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# NOTES ON REFINABLE FUNCTIONS*1)

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**Abstract**

In this paper some properties of refinable functions and some relationships between the mask symbol and the refinable functions are studied. Especially, it is illustrated by examples that the linear spaces formed by the translates over the lattice points of refinable functions may contain polynomial spaces of degree higher than the smooth order of the corresponding refinable functions.

*Key words*: Mask, Symbol, Refinable Function.

## 1. Introduction

It is well-known that refinable functions play an important role in the studying of wavelet. Usually, one hopes that refinable functions have some particular properties such as smoothness and integrability. In this note, the zeros of an integrable refinable function are obtained. In particalar by examples one shows that the linear space associating the translates over the lattice points of a refinable function could include polynomial space of degree higher than its smooth order.

Let $s$ be a positive integer and let $\mathbf{R}^s$ (resp. $\mathbf{C}^s$) be the $s$-dimensional real (complex) space equipped with the norm $|\cdot|$ given by

$$|x| = \Big(\sum_{j=1}^{s} |x_j|^2\Big)^{\frac{1}{2}} \quad \text{for} \quad x = (x_1, \cdots, x_s) \in \mathbf{R}^s \quad (\text{resp.} \quad \mathbf{C}^s).$$

By a *mask* $\mathbf{a} = \{a_\alpha;\ \alpha \in \mathbf{Z}^s\}$ we mean a mapping of finitely supported from $\mathbf{Z}^s$ to $\mathbf{C}$. For $1 \le p \le \infty$, we use $L_p = L_p(\mathbf{R}^s)$ to denote the Banach space of all functions $f$ on $\mathbf{R}^s$ such that

$$\|f\|_p := \Big(\int_{\mathbf{R}^s} |f(x)|^p\Big)^{\frac{1}{p}} < \infty,$$

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where $f$ could be complex valued.

A function on $\mathbf{R}^s$ is called refinable if it satisfies

$$\varphi(x) = \sum_{\alpha \in \mathbf{Z}^s} a_\alpha \varphi(2x - \alpha), \quad x \in \mathbf{R}^s \tag{1}$$

for a mask $\mathbf{a} = \{a_\alpha, \alpha \in \mathbf{Z}^s\}$, and

$$p(z) = 2^{-s} \sum_{\alpha \in \mathbf{Z}^s} a_\alpha z^\alpha \tag{2}$$

is called the *symbol* of the mask $\mathbf{a}$.

$\varphi \in \mathcal{L}^p$ is called $L_p$-stable if there exists a positive constant $c$ of independent with $\mathbf{a}$ and $\varphi$ such that

$$\|\mathbf{a}\|_p \leq c \Big\| \sum_{\alpha \in Z^s} a_\alpha \varphi(x - \alpha) \Big\|_p,$$

where by $\varphi \in \mathcal{L}^p$ we mean

$$\varphi^0 := \sum_{\alpha \in \mathbf{Z}^s} |\varphi(x - \alpha)| \in L_p([0,1]^s)$$

(see [7]) and by $l_p := l_p(\mathbf{Z}^s)$ we denote the Banach space of all elements $a$ defined on $\mathbf{Z}^s$ for which

$$\|a\|_p = \Big( \sum_{\alpha \in \mathbf{Z}^s} |a_\alpha|^p \Big)^{\frac{1}{p}} < \infty$$

equipped with the norm $\| \cdot \|$.

**Note 1:** Let $\varphi$ be a continuous function with compact support. Then the (see [2] Theorem 4.1 and Theorem 4.2) the $L_p-$stability of $\varphi$ for some $1 \leq p \leq \infty$ implies that of $\varphi$ for all $1 \leq p \leq \infty$. Furthermore, it is easy to show that $L_p$-stability of the function $\varphi$ is equivalent to the $l_\infty$ linear independence, where by $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ being $l_p$ linearly dependent we mean that there exists $0 \neq \lambda \in l_p$ such that

$$\sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha \varphi(x - \alpha) \equiv 0, \ x \in \mathbf{R}^s.$$

In fact, if $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ are $l_\infty$ linearly independent, then $\varphi$ is $L_\infty$-stable (see [2] page 24). In other way round, it is easy to see that $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ being $l_\infty$ linearly dependent implies $\sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha \varphi(x - \alpha) \equiv 0$ for some $0 \neq \lambda \in l_\infty$. Therefore, $\varphi$ could not be $L_\infty$-stable.

