convergent. Since $D^{-1}A = I - B$ is a singular Z-matrix, we obtain $\rho(B) \geq 1$. Hence $a_i, b \in [0, 1)$ and $b \neq 0$ from (3.1). It is easy to show that $\rho\Big(\Big(I - \sum_{i=1}^{k} a_i L_i\Big)^{-1}\Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)\Big) = 1$ from Lemma 2.1. It follows from Lemma 4.1 that $\tilde{A} \doteq I - \Big(I - \sum_{i=1}^{k} a_i L_i\Big)^{-1}\Big(\sum_{i=1}^{k}(1-a_i)L_i + U\Big)$ is an M-matrix with property c. Since $\Big(I - \sum_{i=1}^{k} a_i L_i\Big)\tilde{A} = \Big(I - \sum_{i=1}^{k} a_i L_i\Big) - \Big(\sum_{i=1}^{k}(1 - a_i)L_i + U\Big)$ is an M-splitting and $\tilde{A}$ is an M-matrix with property c, from Lemma 4.2 we conclude that $\Big(I - \sum_{i=1}^{k} a_i L_i\Big)\tilde{A}$ is also an M-matrix. This means that $D^{-1}A$ is an M-matrix since $D^{-1}A = \Big(I - \sum_{i=1}^{k} a_i L_i\Big)\tilde{A}$. It follows from Theorem 4.5[8] that $ind_0(D^{-1}A) = ind_0(\tilde{A}) = 1$. Hence $D^{-1}A$ is an M-matrix with property c, which completes the proof of the theorem.

Let $A = D - E - F$ be a splitting of A. For simplicity, by J_b and S_b we denote the (generalized) extrapolated Jocobi (JOR) iteration matrix L_{GJOR} and the (generalized) successive overrelaxation (SOR) iteration matrix L_{GSOR}, respectively. Now, we deal with the common convergence inteval of the GJOR method and the GSOR method.

Theorem 4.2. *Let A be a Z-matrix and $A = D - (E + F)$ be a regular splitting with $E \geq 0$ and $F \geq 0$. If A is an M-matrix with property c, then for all b in the real inteval* $[0, 1)$,

(1) $\rho(bD^{-1}E) < 1$ *and*

(2) S_b *and* J_b *simultaneously converge.*

Proof. If $b = 0$, then $S_b = J_b = I$. The proof of the theorem is trivial. Now assume that $b \neq 0$.Let A be an M-matrix with property c. If A is nonsingular, then Theorem 2.2.3[13] guarantees $\rho(D^{-1}(E + F)) < 1$, hence $D^{-1}A$ is a nonsingular M-matrix. Since

$$J_b = (1 - b)I + bJ_1 \tag{4.2}$$

for any $b \in (0, 1)$ we have $\rho(J_b) < 1$. Because of $D^{-1}E \leq D^{-1}(E + F)$, we obtain

$\rho(D^{-1}E) \leq \rho(D^{-1}(E+F)) < 1$, and hence $\rho(bD^{-1}E) < 1$. From $S_b = L(b,b)$ and Theorem 3.1 it follows that S_b converges for $b \in (0,1)$. If A is singular, then from Lemma 2[7] one can deduce that $D^{-1}A$ is a singular M-matrix (and hence $\rho(D^{-1}(E+F)) = 1$) and $ind_0(D^{-1}A) = 1$. It follows from Lemma 4.1 and (4.2) that J_b is convergent. Since $bD^{-1}E \leq bD^{-1}(E+F)$, $\rho(bD^{-1}E) \leq b\rho(D^{-1}(E+F)) \leq b < 1$ for all $b \in (0,1)$, this proves that (1) holds. It is readily to obtain $S_b = (1-b)I + bH$, where $H = (I-bL)^{-1}((1-b)L+U)$, $L = D^{-1}E$ and $U = D^{-1}F$. From (1) and the hypothesis it follows that $D^{-1}A = (I-bL) - ((1-b)L+U)$ is an M-splitting of an M-matrix with property c, and hence $\rho(H) = 1$ and $ind_0(I-H) = ind_0(D^{-1}A) = 1$ from Theorem 4.5[8]. This implies that $I-H$ is an M-matrix with property c. It follows from Lemma 4.1 that S_b converges. Therefore, for any $b \in [0,1)$ S_b and J_b simultaneously converge, which proves (2).

Remark: Theorem 4.2 is a generalization of Theorem 3.4[1], in which it only consider the specific cases that D, E and F in the splitting of A are respectively the diagonal, the strictly lower triangular and the strictly upper triangular parts of A. We also remark that the real inteval $[0,1)$ such that S_b and J_b simultaneously converge is sharp (see [1, p.194]).

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