Finally, in this note we will use $\hat{f}(u) = \int_{\mathbf{R}^s} f(x) e^{-iu \cdot x} dx$ ($u \in \mathbf{C}^s$) to denote the Fourier-Laplace transform of $f$, where for $u = (u_1, \cdots, u_s) \in \mathbf{C}^s$ and $x = (x_1, \cdots, x_s) \in \mathbf{R}^s$, $u \cdot x = \sum_{j=1}^s u_j x_j$. Restricted to $\mathbf{R}^s$, $\hat{f}$ become the Fourier transform of $f$.

## 2. Main Results

Our first result will be about the Fourier-Laplace transform of a refinable function. From [2], we know that if $\varphi \in L_1$ is refinable, then $\hat{\varphi}$ is an entire function and

$\sum_{\alpha\in\mathbf{Z}^s} a_\alpha = 2^s$ if $\hat{\varphi}(0)\neq 0$, where $\mathbf{a}=\{a_\alpha, \alpha\in\mathbf{Z}^s\}$ is the mask of $\varphi$, and that for a mask $\mathbf{a}$, there exists at most one function $\varphi\in L_1$, $\hat{\varphi}(0)=1$, such that (1) holds. Unless otherwise stated we assume that $\hat{\varphi}(0)=1$ if $\hat{\varphi}\in L_1$. Using these results, we could get the following

**Theorem 1.** *If $\varphi$ is refinable and for some $p, 1\le p\le 2$, it holds*

$$\prod_{j=1}^{\infty} p(e^{-\frac{u}{2^j}i})\in L_p, \tag{3}$$

*then $\varphi\in L_1$ and $\hat{\varphi}=\prod_{j=1}^{\infty} p(e^{-\frac{u}{2^j}i})$.*

Before proving Theorem 1, we introduce a result from [8] (Theorem 3.2.5).

**Lemma 1.** *If entire function $g(u)=g(x+iy)$ of exponential type $\mu=(\mu_1,\cdots,\mu_s)>0$, i.e., for every $\varepsilon>0$ there exists a positive number $A_\varepsilon$ such that $|g(z)|\le A_\varepsilon e^{\sum_{j=1}^{s}(\delta_j+\varepsilon)|z_j|}$ holds for all $z=(z_1,\cdots,z_s)\in\mathbf{C}^s$, belongs to the class $L_p(\mathbf{R}^s)$ where $1\le p<\infty$, then*

$$\lim_{|x|\to\infty} g(x)=0, \quad x\in\mathbf{R}^s.$$

**The proof of Theorem 1:** Firstly, it is easy to prove that

$$f:=\prod_{j=1}^{\infty} p(e^{-\frac{u}{2^j}i}) \tag{4}$$

is an entire function of exponential type $\Delta=(\delta_1,\cdots,\delta_s)$, where $\delta_j=\max\{|\alpha_j|, \alpha=(\cdots,\alpha_j,\cdots)\in \text{supp}\mathbf{a}\}$. In fact, set $u_j=x_j+y_ji$, $x_j,y_j\in\mathbf{R}$. Noting $\sum_{\alpha\in\mathbf{Z}^s} a_\alpha=2^s$, it holds

$$|p(e^{-\frac{u}{2^j}i})|\le e^{\frac{\sum_{j=1}^{s}\delta_j|y_j|}{2^j}}\min\Big\{2^{-s}\sum_{\alpha\in\mathbf{Z}^s}|a_\alpha|,\quad 1+2^{1-s-j}\delta|u|\sum_{\alpha\in\mathbf{Z}^s}|a_\alpha|\Big\}, \tag{5}$$

where $\delta=\max\limits_j \delta_j$. From (4) and (5), we obtain

$$|f(u)|\le ce^{\sum_{j=1}^{s}\delta_j|y_j|}\Big(2^{-s}\sum_{\alpha\in\mathbf{Z}^s}|a_\alpha|\Big)^{\log_2(2^{1-s}\delta|u|\sum_{\alpha\in\mathbf{Z}^s}|a_\alpha|)}\le ce^{\sum_{j=1}^{s}\delta_j|y_j|}|u|^\beta, \tag{6}$$

where $c$ and $\beta$ are some positive constants depending only on the mask $\mathbf{a}$ and they could be different when they appear in different places. This shows that $f$ is an entire function of type $\Delta$. In another hand, since $f=\prod_{j=1}^{\infty}p(e^{-\frac{u}{2^j}i})\in L_p$ for some $p, 1\le p\le 2$, we know from Lemma 1 that

$$f=\prod_{j=1}^{\infty} p(e^{-\frac{u}{2^j}i})\in L_2. \tag{7}$$

Thus, there is $g\in L_2$ such that $\hat{g}=f$. By the uniqueness of Fourier Transform, we know

$$\varphi=g\in L_2, \tag{8}$$

i.e., $\hat{\varphi} = f$. By (8), to complete Theorem 1, we only need to prove that $\varphi$ has a bounded support, and this could be obtained from L. Schwartz Theorem [8] (Theorem 3.1.5), since $\hat{\varphi} = f$ is an entire function of exponential type $\Delta = (\delta_1, \cdots, \delta_s)$.■

Our next result is about the zero point set of the Fourier-Laplace transform of a refinable function. From the well-known result, there exists unique irreducible algebraic polynomials $p_j, j = 1, \cdots, m$ with $p_j(1) = 1$ such that

$$p(z) = z^{\sigma} \prod_{j=1}^{m} p_j^{r_j}(z), \tag{9}$$

where $\sigma = (\alpha_1^m, \cdots, \alpha_s^m)$, $\alpha_j^m = \min\{\alpha_j, (\cdots, \alpha_j, \cdots) \in \text{supp}\mathbf{a}\}$, $r_j \geq 1$, $j = 1, \cdots, m$, are integers. In the following, we denote by $Z_f := \{z \in \mathbf{C}^s | \ f(z) = 0\}$ the zero point set of $f$. For $\hat{\varphi}$, it holds

**Theorem 2.** *If $\varphi \in L_1$ is refinable, then*

*i.* $\hat{\varphi}(u) = e^{iu\cdot\sigma} \hat{\varphi}_1^{r_1}(u) \cdots \hat{\varphi}_m^{r_m}(u)$, *where* $\hat{\varphi}_l(u) = \prod_{j=1}^{\infty} p_l(e^{-\frac{u}{2^j}i})$.

*ii.* $Z_{\hat{\varphi}_l} = \{u = 2^j(2k\pi - \theta) | \ j \geq 1, k \in \mathbf{Z}^s, e^{\theta i} \in Z_{p_l}\}$, $l = 1, \cdots, m$, *and as a clear corollary, we have* $Z_{\hat{\varphi}} = \bigcup_{j=1}^{m} Z_{\hat{\varphi}_j}$.

*Proof.* Since $p_l(1) = 1$, it is easy to prove that the sequence $\hat{\varphi}_{l,k}(u) = \prod_{j=1}^{k} p_l(e^{-\frac{u}{2^j}i})$, $k = 1, 2, \cdots$ converge uniformly to $\hat{\varphi}_l(u) = \prod_{j=1}^{\infty} p_l(e^{-\frac{u}{2^j}i})$ on each compact domain of $\mathbf{C}^s$. Therefore,

$$\begin{aligned}\hat{\varphi}(u) &= \lim_{k\to\infty} \prod_{j=1}^{k} p(e^{-\frac{u}{2^j}i}) = \lim_{k\to\infty} e^{-i(\sum_{j=1}^{k} \frac{u}{2^j})\cdot\sigma} \hat{\varphi}_{1,k}^{r_1}(u) \cdots \hat{\varphi}_{m,k}^{r_m}(u) \\ &= e^{-iu\cdot\sigma} \hat{\varphi}_1^{r_1}(u) \cdots \hat{\varphi}_m^{r_m}(u)\end{aligned}$$

and $i$ follows. The proof of $ii$. If $\hat{\varphi}_l(u) = 0$, we will prove that there exists some index $j$ such that

$$p_l(e^{-\frac{u}{2^j}i}) = 0. \tag{10}$$

To this end, we assume that $p_l(z) = \sum_{|\alpha|\leq N} b_\alpha z^\alpha$, where $N$ is a positive integer. Supposing that $u_0$ is a zero of $\hat{\varphi}(u)$, similar to (6), for sufficiently large number $k$ and $|u| \leq |u_0|$, we have

$$\Big| \prod_{j=k}^{\infty} p(e^{-\frac{u}{2^j}i}) \Big| = \prod_{j=k}^{\infty} \Big| 1 + \sum_{|\alpha|\leq N} b_\alpha (e^{-\frac{u\cdot\alpha}{2^j}i} - 1) \Big| \geq \prod_{j=k}^{\infty} \Big( 1 - \frac{N'}{2^j} \Big) \geq 1 - \frac{2N'}{2^k}, \tag{11}$$

where $N' = |u_0| N e^{|u_0|N} \sum_{|\alpha|\leq N} |b_\alpha|$. Since $|\prod_{j=k}^{\infty} p(e^{-\frac{u}{2^j}i})| > 0$ when $2^k > 2N'$, from (11) we know (10) holds for some $j$, and $ii$ is an obvious conclusion.■

From Theorem 2, we know that if $2\pi u \in Z_{\hat{\varphi}_{l_1}} \bigcup \cdots \bigcup Z_{\hat{\varphi}_{l_t}}$ and if $z = 2\pi u$ is a zero point of order $q_{u,j}$ of $\hat{\varphi}_{l_j}$, then $z = 2\pi u$ is a zero point of order at least $r_u := \sum_{j=1}^{t} q_{u,j} r_{l_j}$ of $\hat{\varphi}$. By setting

$$r_\varphi := \min_{\alpha\in\mathbf{Z}^s} r_\alpha, \tag{12}$$

we obtain the following usefulness

**Corollary 1.** *If $\varphi \in L_1$ is refinable, then*

$$D^\beta \hat{\varphi}(2\alpha\pi) = 0, \quad |\beta| \le r_\varphi - 1, 0 \ne \alpha \in \mathbf{Z}^s. \tag{13}$$

In fact, according to [2], we know

$$\hat{\varphi}(2\alpha\pi) = 0, \quad 0 \ne \alpha \in \mathbf{Z}^s. \tag{14}$$

Thus, Corollary 1 could be directly obtained from Theorem 2. The following lemma ([4]) is usful in this paper.

**Lemma 2.** *Let $\phi$ is of compact support and integrable, and let $V_m := span\{\phi(2^m \cdot -\alpha);\ \alpha \in \mathbf{Z}^s\}$. Then $\Pi_r \subset V_m$, $m \ge 0$, if $D^\beta\hat{\varphi}(2\alpha\pi) = 0$, $|\beta| \le r$, $0 \ne \alpha \in \mathbf{Z}^s$, where $\Pi_d$ stands for all polynomials of total degree $\le d$.*

From corollary 1 and Lemma 2, we have

$$\Pi_{r_\varphi - 1} \subset V_m := \text{span}\ \{\varphi(2^m \cdot -\alpha), \alpha \in \mathbf{Z}^s\}, \quad m \ge 0.$$

Recall from [2] Theorem 8.4 that, whenever $\varphi \in W^{r,1}$ (Sobolev space) is refinable, then the space span $\{\varphi(\cdot - \alpha), \alpha \in \mathbf{Z}^s\}$ contains all polynomials of degree $\le r$. We will illustrate by examples that there does exist refinable function $\varphi$ such that $\varphi \notin W^{r_\varphi - 1,1}$. Before giving out such examples, we will particularly study some properties of refinable functions in the case of $s = 1$. For $s = 1$, we denote by

$$p(z) = \frac{1}{2} z^{-a} \sum_{j=-a}^{b} a_j z^{j+a} = z^{-a}\Big(\frac{z - z_1}{1 - z_1}\Big)^{r_1} \cdots \Big(\frac{z - z_m}{1 - z_m}\Big)^{r_m} := z^{-a} p_1^{r_1}(z) \cdots p_m^{r_m}(z),$$

where $z_i \ne z_j$ when $i \ne j$.

It is convenient to set

$$z_j = e^{\theta_j i}, \quad \theta_j = \theta_j' + \theta_j'' i, \quad \theta_j', \theta_j'' \in \mathbf{R},\ 0 \le \theta_j' < 2\pi. \tag{15}$$

From Theorem 2, we have

**Corollary 2.** *If $\varphi \in L_1$ is refinable, then*

*i.* $\hat{\varphi}(u) = e^{\frac{a-b}{2} ui} \hat{\varphi}_1^{r_1}(u) \cdots \hat{\varphi}_m^{r_m}(u)$ *where* $\hat{\varphi}_l(u) = \prod_{j=1}^{\infty} \frac{\sin\left(\frac{\theta_l}{2} + \frac{u}{2^{j+1}}\right)}{\sin \frac{\theta_l}{2}}$.

*ii.* $Z_{\hat{\varphi}_l} = \{u = 2^j(2k\pi - \theta_l) |\ j \ge 1, k \in \mathbf{Z}\}$, $l = 1, \cdots, m$ *and* $Z_{\hat{\varphi}} = \bigcup_{j=1}^m Z_{\hat{\varphi}_j}$.

*iii.* *If $z = -1$ is a zero of order $r$ of $p(z)$, then*

$$D^\beta \hat{\varphi}(2\pi k) = 0, 0 \le \beta \le r - 1, 0 \ne k \in \mathbf{Z}.$$

*ii can be immediately obtained from Theorem 2. To prove i by using Theorem 2, we need only to note*

$$\frac{e^{-\frac{u}{2^j}} - z_j}{1 - z_j} = \frac{\sin\left(\dfrac{\theta_j}{2} + \dfrac{u}{2^{j+1}}\right)}{\sin \dfrac{\theta_j}{2}} e^{-\frac{u}{2^{j+1}}}.$$

*Now we prove iii. Assume $z_m = -1$, then $\theta_m = \pi$. Therefore,*

$$\prod_{j=1}^{\infty} \frac{\sin\left(\frac{\theta_m}{2} + \frac{u}{2^{j+1}}\right)}{\sin\frac{\theta_m}{2}} = \lim_{k\to\infty} \prod_{j=1}^{k} \cos\frac{u}{2^{j+1}} = \lim_{k\to\infty} \frac{\sin\frac{u}{2}}{2^k \sin\frac{u}{2^{k+1}}} = \frac{\sin\frac{u}{2}}{\frac{u}{2}}. \tag{16}$$

*The assertion of iii is held from i and* (16).

**Examples:** From now on, we will use obtained results to construct examples of refinable function which generate linear function spaces of containing polynomials of degree higher than their smooth order, and begin with univariate case.

In $i$ of Corollary 2, we set $r_1 = r$, $m = 2$, $r_2 = 1$, $\theta_1 = \pi$, and $\theta_2/2 = \theta$, and let $\sin\theta = 2^{-p}$, $p = k + 1/2 - \varepsilon$, $0 < \theta < \pi/2$, for some positive integer $k$ and for some positive number $\varepsilon$ of satisfying

$$0 < \varepsilon < \frac{1}{2}\left(1 + \log_2\left(\frac{3}{4} - \frac{\sqrt{2}}{8}\right)\right) < \frac{1}{4}. \tag{17}$$

Then, $\hat{\varphi}(2u) = e^{(a-b)ui}\left(\frac{\sin u}{u}\right)^r \prod_{j=1}^{\infty} \frac{\sin\left(\theta + \frac{u}{2^j}\right)}{\sin\theta}$. Since $|e^{(a-b)ui}| = 1$ when $u$ is real, we need only to discuss

$$f(u) := \left(\frac{\sin u}{u}\right)^r \prod_{j=1}^{\infty} \frac{\sin\left(\theta + \frac{u}{2^j}\right)}{\sin\theta}. \tag{18}$$

According to the assumption of $\theta$, it holds

$$\left|\frac{\sin\left(\theta + \frac{u}{2^j}\right)}{\sin\theta}\right| \le \min\{2^p, \quad 1 + |u|2^{p-j}\}. \tag{19}$$

Thus, similar to (6), we derive

$$\left|\prod_{j=1}^{\infty} \frac{\sin\left(\theta + \frac{u}{2^j}\right)}{\sin\theta}\right| \le c|u|^p, \tag{20}$$

where $c$ is a constant of depending only on $p$. From (18) and (20), it is concluded that

$$|f(u)| \in L_2 \tag{21}$$

when $r = k + 1$, since $|f(u)| \le c\frac{|\sin u|}{|u|^{\frac{1}{2}+\varepsilon}}$ for real number $u$, where $c$ is also a constant of depending only on $p$. From Theorem 1, $\varphi \in L_1$. Next, we will prove $f(u) \notin H^1$ ($H^1$ is the Sobolev space) for all positive integer $k$. It is well-known that

$$f \in H^\alpha \Longleftrightarrow (1 + |u|)^\alpha \hat{f}(u) \in L_2. \tag{22}$$

From Lemma 1, to prove $f(u) \notin H^1$, we need only to prove that there is some sequence $\{u_n\}_1^\infty (u_n \to \infty)$ such that

$$|u_n f(u_n)| \to +\infty, \quad n \to \infty, \tag{23}$$

since $uf(u)$ is also an entire function and has the same exponential type as $f(u)$.

To this end, we choose

$$u_n = \frac{2^{2n}\pi}{3}. \tag{24}$$

Then

$$\left|\sin\left(\theta + \frac{u_n}{2^j}\right)\right| = \begin{cases} \cos\left(\theta + \frac{\pi}{6}\right), & j = 2d+1, \quad 0 \le d \le n-1, \\ \sin\left(\theta + \frac{\pi}{3}\right), & j = 2d, \qquad 1 \le d \le n, \\ \sin\left(\theta + \frac{\pi}{3 \times 2^{j-2n}}\right), & j \ge 2n+1, \end{cases} \tag{25}$$

From above equality, we further obtain

$$\begin{aligned} \left|\prod_{j=1}^{\infty} \frac{\sin\left(\theta + \frac{u_n}{2^j}\right)}{\sin\theta}\right| &= \left(2^{2p}\cos\left(\theta + \frac{\pi}{6}\right)\sin\left(\theta + \frac{\pi}{3}\right)\right)^n \prod_{j=0}^{\infty} \frac{\sin\left(\theta + \frac{\pi}{3 \cdot 2^j}\right)}{\sin\theta} \\ &\ge \left(2^{2p}\cos\left(\theta + \frac{\pi}{6}\right)\sin\left(\theta + \frac{\pi}{3}\right)\right)^n \ge c u_n^\tau, \end{aligned} \tag{26}$$

where $\tau = p + \frac{1}{2}\log_2\left[\cos\left(\theta + \frac{\pi}{6}\right)\sin\left(\theta + \frac{\pi}{3}\right)\right]$ and $c$ is a positive constant depending only on $p$. In addition, in the first step we have used the assumption of $\theta$ that $\sin\theta = 2^{-p}$. According to the definations of $\tau, p$ and $\theta$, we have

$$\begin{aligned} \tau &= p + \frac{1}{2}\log_2 \cos\left(\theta + \frac{\pi}{6}\right)\sin\left(\theta + \frac{\pi}{3}\right) = k + \frac{1}{2} - \varepsilon + \frac{1}{2}\log_2\left(\frac{3}{4} - \sin^2\theta\right) \\ &= k + \frac{1}{2} - \varepsilon + \frac{1}{2}\log_2\left(\frac{3}{4} - 2^{-2k-1+2\varepsilon}\right) > k + \frac{1}{2} - \varepsilon + \frac{1}{2}\log_2\left(\frac{3}{4} - 2^{-2k-\frac{1}{2}}\right) > k. \end{aligned} \tag{27}$$

Therefore, from (18) and (26) we conclude

$$|u_n f(u_n)| > c'|u_n|^{\tau - k} \longrightarrow +\infty, \quad n \to \infty,$$

since, according to (27), $\tau - k > 0$, where $c'$ is still a positive constant depending only on $p$. And the proof of (23) is completed, i.e., $\varphi \notin H^1$.

**Note 2:** From (18) and Corollary 1, this example demonstrates that for any positive integer $k$, no matter how large it is, there always exists a refinable function $\varphi$, $\varphi \in L_2$ ($\varphi \in L_1$ as well according to Theorem 1) and $\varphi \notin H^1$, such that the space $\text{span}\{\varphi(\cdot - \alpha) \colon \alpha \in \mathbf{Z}\}$ contains all polynomials of degree $\le k$.

Let $\varphi$ be the same as above. Then, $f(x_1, \cdots, x_s) = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_s)$, $f \in L_2$ but $f \notin H^1$. It is clear that the space $\text{span}\{f(\cdot - \alpha) \colon \alpha \in \mathbf{Z}^s\}$ contains all polynomials of degree $\leq k$ no matter how large $k$ is. Certainly, the more important case is of non-product form. The following example shows that there even exists non-product form refinable functions which belong to $L_2$ but not in $H^1$, such that the space span$\{f(\cdot-\alpha):\alpha\in\mathbf{Z}^s\}$ contains all polynomials of degree $\leq k$ no matter how large $k$ is. In fact, we set

$$\hat{f}(u_1,\cdots,u_s)=\Big(\frac{\sin u_1}{u_1}\frac{\sin u_2}{u_2}\cdots\frac{\sin u_s}{u_s}\Big)^{k+1}\prod_{j=1}^{\infty}\frac{\sin\Big(\theta+\dfrac{u_1+u_2+\cdots+u_s}{2^j}\Big)}{\sin\theta},$$

where $\sin\theta=2^{-p}$, $p=k+\frac{1}{2}-\varepsilon$, the same as the first example. Obviously, we have $D^\beta\hat{f}(2\alpha\pi)=0$, $0\neq\alpha\in\mathbf{Z}^s$, $|\beta|\leq k$. By Corollary 1 and the results in [4], we know that the space span$\{f(\cdot-\alpha)$: $\alpha\in\mathbf{Z}^s\}$ contains all polynomials of degree $\leq k$. What left for us is to prove $f\in L_2$ but $f\notin H^1$. $f\in L_2$ could be derived from $\hat{f}\in L_2$, since

$$\Big|\prod_{j=1}^{\infty}\frac{\sin\Big(\theta+\dfrac{u_1+u_2+\cdots+u_s}{2^j}\Big)}{\sin\theta}\Big|\leq c(|u_1|^p+|u_2|^p+\cdots+|u_s|^p),$$

where $c$ is constant depending only on $p$ and $s$. And $f\notin H^1$ can be obtained similarly from $U_n\hat{f}(U_n)\longrightarrow+\infty$, $n\to\infty$, where $U_n=(u_n,0,\cdots,0)$ and $u_n$ is defined as (24).

**Note 3:** All the above example functions don't belong to the Sobolev space $W^{1,1}$. If $f'\in L_1$, then

$$\hat{f'}(u)=iu\hat{f}(u).$$

This means $u\hat{f}(u)\in L_\infty$. This is contradicting with (23).

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